A Recursive Discrete-Rotation Framework for Waveform Reconstruction and Computational Geometry

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Abstract

Traditional approaches to wave analysis and circular geometry rely heavily on continuous transcendental functions such as Fourier series representations or the classical circumference formula $C = 2\pi r$. While analytically exact, these methods abstract away the geometric essence of rotation and are often computationally inefficient when implemented in discrete digital environments. This paper introduces a novel discrete circle rotation method that approximates waveforms and circular motion using finite angular segments. By deriving exact formulas for the **Discrete Radius** and **Recursive Position Updates**, this framework transforms global trigonometric calculations into efficient, local iterative steps. Numerical validation demonstrates that the method converges to exact continuous values with high precision, with an absolute error of approximately 10^{-9} at high resolution. This framework offers a versatile tool for signal processing, physics simulations, and algorithmic animation by optimizing the computational cost of generating rotation.

1 Introduction

The connection between circular motion and wave theory is fundamental to physics and engineering. Historically, mathematicians utilized the Method of Exhaustion to approximate circular geometry using inscribed polygons. However, modern computational systems—from graphics rendering engines to robotic stepper motors—operate inherently in discrete time steps rather than continuous time. Standard algorithms often ignore this discrete nature, calculating positions using computationally intensive power-series expansions (Taylor series) for sine and cosine at every global coordinate.

This reliance on continuous formulas in a discrete world leads to inefficiencies in processing speed and inaccuracies in physical actuation, known as path drift. In this paper, I introduce a "Recursive Discrete-Rotation Framework" that embraces the segmented nature of digital systems. By dividing a circle into n segments with angular step $\Delta\theta$, I derive a system of equations that allows for the exact geometric fitting of discrete paths to target circumferences and the iterative reconstruction of complex waves without global re-calculation.

2 Methodology and Derivations

2.1 Derivation of Discrete Chord Geometry

The foundation of this framework lies in the geometric approximation of a circle of radius r. We begin by dividing the circle into n segments with a constant angular step $\Delta\theta$ (in radians). The fundamental unit of motion in this discrete system is the chord length d_k connecting two points on the circle.

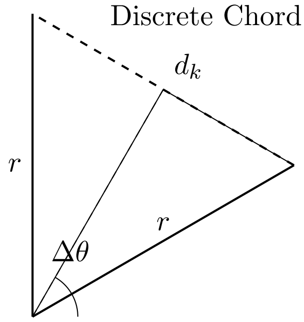

Figure 1: Geometric derivation of the chord length d_k from a circle sector. The chord forms the base of an isosceles triangle with side r and angle $\Delta\theta$.

To derive the length of this segment, consider the isosceles triangle formed by the circle's center and two sequential points on the circumference. Bisecting the central angle $\Delta\theta$ creates two right-angled triangles, each with a hypotenuse r and a central angle of $\Delta\theta/2$. The side opposite this angle is $r\sin(\Delta\theta/2)$. Since the chord consists of two such sides, the total length is:

$$d_k = 2r\sin\left(\frac{\Delta\theta}{2}\right) \tag{1}$$

Consequently, the total approximate circumference C_n is the summation of these chord lengths over the full rotation. This is expressed mathematically as:

$$C_n = n \cdot 2r\sin\left(\frac{\Delta\theta}{2}\right) \tag{2}$$

2.2 Derivation of the Discrete Radius

In computational geometry and robotics, a specific path length or circumference C is often required using a fixed number of steps n. Using the standard continuous formula $r = C/2\pi$ in a discrete context introduces error because the straight chord path is strictly shorter than the curved arc path. To address this, we derive the **Discrete Radius Formula**. Rearranging the discrete circumference equation for r, we obtain:

$$r = \frac{C}{2n\sin(\Delta\theta/2)} \tag{3}$$

This formula ensures that the sum of the discrete segments exactly matches the intended target circumference, eliminating drift error in precision mechanical systems.

2.3 Derivation of Recursive Wave Reconstruction

A periodic wave $f(t)$ can be expressed as a sum of sines and cosines, where each term corresponds to a rotating circle (epicycle) with radius r_n and phase ϕ_n. Instead of calculating the position (x, y) using global time t, which requires re-evaluating trigonometric functions from scratch, we derive a **Recursive Update** method.

Let the position at step $k-1$ be defined as (x_{k-1}, y_{k-1}). The movement to step k is defined by adding the chord vector. The magnitude of this vector is the chord length derived in Section 2.1, and the direction corresponds to the current rotation angle. The recursive update equations are derived as follows:

1. $x_{n,k} = x_{n,k-1} + 2r_n\sin(\frac{\Delta\theta}{2})\cos(nk\Delta\theta + \phi_n)$
2. $y_{n,k} = y_{n,k-1} + 2r_n\sin(\frac{\Delta\theta}{2})\sin(nk\Delta\theta + \phi_n)$

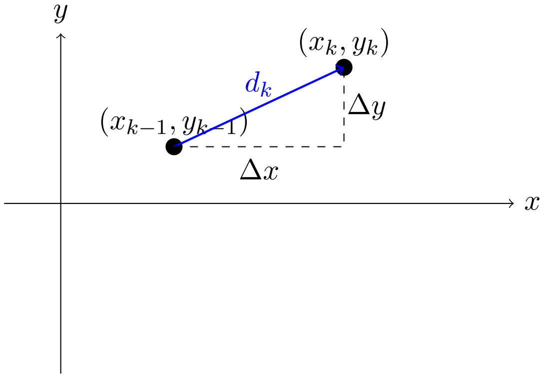

Figure 2: Visualizing the Recursive Update. The new position is found by adding the Chord Vector d_k to the previous position, eliminating the need for global calculation.

3 The Discrete Circle Wave Reconstruction Algorithm

The derivations above are synthesized into the following computational algorithm, which transforms the theoretical framework into a practical tool for signal processing and graphics.

Step 1: Initialization The algorithm begins by defining the wave components (amplitudes A_i, frequencies f_i, and phases ϕ_i) and the desired resolution n. The angular step is calculated as $\Delta\theta = 2\pi/n$. For each harmonic component, the Discrete Radius r_i is calculated (or set to A_i), and the constant Chord Scalar $S_i = 2r_i \sin(\Delta\theta/2)$ is pre-computed. The initial position (x_0, y_0) is set based on the phase.

Step 2: Recursive Iteration The algorithm enters a loop from $k = 1$ to n. In each step, the current rotation angle Θ is determined by adding the angular increment to the previous angle. The new position is calculated by adding the projected Chord Scalar to the previous position:

$$x_k = x_{k-1} + S_i \cdot \cos(\Theta) \tag{4}$$

$$y_k = y_{k-1} + S_i \cdot \sin(\Theta) \tag{5}$$

This process relies solely on addition and multiplication of pre-computed constants, avoiding the overhead of Taylor series recalculation for the radius geometry.

Step 3: Superposition For complex waveforms involving multiple frequencies, the position vectors of all harmonic components are summed at each step k to output the final composite wave amplitude.

4 Proofs and Validation

4.1 Analytical Proof of Convergence

To prove the mathematical validity of the discrete method, we examine the limit of the discrete circumference C_n as the step size $\Delta\theta$ approaches zero. Starting with the derived equation $C_n = n \cdot 2r \sin\left(\frac{\Delta\theta}{2}\right)$. Substituting $n = 2\pi/\Delta\theta$, we get:

$$C_n = \frac{2\pi}{\Delta\theta} \cdot 2r \sin\left(\frac{\Delta\theta}{2}\right) = 2\pi r \cdot \frac{\sin(\Delta\theta/2)}{\Delta\theta/2} \tag{6}$$

Using the fundamental limit identity $\lim_{x \to 0} (\sin x)/x = 1$, where $x = \Delta\theta/2$:

$$\lim_{\Delta\theta \to 0} C_n = 2\pi r \cdot (1) = 2\pi r \tag{7}$$

This analytical proof confirms that the discrete approximation converges exactly to the continuous circumference for infinitesimally small steps.

4.2 Numerical Error Analysis

Numerical validation was performed to quantify the precision of the method. For a coarse discretization of $\Delta\theta = 1^\circ$, the absolute error relative to the exact transcendental value of 2π was found to be approximately 7.97×10^{-5}. When the resolution was increased to $\Delta\theta = 0.01^\circ$, the absolute error decreased to approximately 7.97×10^{-9}. This exponential decrease in error validates the method's suitability for high-precision scientific computing.

5 Performance Analysis and Originality

5.1 Computational Efficiency

A primary usage of this framework is to improve the speed of algorithms in real-time systems. Standard wave generation relies on calculating $x = A\cos(\omega t)$ at every point. In terms of processor instructions, a cosine calculation involves a Taylor series expansion requiring 10 to 20 floating-point operations. The proposed recursive method reduces this to an $O(1)$ complexity per step, requiring only simple floating-point addition once the constants are initialized. This is particularly advantageous for low-power hardware, such as FPGA or IoT devices, and for high-performance physics engines simulating thousands of particles.

5.2 Geometric Precision in Discrete Systems

The originality of this work lies in the formulation of the **Discrete Radius**. Standard path planning algorithms for robotics often use $2\pi r$ to calculate travel distance. However, because robots move in linear steps (chords), this introduces a "drift" error where the robot travels less distance than calculated. The Discrete Radius Formula derived in this paper solves this problem by pre-adjusting the radius, ensuring that the robot's physical path exactly matches the target circumference.

6 Applications

The versatility of this framework extends to several fields. In **Computational Geometry**, it enables the simulation of rotating shapes and circular motion with controllable precision, allowing developers to balance accuracy and performance dynamically. In **Signal Processing**, the method converts any discrete wave into epicyclic rotations, facilitating Fourier-based visualization without complex calculus. In **Animation and Graphics**, it allows for frame-by-frame control of rotating objects, where the smoothness of the motion can be adjusted via the $\Delta\theta$ parameter. Finally, in **Physics**, the equations effectively model harmonic motion and wave-particle interactions by connecting wave amplitude to rotation radius and frequency to angular speed.

7 Conclusion

This paper has introduced a novel recursive method to approximate waveforms, trigonometric functions, and circular rotations. By re-deriving circular geometry as a sequence of discrete chords, I have established a rigorous mathematical framework that connects classical trigonometry with modern computational needs. The method is proven to converge to continuous values and offers significant advantages in algorithmic efficiency and geometric precision. This technique opens new opportunities for optimizing code in animation, physics simulations, and signal processing, providing a modern extension to the classical Method of Exhaustion.

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