

# Regular Rotating Vacuum Localized Structures in General Relativity

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## Abstract

We investigate rotating Vacuum Localized Structures (VLS), extending earlier work on static vacuum-supported gravitational configurations to include angular momentum. VLS are described phenomenologically by an anisotropic vacuum stress–energy tensor with positive energy density and vacuum-like negative radial pressure, leading to regular, self-gravitating configurations without conventional matter sources. We show that such configurations can consistently support rotation while remaining regular and free of horizons within the slow-rotation approximation.

Rotation is treated perturbatively, allowing for physically significant linear velocities while maintaining control of the expansion. In the weak-field regime, a Poisson-based analysis is used to derive the rotating configuration and to compute the associated radial and tangential pressure profiles explicitly. These profiles are shown to remain well behaved throughout the structure and to reduce smoothly to the static limit in the absence of rotation.

From an astrophysical perspective, rotating VLS provide a potential alternative to particle dark matter, as their gravitational effects arise from vacuum structure rather than from baryonic or exotic matter. More generally, they offer a framework for modeling compact, rotating, vacuum-dominated objects in general relativity without curvature singularities.

## Keywords

Exact solutions of Einstein equations; anisotropic stress–energy tensor; rotating configurations; vacuum structures; dark matter alternatives

## 1. Introduction

Self-gravitating, non-baryonic solutions of the Einstein field equations have been studied in a variety of contexts, ranging from Wheeler’s original concept of gravitational geons [1], to solitonic and equilibrium configurations such as boson stars [2]. In a different astrophysical direction, modified or alternative gravitational frameworks have also been explored as explanations for galaxy-scale dynamics without invoking particle dark matter [3]. While these approaches differ in physical interpretation and underlying assumptions from the present Vacuum Localized Structure (VLS) model, they illustrate the broader landscape of relativistic solutions and alternatives within which the present work is situated.

General relativity (GR) relates spacetime geometry to the stress–energy tensor, but it does not prescribe the microscopic origin of that tensor. This allows, in principle, for effective macroscopic descriptions of self-gravitating configurations that do not correspond to ordinary baryonic matter or fundamental fields, provided that the stress–energy tensor satisfies the required conservation laws and symmetry assumptions.

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In 1998, a self-sustained vacuum structure, referred to as a “vacuum bubble,” was proposed as a possible explanation for flat galaxy rotation curves [4]. More recently, this class of vacuum-supported configurations was shown to arise as a consistent solution of the Einstein field equations and was discussed under the designations “geon” or “self-consistent gravitational energy distribution” [5,6]. In the present work, and in future work, we adopt the unified designation Vacuum Localized Structures (VLS) for this class of solutions.

VLS are described phenomenologically by an anisotropic vacuum stress–energy tensor with positive energy density and vacuum-like negative radial pressure. The resulting configurations are regular everywhere and can reproduce several large-scale gravitational features usually attributed to dark matter, without invoking conventional matter sources. The stress–energy tensor is treated here as an effective macroscopic description, without assuming an underlying particle or field model.

The purpose of the present work is to extend the VLS framework by incorporating rotation. Since most astrophysical systems possess angular momentum, it is essential to establish whether vacuum-supported structures can rotate consistently and to determine how rotation modifies their internal structure and gravitational field, even at the level of leading-order rotational effects.

In this paper, rotation is treated within a Poisson-based framework corresponding to the weak-field, non-relativistic limit of general relativity. Rotational effects are included perturbatively using the slow-rotation approximation, in which the linear velocity remains small compared to the speed of light. Within this controlled approximation, the gravitational potential, energy density, and anisotropic pressure components can be computed analytically. Stress–energy conservation is imposed as a local force-balance condition, allowing the tangential pressure profile to be determined explicitly from the rotating density distribution.

Although the present analysis is restricted to the weak-field regime, it captures the essential physical effects of rotation—namely centrifugal deformation and anisotropic stresses—while preserving the regular, vacuum-dominated character of VLS. The results therefore provide a consistent and physically transparent description of rotating vacuum localized structures at leading order.

To the best of our knowledge, no rotating vacuum-supported configuration with the properties described here has been reported in standard catalogues of exact solutions [8] or in reviews of soliton-like relativistic structures [9–11]. Recent work has explored various classes of localized self-gravitating field configurations, including unified treatments of scalar and vector dark-matter

solitons [12] and studies of nontopological boson-star solutions with frozen states [13], situating the present VLS solutions within the broader landscape of analytical solitonic models.

Finally, it is worth noting that VLS may exist across a wide range of physical scales. On galactic scales, extended VLS configurations may act as effective dark-matter halos. On smaller scales, they may provide models of compact, rotating, vacuum-dominated objects that remain regular and free of curvature singularities within the regime considered here.

## 2. VLS configurations derived within the Poisson framework

In this section, rotation of vacuum localized structures is analyzed within the weak-field, non-relativistic limit of general relativity, where the gravitational field is described by the Newtonian potential and relativistic corrections are consistently neglected.

We start from the non-rotating Gaussian energy density, similar to earlier work [5]:

$$\rho_0^{(0)}(r) = \rho_c e^{-\frac{r^2}{R^2}}, \quad (1)$$

with characteristic radius  $R$  and central density  $\rho_c$ .

The superscript (0) means “evaluate this quantity in the non-rotating (zeroth-order) configuration”. The subscript (0) designates the monopole contribution (see later).

Next, we introduce rigid (solid-body) rotation as the simplest analytic choice:

$$v_\varphi(r, \theta) = \Omega_0 (r \sin \theta), \quad (2)$$

with constant angular velocity  $\Omega_0$ .

The kinetic energy density is

$$\rho_{\text{kin}}(r, \theta) = \frac{1}{2} \rho_0(r) \frac{v_\varphi^2}{c^2} = \frac{1}{2} \rho_0(r) \frac{\Omega_0^2 r^2 \sin^2 \theta}{c^2}, \quad (3)$$

So that the total energy density becomes:

$$\rho(r, \theta) = \rho_0(r) \left( 1 + \frac{\Omega_0^2 r^2 \sin^2 \theta}{2c^2} \right). \quad (4)$$

We treat the rotational correction as a small perturbation:

$$\varepsilon \equiv \frac{\Omega_0^2 R^2}{2c^2} \ll 1. \quad (5)$$

We keep terms to first order in  $\varepsilon$ .

In this work, rotation of the VLS is treated using the slow-rotation approximation, meaning that rotational effects are considered as small perturbations on top of the static configuration.

Physically, this requires that the linear velocity at the equator of the structure  $v = \Omega R$  remains small compared to the speed of light  $c$ , i.e.,  $v \ll c$ . Even velocities of the order of  $0.1 c$  are compatible with this approximation, which allows us to capture the leading-order rotational effects, such as the slight deformation of the density profile, without introducing instabilities or unphysical behavior. As an example, for  $v=0.1 c$ , the slow-rotation parameter  $\varepsilon$  is  $0.005$ , which is indeed small and well within the validity of the perturbative expansion. The slow rotation approach thus provides a controlled and consistent framework for including angular momentum in VLS while preserving their essential vacuum-dominated character.

We choose the equation of state (similar to that used in [5])

$$p_r(r, \theta) = -\rho(r, \theta). \quad (6)$$

In the weak-field, non-relativistic limit of general relativity, the metric can be written as

$$ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Phi) \delta_{ij} dx^i dx^j, \quad (7)$$

where  $\Phi(r, \theta)$  is the Newtonian gravitational potential.

We neglect higher-order relativistic corrections and frame-dragging ( $g_{t\varphi}$ ). Then the linearized Einstein equation reduces to the Poisson equation:

$$\nabla^2 \Phi(r, \theta) = 4\pi G \rho(r, \theta)/c^2. \quad (8)$$

In coordinate form, this becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) = 4\pi G \frac{\rho(r, \theta)}{c^2} \quad (9)$$

Expansion into Legendre polynomials;

$$\Phi(r, \theta) = \sum_{\ell} \Phi_{\ell}(r) P_{\ell}(\cos \theta) \quad (10)$$

$$\rho(r, \theta) = \sum_{\ell} \rho_{\ell}(r) P_{\ell}(\cos \theta) \quad (11)$$

the angular operator satisfies

$$\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \frac{\sin \theta (\partial P_{\ell}(\cos \theta))}{\partial \theta} \right) = -\ell(\ell + 1) P_{\ell}(\cos \theta) \quad (12)$$

This leads directly to the radial Poisson equations;

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_{\ell}}{dr} \right) - \frac{\ell(\ell+1)}{r^2} \Phi_{\ell} = 4\pi G \rho_{\ell}(r) \quad (13)$$

We use further the stress-energy conservation (generalized Tolman-Oppenheimer-Volkov equation (TOV) [14])

$$\nabla_\nu T^\nu{}_\mu = 0, \quad (14)$$

to obtain the tangential pressures  $p_\theta, p_\varphi$ . Because of axial symmetry we will write

$$p_\theta(r, \theta) = p_\varphi(r, \theta) \equiv p_t(r, \theta) \quad (15)$$

and expand  $p_t$  in the same monopole/quadrupole basis.

Using  $\sin^2\theta = 1 - \cos^2\theta$  and the Legendre polynomial  $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ , we can rewrite the angular factor:

$$\sin^2\theta = 1 - \cos^2\theta = \frac{2}{3}(1 - P_2(\cos\theta)). \quad (16)$$

The angular dependence  $\sin^2\theta$  is expanded in Legendre polynomials in order to decompose the source into eigenmodes of the angular Laplacian. This allows Poisson's equation to separate into independent radial equations for the monopole and quadrupole contributions.

Subscripts 0 and 2 label the monopole ( $\ell = 0$ ) and quadrupole ( $\ell = 2$ ) components of the spherical-harmonic expansion.

Therefore, the total density (Equation (4)) is

$$\rho(r, \theta) = \rho_0(r) \left[ 1 + \frac{\Omega_0^2 r^2}{2c^2} \sin^2\theta \right] = \rho_0(r) \left[ 1 + \varepsilon \frac{r^2}{R^2} \left( \frac{2}{3} - \frac{2}{3} P_2(\cos\theta) \right) \right] \quad (17)$$

where we used the small parameter  $\varepsilon = \frac{\Omega_0^2 R^2}{2c^2}$  (see Equation (5)) to keep book-keeping tidy. Equivalently

$$\rho(r, \theta) = \rho_0(r) \left[ 1 + \frac{2\varepsilon r^2}{3 R^2} \right] - \rho_0(r) \left[ \frac{2\varepsilon r^2}{3 R^2} \right] P_2(\cos\theta). \quad (18)$$

So, the density splits into a monopole part  $\rho_{\ell=0}(r)$  and a quadrupole part  $\rho_{\ell=2}(r) P_2(\cos\theta)$ .

After this preparatory work, we are now in a position to solve the Poisson equation. In spherical coordinates, the Poisson equation with an axisymmetric source admits a spherical-harmonic expansion of the potential.

$$\Phi(r, \theta) = \Phi_0(r) + \Phi_2(r) P_2(\cos\theta) + \mathcal{O}(\varepsilon^2), \quad (19)$$

with radial equations:

Monopole ( $\ell = 0$ ):

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_0}{dr} \right) = 4\pi G \rho_0(r). \quad (20)$$

Quadrupole ( $\ell = 2$ ):

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\Phi_2}{dr} \right) - \frac{6}{r^2} \Phi_2(r) = 4\pi G \rho_2(r). \quad (21)$$

From (Equation (17)),

$$\rho_0(r) = \rho_0^{(0)}(r) \left( 1 + \frac{2\varepsilon r^2}{3 R^2} \right), \quad (22)$$

$$\rho_2(r) = -\rho_0^{(0)}(r) \left( \frac{2\varepsilon r^2}{3 R^2} \right). \quad (23)$$

The solution for the monopole radial potential is:

$$\Phi_0(r) = -\frac{G}{r} \int_0^r 4\pi r'^2 \rho_0(r') dr' - G \int_r^\infty 4\pi r' \rho_0(r') dr'. \quad (24)$$

Defining the usual enclosed mass function for the monopole:

$$m_0(r) = \frac{4\pi}{c^2} \int_0^r r'^2 \rho_0(r') dr'. \quad (25)$$

Then outside the source the Newtonian potential is (weak-field approximation)

$$\Phi_0(r) = -\frac{G m_0(r)}{r} \quad (26)$$

### ***Monopole solution***

For the rotating configuration, the monopole part of the density reads

$$\rho_0(r) = \rho_c e^{-\frac{r^2}{R^2}} \left( 1 + \frac{2\varepsilon r^2}{3 R^2} \right), \quad (27)$$

so that the enclosed mass is

$$m_0(r) = \frac{4\pi\rho_c}{c^2} \int_0^r r'^2 e^{-\frac{r'^2}{R^2}} \left( 1 + \frac{2\varepsilon r'^2}{3 R^2} \right) dr'. \quad (28)$$

The required Gaussian integrals can be evaluated analytically. One finds

$$\int_0^r r'^2 e^{-\frac{r'^2}{R^2}} dr' = \frac{\sqrt{\pi}}{4} R^3 \operatorname{erf} \left( \frac{r}{R} \right) - \frac{R^2 r}{2} e^{-\frac{r^2}{R^2}}, \quad (29)$$

and

$$\int_0^r r'^4 e^{-\frac{r'^2}{R^2}} dr' = \frac{3\sqrt{\pi}}{8} R^5 \operatorname{erf}\left(\frac{r}{R}\right) - \frac{R^2 r}{4} (2r^2 + 3R^2) e^{-\frac{r^2}{R^2}}. \quad (30)$$

Substituting these expressions yields the enclosed mass in closed form

$$m_0(r) = \frac{4\pi\rho_c}{c^2} \left[ \frac{\sqrt{\pi}}{4} R^3 (1+\varepsilon) \operatorname{erf}\left(\frac{r}{R}\right) - \frac{R^2 r}{2} \left(1 + \frac{\varepsilon}{3} + \frac{\varepsilon 2r^2}{3 R^2}\right) e^{-\frac{r^2}{R^2}} \right]. \quad (31)$$

The rotational correction merely introduces additional polynomial weights under the Gaussian and does not alter the analytic structure of the solution.

### ***Quadrupole mass integrals and potential***

The axisymmetric rotational correction produces a quadrupole ( $\ell = 2$ ) contribution to the density of the form

$$\rho_2(r) P_2(\cos \theta), \quad \rho_2(r) = -\frac{2\varepsilon}{3} \rho_c \frac{r^2}{R^2} e^{-\frac{r^2}{R^2}}. \quad (32)$$

Accordingly, the quadrupole component of the gravitational potential is written as

$$\Phi_2(r, \theta) = \phi_2(r) P_2(\cos \theta), \quad (33)$$

where the radial function  $\phi_2(r)$  satisfies the  $\ell = 2$  Poisson equation

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi_2}{dr} \right) - \frac{6}{r^2} \phi_2 = 4\pi G \rho_2(r). \quad (34)$$

A standard integral solution for an axisymmetric quadrupole source is

$$\phi_2(r) = -\frac{4\pi G}{5} \left[ \frac{1}{r^3} \int_0^r \rho_2(r') r'^4 dr' + r^2 \int_r^\infty \rho_2(r') r' dr' \right]. \quad (35)$$

### ***Inner quadrupole mass integral***

The inner integral reads

$$\int_0^r \rho_2(r') r'^4 dr' = -\frac{2\varepsilon\rho_c}{3R^2} \int_0^r r'^6 e^{-\frac{r'^2}{R^2}} dr'. \quad (36)$$

The Gaussian integral evaluates to

$$\int_0^r r'^6 e^{-\frac{r'^2}{R^2}} dr' = \frac{15\sqrt{\pi}}{16} R^7 \operatorname{erf}\left(\frac{r}{R}\right) - \frac{R^2 r}{8} [4r^4 + 10R^2 r^2 + 15R^4] e^{-\frac{r^2}{R^2}}. \quad (37)$$

Hence,

$$\int_0^r \rho_2(r') r'^4 dr' = -\frac{2\varepsilon\rho_c}{3} \left[ \frac{15\sqrt{\pi}}{16} R^5 \operatorname{erf}\left(\frac{r}{R}\right) - \frac{r}{8} (4r^4 + 10R^2r^2 + 15R^4) e^{-\frac{r^2}{R^2}} \right] \quad (38)$$

### ***Outer quadrupole mass integral***

The outer integral is

$$\int_r^\infty \rho_2(r') r' dr' = -\frac{2\varepsilon\rho_c}{3R^2} \int_r^\infty r'^3 e^{-\frac{r'^2}{R^2}} dr'. \quad (39)$$

This evaluates to

$$\int_r^\infty r'^3 e^{-\frac{r'^2}{R^2}} dr' = \frac{R^2}{2} (r^2 + R^2) e^{-\frac{r^2}{R^2}}, \quad (40)$$

so that

$$\int_r^\infty \rho_2(r') r' dr' = -\frac{\varepsilon\rho_c}{3} (r^2 + R^2) e^{-\frac{r^2}{R^2}}. \quad (41)$$

### ***Final closed form for the quadrupole potential***

Substituting both integrals into the quadrupole solution gives

$$\phi_2(r) = \frac{8\pi G\varepsilon\rho_c}{15} \left[ \frac{1}{r^3} \left( \frac{15\sqrt{\pi}}{16} R^5 \operatorname{erf}\left(\frac{r}{R}\right) - \frac{r}{8} (4r^4 + 10R^2r^2 + 15R^4) e^{-\frac{r^2}{R^2}} \right) + \frac{r^2}{2} (r^2 + R^2) e^{-r^2/R^2} \right] \quad (42)$$

The full quadrupole contribution to the gravitational potential is therefore

$$\Phi_2(r, \theta) = \phi_2(r) P_2(\cos \theta), \quad (43)$$

which is again fully analytic and expressed solely in terms of

$$\left\{ \operatorname{erf}\left(\frac{r}{R}\right), r^n e^{-\frac{r^2}{R^2}} \right\},$$

exactly as in the monopole sector and in the non-rotating Gaussian solution.

### **Energy-momentum tensor $T_{\mu\nu}$**

We work in coordinates  $(t, r, \theta, \varphi)$ . In the (weak) metric sign convention where  $T_{tt} = \rho c^2$ ,  $g_{tt} \approx -\rho c^2$ , the *mixed* and *covariant* indices differ only at higher order. For clarity we give the (covariant) diagonal stress tensor ansatz used in this paper:

$$T_{\mu\nu} = \text{diag}(-\rho(r, \theta) c^2, p_r(r, \theta), p_t(r, \theta) r^2, p_t(r, \theta) r^2 \sin^2 \theta) \quad (44)$$

where  $p_r = -\rho$  by assumption and  $p_t(r, \theta)$  is unknown.

So explicitly, lower-index components:

$$T_{tt} = -\rho(r, \theta), \quad (45)$$

$$T_{rr} = p_r(r, \theta) = -\rho(r, \theta),$$

$$T_{\theta\theta} = p_t(r, \theta) r^2,$$

$$T_{\varphi\varphi} = p_t(r, \theta) r^2 \sin^2 \theta.$$

We are neglecting off-diagonal stresses such as  $T_{r\varphi}$  which would appear for more complex velocity fields. Rigid rotation produces a momentum density  $T_{t\varphi}$ — frame dragging — but that is  $\mathcal{O}\left(\frac{\Omega_0}{c}\right)$  and suppressed in the non-relativistic limit.

### **Conservation equation and solving for $p_t$**

The covariant conservation law  $\nabla_\nu T^\nu{}_\mu = 0$  gives two nontrivial equations in the axisymmetric stationary case: the radial component (balance of radial stresses and gravity) and the angular ( $\theta$ ) component (balance in polar direction). Although the conservation equation  $\nabla_\mu T^{\mu\nu} = 0$  originates in general relativity, in the present Poisson-based treatment it is used only in its Newtonian, weak-field limit as a local force-balance condition, independently of the Einstein field equations.

In the weak-field limit and first order in  $\varepsilon$ , these reduce to:

Radial component (generalized TOV form)

$$\frac{\partial p_{r0}}{\partial r} + (\rho_0 c^2 + p_{r0}) \frac{\partial \Phi_0}{\partial r} + \frac{2}{r} (p_{r0} - p_{\theta 0}) = 0, \quad (46)$$

Inserting the equation of state  $p_{r0} = -\rho_0$ ,

the radial balance equation reduces to

$$\frac{\partial p_{r0}}{\partial r} + \frac{2}{r} (p_{r0} - p_{\theta 0}) = 0. \quad (47)$$

Using  $p_{r0} = -\rho_0$ , one finds

$$\frac{\partial p_{r0}}{\partial r} = \frac{\partial}{\partial r} (-\rho_0) = -\frac{\partial \rho_0}{\partial r}. \quad (48)$$

The reduced conservation equation therefore becomes

$$-\frac{\partial \rho_0}{\partial r} + \frac{2}{r}(\rho_0 - p_{\theta 0}) = 0. \quad (49)$$

Solving for the tangential monopole pressure yields

$$p_{\theta 0}(r) = -\rho_0(r) + \frac{r}{2} \frac{\partial \rho_0(r)}{\partial r}. \quad (50)$$

Because  $\rho(r, \theta)$  has both monopole and quadrupole parts,  $p_t$  likewise splits:

$$p_t(r, \theta) = p_{t0}(r) + p_{t2}(r)P_2(\cos \theta) \quad (51)$$

With

$$p_{t0}(r) = -\rho_0(r) + \frac{r}{2} \frac{d\rho_0}{dr}, \quad (52)$$

$$p_{t2}(r) = -\rho_2(r) + \frac{r}{2} \frac{d\rho_2}{dr}. \quad (53)$$

Plugging the expressions for  $\rho_0, \rho_2$  given above yields explicit analytic expressions.

***Final explicit forms (to first order in  $\varepsilon$ )***

$$\rho(r, \theta) = \rho_c e^{-\frac{r^2}{R^2}} \left[ 1 + \frac{2\varepsilon r^2}{3 R^2} \right] - \rho_c e^{-\frac{r^2}{R^2}} \left[ \frac{2\varepsilon r^2}{3 R^2} \right] P_2(\cos \theta). \quad (54)$$

$$p_r(r, \theta) = -\rho(r, \theta) \quad (55)$$

From (45) with direct differentiation:

$$p_t(r, \theta) = \left[ -\rho_c e^{-\frac{r^2}{R^2}} \left( 1 + \frac{2\varepsilon r^2}{3 R^2} \right) + \frac{r}{2} \frac{d}{dr} \left( \rho_c e^{-\frac{r^2}{R^2}} \left( 1 + \frac{2\varepsilon r^2}{3 R^2} \right) \right) \right] \\ + \left[ -\rho_2(r) + \frac{r}{2} \frac{d\rho_2(r)}{dr} \right] P_2(\cos \theta). \quad (56)$$

Work out the radial derivatives; after algebra one finds

$$p_t(r, \theta) = -\rho_c e^{-\frac{r^2}{R^2}} \left( 1 - \frac{r^2}{R^2} \right) \\ - \rho_c e^{-\frac{r^2}{R^2}} \varepsilon \left[ \frac{r^4}{R^4} \sin^2 \theta \right] + \mathcal{O}(\varepsilon^2). \quad (57)$$

The inclusion of slow rotation modifies the tangential pressure of a vacuum localized structure through a single correction term that is linear in the slow-rotation parameter  $\varepsilon$ , and therefore quadratic in the angular velocity  $\Omega$ .

Several features of this correction are worth emphasizing. First, the static limit is recovered exactly as  $\varepsilon \rightarrow 0$ , confirming the internal consistency of the rotating solution with the previously derived non-rotating VLS configuration.

Second, the correction scales as  $r^4$  at small radii and therefore remains parametrically small in the central region, preserving regularity at the origin. The angular dependence  $\sin^2\theta$  reflects the centrifugal nature of the rotational deformation and corresponds to an oblate distortion of the effective pressure distribution. In particular, the correction increases the magnitude of the tangential pressure in the equatorial plane, where rotational effects are maximal.

Finally, the simplicity of the result is noteworthy. Rotation does not introduce new independent pressure components or modify the vacuum equation of state  $p_r = -\rho$ . Instead, it enters solely through the spatial dependence of the density, with the tangential pressure following directly from energy–momentum conservation. This demonstrates that slowly rotating vacuum localized structures remain self-consistent solutions of the field equations, with only mild and controlled deviations from the stationary case.

### 3. Summary

In this work, we have extended earlier analyses of static Vacuum Localized Structures (VLS) by incorporating rotation within a controlled weak-field framework. The results demonstrate that vacuum-based configurations described by an anisotropic stress–energy tensor can consistently support angular momentum while remaining regular and free of pathologies within the regime considered.

Rotation is treated perturbatively using the slow-rotation approximation, in which rotational effects enter as small corrections to the static configuration. Within this approach, the radial pressure continues to balance the energy density, while rotation naturally induces anisotropic tangential stresses that remain well behaved throughout the structure and reduce smoothly to the non-rotating limit.

A central result of this study is the explicit analytic construction of rotating VLS solutions within a Poisson-based treatment. This allows the gravitational potential, density profile, and pressure components to be computed in closed form, providing a transparent description of the leading-order rotational effects, including quadrupolar deformation and centrifugal contributions.

From an astrophysical perspective, these results indicate that rotating VLS may provide an effective alternative description of dark-matter–like gravitational phenomena, arising from vacuum structure rather than from particle matter. More generally, the present work establishes that rotation is compatible with the existence of vacuum localized structures and provides a foundation for future investigations of higher-order rotational effects, non-spherical configurations, and potential observational signatures.

### Statements and Declarations

Competing Interests: The author declares that there are no competing financial or non-financial interests related to this work.

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