Simplified Master Formula for a Right Triangular Plane and Solid-Angle Corollaries in the Theory of Polygon

Harish Chandra Rajpoot

Department of Mechanical Engineering, Indian Institute of Technology Bombay, Mumbai 400076, Maharashtra, India

ABSTRACT

A generalized framework from HCR's Theory of Polygon is presented for computing the solid angle subtended by an arbitrary polygonal plane, regular or irregular, at any point in three-dimensional space. The approach is unified and systematic, relying on a single master formula derived for a right triangular plane. This formula is simplified and equivalently expressed in terms of inverse trigonometric functions, including arcsine, arccosine, and arctangent. The variation of the solid angle with respect to the orthogonal sides of the triangle and the distance of the observation point is illustrated graphically. In addition, several corollaries are established for the solid angle subtended by planar surfaces, both polygonal and non-polygonal, at different coplanar locations of the observation point. The results are derived using the standard formula for right-triangle geometry and the concept of the angle of vision for observation of two-dimensional figures.

Keywords: HCR’s Theory of Polygon, solid angle, Master formula, right triangular plane, solid-angle corollaries

1. Introduction

The Theory of Polygon uses an "element method" to break down complex polygonal shapes into a certain number of triangles called elementary triangles that have a common vertex at the foot of perpendicular (F.O.P.) drawn from a given point to the polygonal plane, and divide the given polygonal plane internally or externally. Each of these elementary triangles is further divided into two right triangles internally or externally. The solid angle subtended by each right triangle dividing the polygonal plane internally is taken positive, while that of each externally dividing right triangle is taken negative. The total solid angle subtended by the original polygonal plane at a given point in space is the algebraic sum of the solid angles subtended by all these individual elementary right triangles. The solid angle subtended by a right triangular plane at any point lying on the perpendicular passing through one of the acute-angled vertices is determined by HCR’s Master Formula or Standard Formula-1 [1]. In this paper, the master formula is simplified and expressed in terms of different inverse trigonometric functions, such as arcsine, arccosine, and arctangent, which is applicable to any polygonal plane and 3D surfaces with polygonal faces, such as polyhedra [2,3,4]. The important corollaries for solid angle subtended by planes (polygons & non-polygons) have been proposed and proved for different locations of observation point [5].

2. Simplification of Master formula

The solid angle ($\omega_{\triangle}$), subtended by a right triangle having orthogonal sides p & b at any point lying at a normal distance h on the perpendicular passing through the common vertex of the side p & hypotenuse (as shown in Figure-1 below), is given by HCR’s Master/Standard Formula-1 [1] as follows

$$\omega_{\triangle} = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\} \quad (1)$$

where, $\omega_{\triangle} \in \left[0, \frac{\pi}{2}\right] \ \forall \ b, p, h \geq 0$.

Corresponding Author: H C Rajpoot,
Email Address: hcrajpoot.iitb@gmail.com

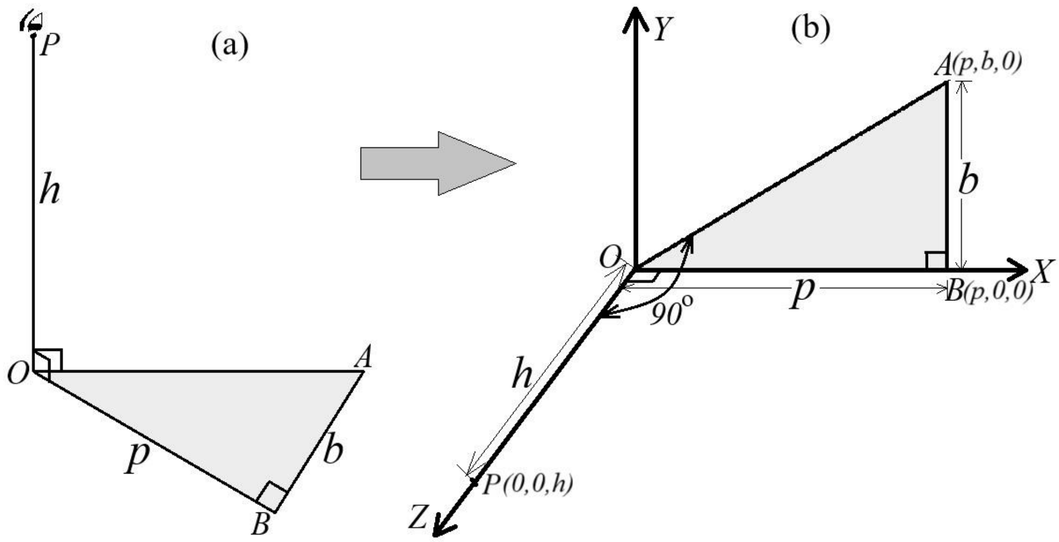

Figure 1: (a) Observation point P is lying on the perpendicular passing through acute-angled vertex O of right triangular plane ABO (b) orientation of triangle ABO on XY plane and given point P on Z-axis in 3D Euclidean space.

The Eq. (1) can be simplified using trig equation; $\sin^{-1}x-\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}-y\sqrt{1-x^2}\right)$ as follows

$$\omega_{\triangleright}=\sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\sqrt{1-\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}^2}-\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\sqrt{1-\left(\frac{b}{\sqrt{b^2+p^2}}\right)^2}\right)$$

$$\omega_{\triangleright}=\sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\sqrt{\frac{b^2h^2+p^2h^2+b^2p^2+p^4-b^2h^2}{(b^2+p^2)(h^2+p^2)}}-\frac{bh}{\sqrt{b^2+p^2}\sqrt{h^2+p^2}}\sqrt{\frac{b^2+p^2-b^2}{b^2+p^2}}\right)$$

$$\omega_{\triangleright}=\sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\frac{p\sqrt{b^2+p^2+h^2}}{\sqrt{b^2+p^2}\sqrt{h^2+p^2}}-\frac{bh}{\sqrt{b^2+p^2}\sqrt{h^2+p^2}}\frac{p}{\sqrt{b^2+p^2}}\right)$$

$$\omega_{\triangleright}=\sin^{-1}\left(\frac{bp\sqrt{b^2+p^2+h^2}}{(b^2+p^2)\sqrt{p^2+h^2}}-\frac{bph}{(b^2+p^2)\sqrt{p^2+h^2}}\right)$$

$$\omega_{\triangleright}=\sin^{-1}\left(\frac{bp\sqrt{b^2+p^2+h^2}-bph}{(b^2+p^2)\sqrt{p^2+h^2}}\right)\qquad(2)$$

Similarly, Eq. (1) can be simplified using trig equation; $\sin^{-1}x-\sin^{-1}y=\cos^{-1}\left(\sqrt{1-x^2}\sqrt{1-y^2}+xy\right)$ as follows

$$\omega_{\triangleright}=\cos^{-1}\left(\sqrt{1-\left(\frac{b}{\sqrt{b^2+p^2}}\right)^2}\sqrt{1-\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}^2}+\frac{b}{\sqrt{b^2+p^2}}\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right)$$

$$\omega_{\triangleright}=\cos^{-1}\left(\sqrt{\frac{b^2+p^2-b^2}{b^2+p^2}}\sqrt{\frac{b^2h^2+p^2h^2+b^2p^2+p^4-b^2h^2}{(b^2+p^2)(h^2+p^2)}}+\frac{b^2h}{(b^2+p^2)\sqrt{h^2+p^2}}\right)$$

$$\omega_{\triangleright}=\cos^{-1}\left(\frac{p}{\sqrt{b^2+p^2}}\frac{p\sqrt{b^2+p^2+h^2}}{\sqrt{b^2+p^2}\sqrt{h^2+p^2}}+\frac{b^2h}{(b^2+p^2)\sqrt{h^2+p^2}}\right)$$

$$\omega_{\triangleright}=\cos^{-1}\left(\frac{p^2\sqrt{b^2+p^2+h^2}}{(b^2+p^2)\sqrt{p^2+h^2}}+\frac{b^2h}{(b^2+p^2)\sqrt{p^2+h^2}}\right)$$

$$\omega_{\triangle} = \cos^{-1}\left(\frac{p^2\sqrt{b^2+p^2+h^2}+b^2h}{(b^2+p^2)\sqrt{p^2+h^2}}\right) \quad (3)$$

Similarly, Eq. (1) can be expressed in term of arctangent function as follows

$$\tan\omega_{\triangle} = \frac{\sin\omega_{\triangle}}{\cos\omega_{\triangle}}$$

$$\omega_{\triangle} = \tan^{-1}\left(\frac{\sin\omega_{\triangle}}{\cos\omega_{\triangle}}\right)$$

$$\omega_{\triangle} = \tan^{-1}\left(\frac{\frac{bp\sqrt{b^2+p^2+h^2}-bph}{(b^2+p^2)\sqrt{p^2+h^2}}}{\frac{p^2\sqrt{b^2+p^2+h^2}+b^2h}{(b^2+p^2)\sqrt{p^2+h^2}}}\right) \quad \text{(setting values of } \sin\omega_{\triangle} \text{ and } \cos\omega_{\triangle} \text{ from (2) \&(3))}$$

$$\omega_{\triangle} = \tan^{-1}\left(\frac{bp\sqrt{b^2+p^2+h^2}-bph}{p^2\sqrt{b^2+p^2+h^2}+b^2h}\right) \quad (4)$$

The Eq. (2), (3) and (4) are the simplified forms of original derived HCR's Master formula Eq. (1) which are equally applicable to find the solid angle by a right triangular plane.

3. Effect of parameters on solid angle ($\omega_{\triangle}$)

The solid angle ($\omega_{\triangle}$) (given from above Eq. (1), (2), (3) and (4)) subtended by a right triangular plane at any point lying on the perpendicular passing through one of acute-angled vertices depends on three key parameters, b, p, and h such that the orthogonal side b is opposite to foot of perpendicular (F.O.P.) O (drawn from the given/observation point P to the right triangular plane ABO), and orthogonal side p is adjacent to the F.O.P. (as shown in the above Figure 1). It is worth noticing that as the orthogonal side b opposite to F.O.P. O increases, the solid angle ($\omega_{\triangle}$) increases sharply keeping p and h constant and becomes parallel to x-axis at very large value of b (as shown in Figure 2 (a)). While as the orthogonal side p adjacent to F.O.P. O increases, solid angle ($\omega_{\triangle}$) first increases sharply for a short range and then decreases keeping b and h constant and approaches zero at very large value of p (as shown in Figure 2 (b)).

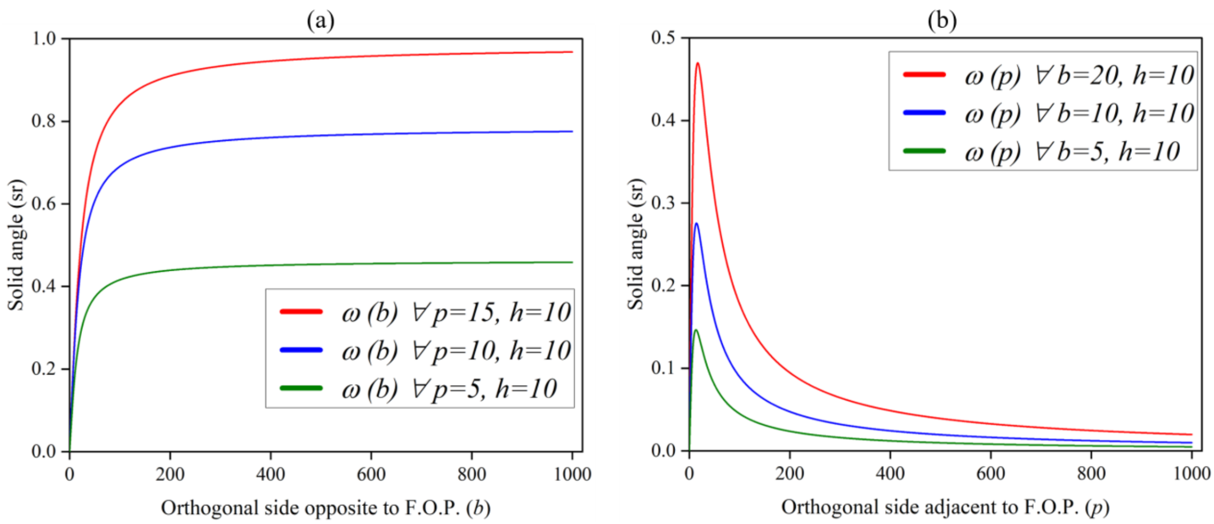

Figure 2: Variation of solid angle subtended by right triangular plane ABO at given observation point P with respect to (a) side *b* opposite to F.O.P. (b) side *p* adjacent to F.O.P..

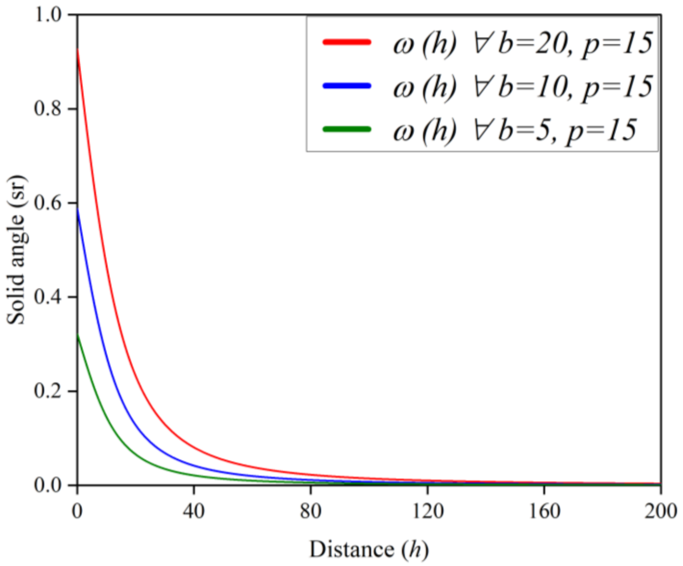

Figure 3: Variation of solid angle subtended by right triangular plane ABO at given point P with respect to normal distance h of point P from acute-angled vertex O.

As the normal distance h of given observation point P from acute-angled vertex O increases, solid angle ($\omega_{\triangle}$) decreases sharply keeping b and p constant, and approaches zero at very large value of h (as shown in Figure-3).

4. Solid-Angle Corollaries from HCR's Master formula:

4.1. Corollary-1: The solid angle (in steradian) subtended by a right triangular plane at its acute-angled vertex is equal to that vertex angle (in radian).

Proof. Consider an observation point P lying on an acute-angled vertex O of right triangular plane ABO (as shown in Figure 4). The solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at the point P (i.e. coincident with vertex O) is obtained by substituting $h = 0$ in the above Eq. (1) as follows

$$\omega_{\triangle} = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{0}{\sqrt{0^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) \quad (5)$$

$$= \angle AOB = \text{vertex angle at vertex O}$$

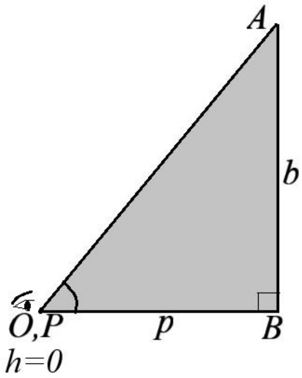

Figure 4: Observation point P lies on the vertex O ($h = 0$).

or substituting $h = 0$ into Eq. (2),

$$\omega_{\triangle} = \sin^{-1}\left(\frac{bp\sqrt{b^2+p^2+0^2} - bp(0)}{(b^2+p^2)\sqrt{p^2+0^2}}\right) = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) = \angle AOB = \text{vertex angle at vertex O}$$

or substituting $h = 0$ into Eq. (3),

$$\omega_{\triangle} = \cos^{-1}\left(\frac{p^2\sqrt{b^2+p^2+0^2} + b^2(0)}{(b^2+p^2)\sqrt{p^2+0^2}}\right) = \cos^{-1}\left(\frac{p}{\sqrt{b^2+p^2}}\right) = \angle AOB = \text{vertex angle at vertex O}$$

or substituting $h = 0$ into Eq. (4),

$$\omega_{\triangle} = \tan^{-1}\left(\frac{bp\sqrt{b^2+p^2+0}-bp(0)}{p^2\sqrt{b^2+p^2+0^2}+b^2(0)}\right) = \tan^{-1}\left(\frac{b}{p}\right) = \angle AOB = \text{vertex angle at vertex O}$$

It is proven from the above results that if the observation point P lies on the right triangular plane ABO and coincide with vertex O, solid angle $\omega_{\triangle}$ by right triangular plane ABO becomes equal to the (internal) vertex angle ∠AOB at vertex O.

4.2. Corollary-2: The solid angle subtended at acute-angled vertex by a right triangular plane, with infinitely long orthogonal side opposite (and orthogonal side adjacent) to that vertex, is $\frac{\pi}{2}$ sr.

Proof. Consider an observation point P lying on an acute-angled vertex O of right triangular plane ABO having its orthogonal sides b and p (as shown in Figure 4 above). Consider two possible cases; 1) when orthogonal side $AB = b$ is infinitely long i.e. $b \to \infty$ and other orthogonal side p is finite, 2) when both the orthogonal sides $AB = b$ and $OB = p$ become infinitely long i.e. $b \to \infty$ and $p \to \infty$. The two cases are proven as follows

Case-1: If the given observation point P lies on vertex O of right triangular plane ABO i.e. $h = 0$ and the orthogonal side b opposite to F.O.P. becomes infinitely long i.e. $b \to \infty$ (as shown in Figure 5 (a) below), solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at point P (i.e. coincident with vertex O) is obtained by taking limit of $\omega_{\triangle}$ (as $b \to \infty$) given from above Eq. (5) as follows

$$\omega_{\triangle} = \lim_{b\to\infty} \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) = \lim_{b\to\infty} \sin^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{p}{b}\right)^2}}\right) = \sin^{-1}(1) = \frac{\pi}{2}\ sr$$

Case-2: If the given observation point P lies on vertex O of right triangular plane ABO i.e. $h = 0$ and the orthogonal sides b and p become infinitely long i.e. $b \to \infty$ and $p \to \infty$ along y and x directions respectively (as shown in Figure-5 (b)), solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at point P (i.e. coincident with vertex O) is equal to $\frac{\pi}{2}$ sr because the adjacent side p, when extended to infinity, will not change ***angle of vision*** (*i.e. an imaginary angle formed by straight lines starting from the observation point and passing through all the points of a given object in 2D space. It is called* ***cone of vision*** *in 3D space*) and hence the solid angle $\omega_{\triangle}$ will be same as in the previous case-1.

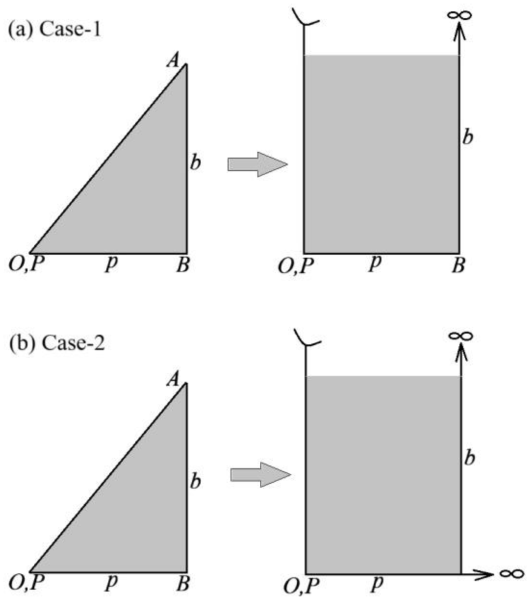

Figure 5: Right triangular plane ABO appearing to an observer at the vertex O when (a) $b \to \infty$ and p is finite, (b) $b \to \infty$ and $p \to \infty$ along y and x orthogonal directions respectively.

It is proven that the solid angle subtended by a right triangular plane at its acute-angled vertex approaches to $\frac{\pi}{2}$ sr when the orthogonal side opposite to that vertex (or F.O.P.) becomes infinitely long irrespective of length of other orthogonal side (i.e. finitely or infinitely long). In these two cases, the right triangle becomes (i.e. appears to an observer) like an infinitely long rectangular plane when adjacent orthogonal side p is finite, or it appears like a quarter infinite plane when adjacent orthogonal side p becomes infinitely long (as shown in Figure 5 (a) and (b)).

4.3. Corollary-3: The solid angle subtended by a right triangular plane at a point lying at infinite distance is always zero.

Proof. Consider an observation point P lying at an infinite distance from the vertex O of right triangular plane ABO i.e. $h \to \infty$ (as shown in above Figure 1), solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at point P is obtained by taking limit of $\omega_{\triangle}$ (as $h \to \infty$) given from Eq. (1), (2), (3), or (4) as follows

$$\omega_{\triangle} = \lim_{h\to\infty} \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \lim_{h\to\infty} \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{1}{\sqrt{1+\left(\frac{p}{h}\right)^2}}\right)\right\}$$

$$\omega_{\triangle} = \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)(1)\right\}$$

$$= 0$$

It is clear from the above result that if the given point P lies at infinite distance, solid angle $\omega_{\triangle}$ by right triangular plane ABO at point P becomes equal to zero because the triangular plane becomes like a point for an infinitely distant point or eye of observer.

4.4. Corollary-4: The solid angle, subtended by a right triangular plane at a point lying on the perpendicular at a distance h from the acute-angled vertex such that orthogonal side adjacent to that vertex is p while other orthogonal side is infinitely long, is given as

$$\omega_{\triangle} = \sin^{-1}\left(\frac{p}{\sqrt{p^2+h^2}}\right)$$

Proof. Assume that the side b of right triangular plane ABO, opposite to the foot of perpendicular (F.O.P.) O, becomes infinitely long i.e. $b \to \infty$ (as shown in above Figure 6), solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at point P is obtained by taking limit of $\omega_{\triangle}$ (as $b \to \infty$) given from Eq. (1), (2), (3), or (4) as follows

$$\omega_{\triangle} = \lim_{b\to\infty} \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \lim_{b\to\infty} \sin^{-1}\left(\frac{1}{\sqrt{1+\left(\frac{p}{b}\right)^2}}\right) - \sin^{-1}\left\{\left(\frac{1}{\sqrt{1+\left(\frac{p}{b}\right)^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \sin^{-1}(1) - \sin^{-1}\left\{(1)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \frac{\pi}{2} - \sin^{-1}\left(\frac{h}{\sqrt{h^2+p^2}}\right)$$

$$\omega_{\triangle} = \sin^{-1}\left(\frac{p}{\sqrt{p^2+h^2}}\right)$$

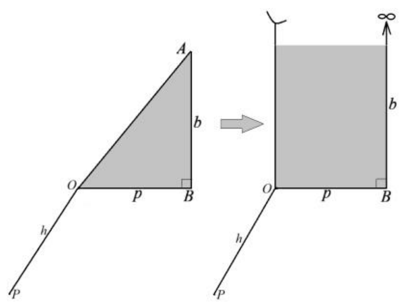

Figure 6: The observation point P lies on perpendicular at a distance *h* from vertex O.

It is clear from the above result that if the side b of right triangular plane ABO, opposite to the foot of perpendicular (F.O.P.) O, becomes infinitely long i.e. $b \to \infty$, solid angle $\omega_{\triangle}$ by right triangular plane ABO at point P becomes equal to solid angle subtended by a thin (width equal to p) semi-infinite rectangular strip at a point lying on the perpendicular passing through one vertex of semi-infinite rectangular strip (as shown in above Figure-5).

4.5. Corollary-5: The solid angle, subtended by a right triangular plane at a point lying on the perpendicular at a finite distance from the acute-angled vertex such that orthogonal side adjacent to that vertex is infinitely long, is zero.

Proof. Let the side p of right triangular plane ABO, adjacent to the foot of perpendicular (F.O.P.) O, be

infinitely long i.e. $p \to \infty$ (as shown in above Figure 5(b)), solid angle $\omega_{\triangle}$ subtended by the right triangular plane ABO at point P is obtained by taking limit of $\omega_{\triangle}$ (as $p \to \infty$) given from Eq. (1), (2), (3), or (4) as follows

$$\omega_{\triangle} = \lim_{p\to\infty} \sin^{-1}\left(\frac{b}{\sqrt{b^2+p^2}}\right) - \sin^{-1}\left\{\left(\frac{b}{\sqrt{b^2+p^2}}\right)\left(\frac{h}{\sqrt{h^2+p^2}}\right)\right\}$$

$$\omega_{\triangle} = \sin^{-1}(0) - \sin^{-1}\{(0)(0)\}$$

$$\omega_{\triangle} = 0$$

It is clear from the above result that if the side p of right triangular plane ABO, adjacent to the foot of perpendicular (F.O.P.) O, becomes infinitely long i.e. $p \to \infty$, solid angle $\omega_{\triangle}$ by right triangular plane ABO at point P becomes equal to zero because the right triangular plane becomes a straight line when observed from the given point P (as shown in above Figure 5(b)).

4.6. Corollary-6: The solid angle (in sr) subtended by any triangular plane at any of its vertices is equal to that vertex angle (in rad).

Proof. Let there be a triangular plane ABC having acute vertex angle α at the vertex A and an obtuse vertex angle β at the vertex B. Consider an observation point P lying on one of the vertices of triangular plane ABC. There are two possible cases; when 1) observation point P lies on the acute-angled vertex A, and 2) observation point P lies on the obtuse-angled vertex B. The two cases are proven as follows

Case-1: If the given observation point P lies on acute-angled vertex A of triangular plane ABC, drop a perpendicular AD to the (extended) opposite side BC to divide ΔABC into two right triangles ΔADC and ΔADB (as shown in Figure 7 (a)). Now, from the above corollary-1, the solid angle $\omega_{\Delta ADC}$ subtended by the right triangular plane ADC at its acute-angled vertex A (i.e. observation point P) is equal to the vertex angle $\angle CAD$ given as follows

$$\omega_{\Delta ADC} = \angle CAD$$

Similarly, the solid angle $\omega_{\Delta ADC}$ subtended by the right triangular plane ADB at its acute-angled vertex A is given as follows

$$\omega_{\Delta ADB} = \angle BAD$$

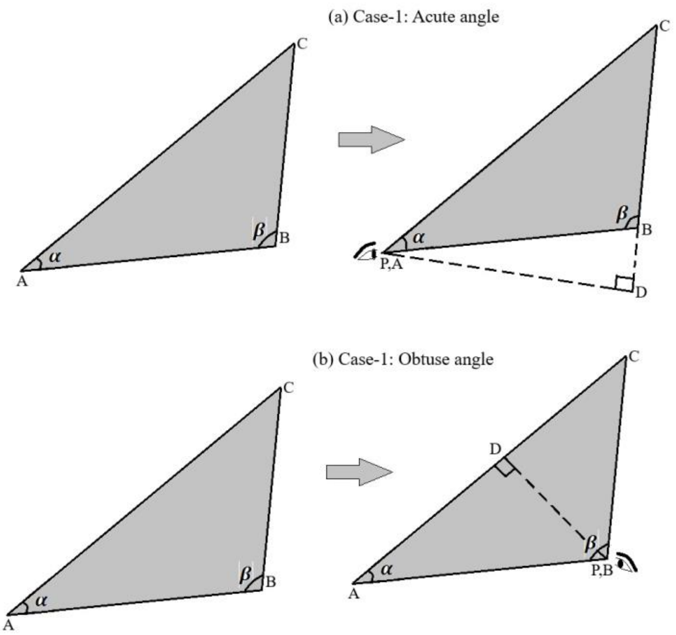

Figure 7: The observation point P lies on a vertex of triangular plane ABC when (a) vertex angle is acute, (b) vertex angle B is obtuse.

Now, the solid angle $\omega_{\Delta ABC}$ subtended by the given triangular plane ABC at its acute-angled vertex A is found by taking the algebraic sum of solid angles $\omega_{\Delta ADC}$ and $\omega_{\Delta ADB}$ subtended by elementary right triangles ADC and ADB respectively (i.e. dividing ΔABC externally) [1] as follows

$$\omega_{\Delta ABC} = \omega_{\Delta ADC} - \omega_{\Delta ADB}$$

$$= \angle CAD - \angle BAD$$

$$= \angle BAC$$

$$= acute\ \angle A$$

Case-2: If the given observation point P lies on obtuse-angled vertex B of triangular plane ABC, drop a perpendicular AD to the opposite side AC to divide ΔABC into two right triangles ΔBDA and ΔBDC (as shown in Figure 7(b)). Now, from the above corollary-1, the solid angle $\omega_{\Delta BDA}$ subtended by the right triangular plane BDA at its acute-angled vertex B (i.e. observation point P) is equal to the vertex angle $\angle ABD$ given as follows

$$\omega_{\Delta BDA} = \angle ABD$$

Similarly, the solid angle $\omega_{\Delta BDC}$ subtended by the right triangular plane BDC at its acute-angled vertex B is given as follows

$$\omega_{\Delta BDC} = \angle CBD$$

Now, the solid angle $\omega_{\Delta ABC}$ subtended by the given triangular plane ABC at its obtuse-angled vertex B is found by taking the algebraic sum of solid angles $\omega_{\Delta BDA}$ and $\omega_{\Delta BDC}$ subtended by elementary right triangles BDA and BDC respectively (i.e. dividing ΔABC internally) [1] as follows

$$\omega_{\Delta ABC} = \omega_{\Delta BDA} + \omega_{\Delta BDC}$$

$$= \angle ABD + \angle CBD$$

$$= \angle ABC$$

$$= \text{Obtuse } \angle B$$

It is proven from the above results that the solid angle subtended by a triangle at any of its vertices is equal to that vertex angle.

4.7. Corollary-7: The solid angle subtended by any triangular plane at any external point (i.e. lying outside the boundary) is always zero.

Proof. Consider an observation point P lying outside the boundary of a triangular plane ABC. Now, join point P to all the vertices of ΔABC to externally divide it into two elementary triangles ΔAPB and ΔBPC (as shown in Figure 8 below). Now, from the above corollary-2, the solid angle $\omega_{\Delta APB}$ subtended by the triangular plane APB at its vertex P (i.e. observation point) is equal to the vertex angle $\angle APB$ given as follows

$$\omega_{\Delta APB} = \angle APB$$

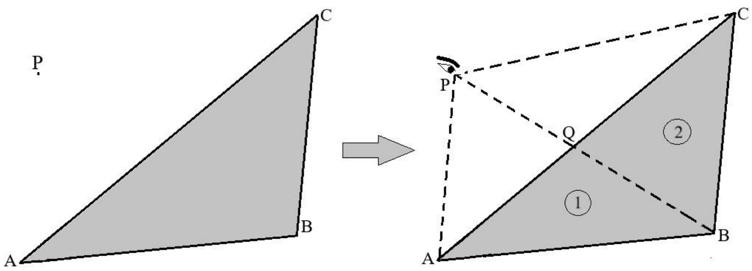

Figure 8: The observation point P is outside the triangular plane ABC from which an observer can only see the boundary line i.e. side AC but not the interior.

Similarly, from the above corollary-2, the solid angles subtended by the triangular planes BPC, APQ and CPQ at their common vertex P (i.e. observation point) are equal to their respective vertex angles given as follows

$$\omega_{\Delta BPC} = \angle BPC,\ \ \omega_{\Delta APQ} = \angle APQ,\ \ \omega_{\Delta CPQ} = \angle CPQ$$

Now, the solid angle $\omega_{\Delta AQB}$ subtended by the triangular plane AQB (i.e. labelled by '1' in Fig. 8) at the external point P (i.e. lying outside triangular plane ABC) is found by taking the algebraic sum of solid angles $\omega_{\Delta APB}$ and $\omega_{\Delta APQ}$ subtended by triangles APB and APQ respectively [1] as follows

$$\omega_{\Delta AQB} = \omega_{\Delta APB} - \omega_{\Delta APQ}$$

$$= \angle APB - \angle APQ \qquad (\because\ \angle APB = \angle APQ)$$

$$= 0$$

Similarly, the solid angle $\omega_{\Delta BQC}$ subtended by the triangular plane BQC (i.e. labelled by '2' in Fig. 8) at the external point P (i.e. lying outside triangular plane ABC) is found by taking the algebraic sum of solid angles $\omega_{\Delta BPC}$ and $\omega_{\Delta CPQ}$ subtended by triangles APB and APQ respectively [1] as follows

$$\omega_{\Delta BQC} = \omega_{\Delta BPC} - \omega_{\Delta CPQ}$$

$$= \angle BPC - \angle CPQ \qquad (\because\ \angle BPC = \angle CPQ)$$

$$= 0$$

It is proven that the solid angle by any triangular plane at any external point is always zero because the triangular plane appears like a line to an observer at that external point.

4.8. Corollary-8: The solid angle (in sr) subtended by any polygonal plane at any of its vertices is equal to that vertex angle (in rad).

Proof. Consider an observation point P lying on a vertex of a polygonal plane $A_1A_2A_3 \dots A_{n-1}A_n$ (i.e. regular, irregular, convex or concave). There are two possible cases when the (internal) vertex angle, 1) $\alpha < \pi$ and 2) $\beta > \pi$. The two cases are proven as follows

Case-1: Consider the observation point P lying on the vertex A_1 of a polygonal plane $A_1A_2A_3 \dots A_{n-1}A_n$ such that the (internal) vertex angle $A_1, \alpha < \pi$ (as shown in Figure 9(a)). Applying the triangulation rule in HCR's Theory of Polygon [1], join all the vertices of given polygon to the F.O.P. i.e. the observation point P (i.e. coincident with vertex A_1) to divide the polygon into elementary triangles all having a common vertex at the vertex A_1 (as shown in Figure 9(a) below). It is worth noticing that given polygonal plane $A_1A_2A_3 \dots A_{n-1}A_n$ (i.e. all its elementary triangles tessellated around vertex A_1) can be completely enclosed by the ***angle of vision*** (*i.e. an imaginary angle formed by straight lines starting from the observation point and passing through all the points of a given object in 2D space. It is called cone of vision in 3D space* [5].) of given polygon [1]. The angles of vision (i.e. angular region for coplanar observation of 2D figures) of given polygon $A_1A_2A_3 \dots A_{n-1}A_n$ (i.e. all elementary triangles) and $\Delta A_1A_3A_n$ are the same (as shown in Figure 9(a) below). Therefore, the solid angles ω_n and $\omega_{\Delta A_3A_1A_n}$, subtended at the vertex A_1, by $\Delta A_3A_1A_n$ and given polygon respectively are equal. Now, the solid angle ω_n subtended by given polygon $A_1A_2A_3 \dots A_{n-1}A_n$ at the vertex A_1 is equal to the algebraic sum of solid angles $\sum_{i=1}^{n}\omega_i$ subtended at the vertex A_1 by all (say n number of) elementary triangles, given as follows

$$\omega_n = \sum_{i=1}^{n}\omega_i$$

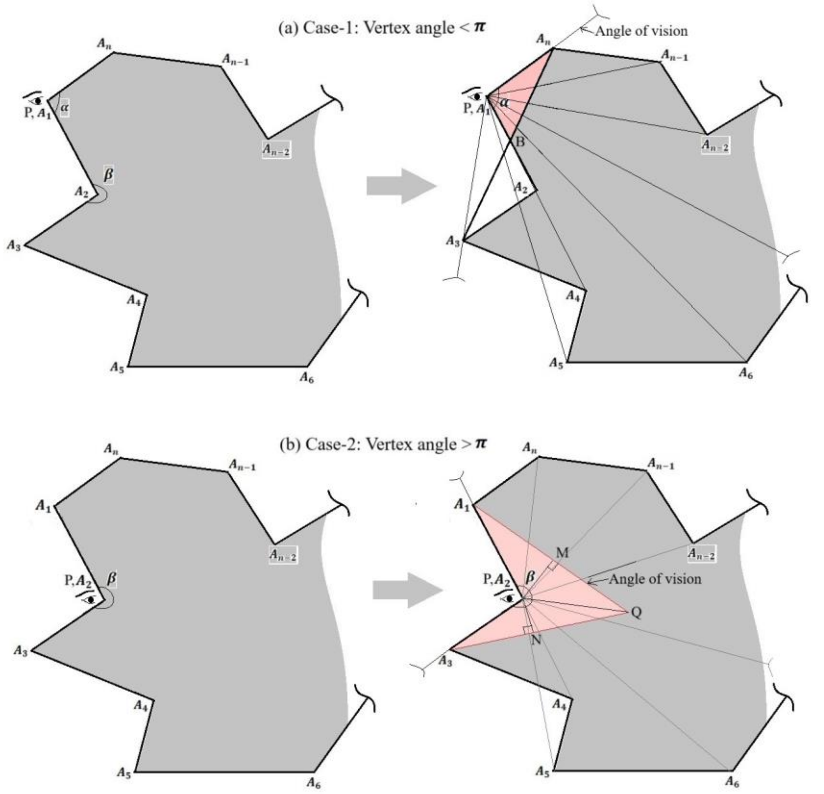

Figure 9: The observation point P lying on the vertex of polygonal plane $A_1A_2A_3 \dots A_{n-1}A_n$ when (a) vertex angle A_1, $\alpha < \pi$ rad, and (b) vertex angle A_2, $\beta > \pi$ rad.

$$\omega_n = \omega_{\Delta A_3A_1A_n} \quad \text{(Angle of vision of polygon)}$$

$$= \omega_{\Delta A_3A_1B} + \omega_{\Delta BA_1A_n} \quad \text{(Algebraic sum)}$$

$$= 0 + \omega_{\Delta BA_1A_n} \quad (\because \Delta A_3A_1B \notin A_1A_2A_3 \dots A_{n-1}A_n \text{ from Fig. 6(a)})$$

$$= \angle BA_1A_n \quad \text{(From Corollary 2)}$$

$$= \alpha \text{ (internal vertex angle of polygon)}$$

Case-2: Consider the observation point P lying on the vertex A_2 of given polygonal plane $A_1A_2A_3 \dots A_{n-1}A_n$ such that the (internal) vertex angle A_2, $\beta > \pi$. Applying the triangulation rule [1], join all the vertices of given polygon to the vertex A_2 to divide the polygon into elementary triangles all having a common vertex at the vertex A_2 (as shown in Figure 9(b)). It is worth noticing that angle of vision (obtained by extending sides A_2A_1 and A_2A_3) of given polygon $A_1A_2A_3 \dots A_{n-1}A_n$ can be represented by concave quadrilateral $A_1A_2A_3Q$ that consists of two triangles ΔA_1A_2Q and ΔA_3A_2Q (i.e. labelled in 'red' in Figure 9(b) above). Therefore, the solid angles ω_n and $\omega_{\square A_1A_2A_3Q}$, subtended at the vertex A_2, by quadrilateral $A_1A_2A_3Q$ and given polygon respectively are equal. Now, the solid angle ω_n subtended by given polygon $A_1A_2A_3 \dots A_{n-1}A_n$ at the vertex A_2 is equal to the algebraic sum of solid angles $\sum_{i=1}^{n} \omega_i$ subtended at vertex A_2 by all (say n number of) elementary triangles, given as follows

$$\omega_n = \sum_{i=1}^{n} \omega_i$$

$$= \omega_{\square A_1A_2A_3Q} \quad \text{(Angle of vision of polygon)}$$

$$= \omega_{\Delta A_1A_2Q} + \omega_{\Delta A_3A_2Q} \quad \text{(Algebraic sum)}$$

$$= \angle A_1A_2Q + \angle A_3A_2Q \qquad \text{(From corollary 2)}$$

$$= \beta \text{ (internal vertex angle of polygon)}$$

It is proven from the above results that solid angle subtended by any polygonal plane (regular, irregular, convex, concave or any combination) at any of its vertices is equal to that vertex angle (measured internally in radians).

4.9. Corollary-9: The solid angle subtended by any plane at its vertex that connects two boundary curves (i.e. linear or non-linear) is always equal to the internal angle (i.e. measured through interior of plane) between the tangents at that point.

Proof. Consider an observation point P lying on the vertex H of a plane ABCDEFGHIJK (i.e. non-polygon) such that the vertex H is the meeting point of two arbitrary curves. The arbitrary curves meeting at the vertex H may both be linear, non- linear or one linear and other non-linear. These cases are proved as follows

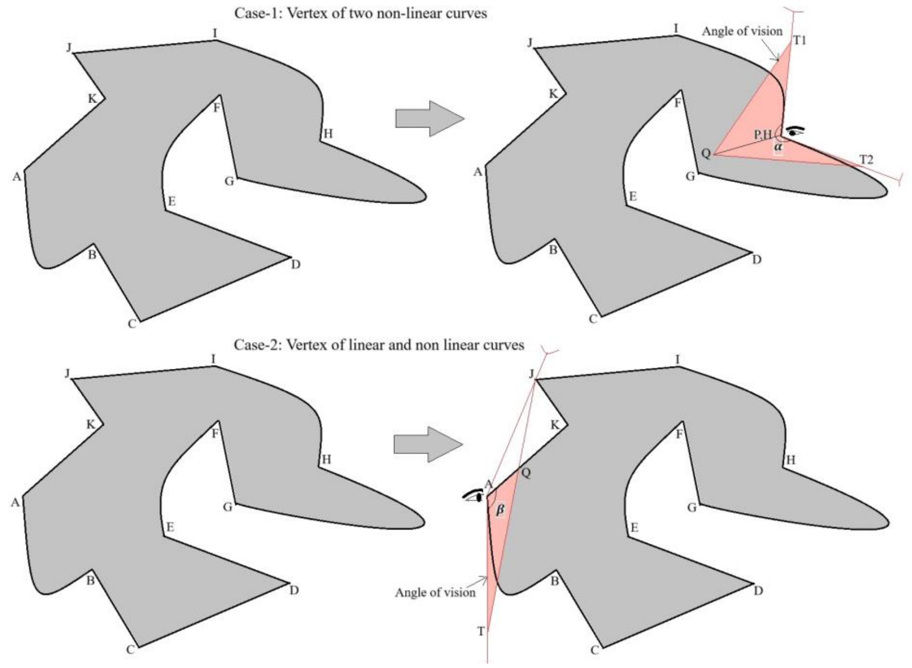

Figure 10: The observation point P lying on a vertex of non-polygonal plane ABCDEFGHIJK when (a) two curves meet at vertex H and (b) straight line and curve meet at the vertex A.

Case-1: Consider the observation point P lying on the vertex H which connects two non-linear boundary curves of non-polygonal plane ABCDEFGHIJK. Draw the tangents lines HT1 and HT2 to both the curves meeting at the vertex H (i.e. observation point P as shown in Figure 10(a)). It is worth noticing that given non-polygonal plane can be completely enclosed by the angular region (through the plane) between the tangent lines HT1 and HT2 ***i.e. angle of vision*** of given non-polygon [1]. In this case, the cone of vision can be represented by concave quadrilateral T1HT2Q that consists of ΔT1HQ and ΔT2HQ (as highlighted in red in Figure 10(a) above). Therefore, the solid angle ω_{non} subtended at the vertex H by non-polygon ABCDEFGHIJK will be equal to the solid angle subtended by cone of vision T1HT2Q, given as follows

$$\omega_n = \omega_{T1HT2Q} \qquad \text{(Angle of vision of non polygon)}$$

$$= \omega_{\Delta T1HQ} + \omega_{\Delta T2HQ} \qquad \text{(Algebraic sum)}$$

$$= \angle T1HQ + \angle T2HQ \qquad \text{(From Corollary 2)}$$

$$= \angle T1HT2$$

$$= \alpha \text{ (Angle between tangents)}$$

Case-2: Consider the observation point P lying on the vertex A which connects one non-linear and other straight line boundary-curves of non-polygonal plane ABCDEFGHIJK. Draw the tangent line AT to the nonlinear curve at the vertex A and extend the straight line AJ by joining the vertex J to vertex A to obtain the cone of vision (i.e. angular region (through the plane) between straight line AJ and tangent AT) that completely encloses the non-polygon (as shown in Figure 10(b)). The angle of vision can be represented by the triangular region JAT that consists of ΔJAQ and ΔTAQ (as shown in Figure 10(b) above). Since ΔJAQ is not a part of non-polygon hence its solid angle subtended at the vertex A will be zero. Therefore, the solid angle ω_{non} subtended at the vertex A by non-polygon ABCDEFGHIJK will be equal to the solid angle subtended by angle of vision JAT, given as follows

$$\omega_n = \omega_{\Delta JAT} \qquad \text{(Angle of vision of non polygon)}$$

$$= \omega_{\Delta \mathrm{JAQ}} + \omega_{\Delta \mathrm{TAQ}} \qquad \text{(Algebraic sum)}$$

$$= 0 + \omega_{\Delta TAQ} \qquad \left(\because \Delta JAQ \notin ABCDEFGHIJK \text{ from Fig. 8(b)}\right)$$

$$= \angle \mathrm{TAQ} \qquad \text{(From Corollary 2)}$$

$$= \beta \text{ (Angle between tangents)}$$

Case-3: If both the boundary curves are linear then the solid angle subtended by the plane at the vertex will be equal to that internal vertex angle from the above corollary-4.

It is proven from the above results that solid angle subtended by any plane (non-polygon) at any of its vertices connecting two curves is always equal to the internal angle between the tangents drawn at that vertex.

4.10. Corollary-10: The solid angle subtended by any plane at any point lying on its single smooth boundary-curve is always π sr.

Proof. Consider an observation point P lying on the boundary of a (confined) plane which is a single smooth curve. There two possible cases when the observation point lies on, 1) linear curve or straight line, 2) non-linear curve. The two cases are proven as follows

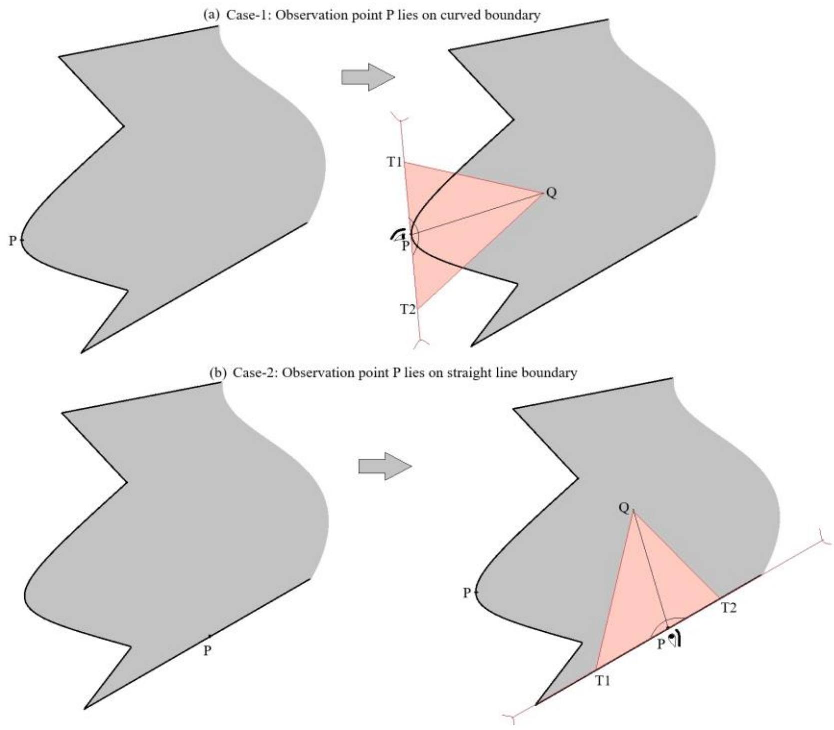

Figure 11: The observation point P lying on the boundary curve when it is (a) single smooth non-linear curve, and (b) straight line.

Case-1: Consider the observation point P lies on the boundary of a (confined) plane such that its boundary is a single-smooth non-linear curve (as shown in Figure 11(a)). It is worth noticing that the observer at the point P can see interior of given plane about the point P through an angle of π. Thus, ***angle of vision*** of given plane will be the region confined by tangent T1T2 (at point P) that extends through the interior of plane. In this case, the angle of vision can be represented by triangle T1T2Q that consists of ΔT1PQ and ΔT2PQ (as highlighted in red in Figure 11(a) above). Therefore, the solid angle $\omega_{boundary}$ subtended by the given plane, at an arbitrary point P lying on the non-linear boundary curve, will be equal to the solid angle subtended by cone of vision T1T2Q given as follows

$$\omega_{boundary} = \omega_{T1T2Q} \qquad \text{(Angle of vision of plane)}$$

$$= \omega_{\Delta T1PQ} + \omega_{\Delta T2PQ} \qquad \text{(Algebraic sum)}$$

$$= \angle T1PQ + \angle T2PQ \qquad \text{(From Corollary 2)}$$

$$= \angle T1PT2$$

$$= \pi$$

Case-2: Consider the observation point P lies on the straight line boundary of a (confined) plane (as shown in Figure 11(b)). It is worth noticing that the observer at the point P can see interior of given plane about the point P through an angle of π. Thus, the ***angle of vision*** of given plane will be the region confined by straight line T1T2 that extends through the interior of plane. In this case, the angle of vision can be represented by triangle T1T2Q that consists of ΔT1PQ and ΔT2PQ (as highlighted in red in Figure 11(b) above). Therefore, the solid angle $\omega_{boundary}$ subtended by the given plane, at an arbitrary point P lying on the non-linear boundary curve, will be equal to the solid angle subtended by cone of vision T1T2Q given as follows

$$\omega_{boundary} = \omega_{T1T2Q} \qquad \text{(Angle of vision of plane)}$$

$$= \omega_{\Delta T1PQ} + \omega_{\Delta T2PQ} \qquad \text{(Algebraic sum)}$$

$$= \angle T1PQ + \angle T2PQ \qquad \text{(From Corollary 2)}$$

$$= \angle T1PT2$$

$$= \pi$$

It is proven from the above results that solid angle subtended by any plane at an arbitrary point lying its boundary in form of a single-smooth curve, is always π sr.

4.11. Corollary-11: The solid angle subtended by any plane at any point lying inside it (i.e. its boundary curve) is always 2π sr.

Proof. Consider an observation point P lying inside the boundary of a plane (bounded or unbounded). In this case, the observer at the point P can see the interior of given plane through an angle of 2π about the point P (as show in Figure 12). Thus, the angle of vision of given plane can be represented by the circle that consists of angles $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}, \alpha_n$ about the centre P. Therefore, the solid angle ω_{inside} subtended by the plane at the internal point P will be equal to the solid angle subtended by the angle of vision given as follows

$$\omega_{inside} = \sum_{i=1}^{n} \omega_i \qquad \text{(Algebraic sum for angle of vision of plane)}$$

$$= \sum_{i=1}^{n} \alpha_i \qquad \text{(From corollary 2)}$$

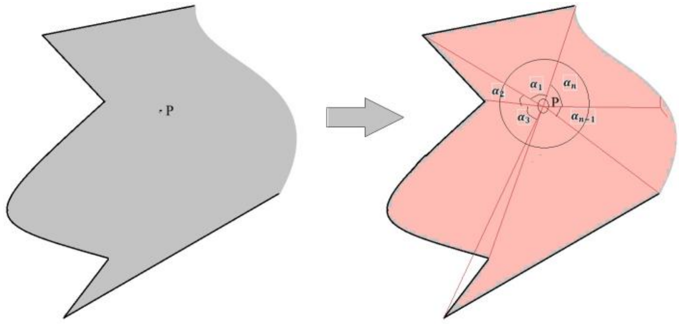

Figure 12: **observation point P lies inside the boundary of a plane.**

$$\omega_{inside} = 2\pi \qquad \text{(Complete angle from Fig. 12)}$$

It is proven from the above result that solid angle subtended by any plane at any point lying inside its boundary is always 2π sr.

Conclusions

In this work, the HCR's Theory of Polygon has been further strengthened through the simplification and reformulation of the standard or master formula i.e. the foundational expression used to compute the solid angle subtended by a right triangular plane at an arbitrary point in 3D space. The simplified forms presented here, expressed using inverse trigonometric functions such as arcsine, arccosine, and arctangent, make the master formula more accessible and easier to apply across a broader range of geometrical configurations. The behaviour of the solid angle with respect to variations in orthogonal side lengths and the observation distance has been illustrated through graphical analyses, offering deeper insight into the geometric dependence of solid-angle evaluation. Additionally, a set of corollaries describing the solid angle subtended by both polygonal and non-polygonal planes at various observation positions has been proposed and rigorously proved using the right-triangle standard formula and the concept of the angle of vision for coplanar observation of 2D figures. These results collectively demonstrate the versatility, generality, and mathematical coherence of the master formula that further established Theory of Polygon as a unified and systematic method for solid-angle computation and lay the groundwork for its broader application in geometry, physics, and engineering analysis.

References

[1] Rajpoot HC. HCR's Theory of Polygon (proposed by harish chandra rajpoot) solid angle subtended by any polygonal plane at any point in the space. Int. J. Math. Phys. Sci. Res. 2014;2:28-56.

[2] Rajpoot HC. HCR's or H. Rajpoot's Formula for Regular Polyhedron. 2014.

[3] Rajpoot H. Mathematical Analysis of Cuboctahedron. MMM University of Technology: Gorakhpur, India. 2014.

[4] Rajpoot HC. Optimum solution of snub dodecahedron (an Archimedean solid) by using HCR's theory of polygon & Newton–Raphson method. Dec. 2014. MMM University of Technology, Gorakhpur-273010 (UP), India [Internet].

[5] Rajpoot CH. Advanced geometry. 1st ed. Chennai: Notion Press Media Pvt. Ltd.; 2013 Apr 3. ISBN: 978-93-83808-15-1.