Thoughts on the generations problem

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Why are there three generations (leptons and quarks)? Are they related to one another? If yes, how? This document is a pioneer work shedding a renewed light on a topic that is currently an open questioning under investigation. It proposes an alternative way of expressing the invariance of the speed of light, which is based on the study of the deformations of the Poynting's vector. This method allows the introduction of trios of deforming matrices which are obliged to respect a very specific constraint. The work examines how to make this constraint compatible with the existence of ratios connecting the masses of three particles according to a formula proposed by Y. Koide.

Content

Context and motivation

The prediction

The attention of the scientific community has been drawn at the beginning of the eighties (20^{th} century) to an empirical prediction made by Y. Koide [01], [02], [03]. At this time, the masses of the (classical) electron (m_e – first generation) and of the muon (m_μ - second generation) were already known. The prediction said that the mass of the tau ($m\tau$ - third generation) can be calculated with a specific formula which is for example recalled in [04; (I.1)]:

$$\text{Ko} = \frac{m_e+m_\mu+m_\tau}{\left(\sqrt[2]{m_e}+\sqrt[2]{m_\mu}+\sqrt[2]{m_\tau}\right)^2} = \frac{2}{3}$$

The first measurements of the mass of the tau were not convincing, but their repetition and refinement ended up proving the prediction right; for now, with m_e = 0,511 MeV/c^2, m_μ = 105,66 MeV/c^2 [05], [20] and m_τ = 1777,09 ± 0,14 MeV/c^2 [06], [20], it can easily be verified that:

$$\text{Ko} = \frac{0{,}511 + 105{,}66 + 1777{,}09}{(0{,}7148 + 10{,}2791 + 42{,}1555)^2} = \frac{1883{,}261}{2824{,}8587} = 0{,}66667 \sim 2/3$$

Hazard, chance, obscure numerology or manifestation of a rational law? This is an outstanding question yet occupying brilliant brains [20], [21], [22], [07].

What can we do with this formula today?

At the end of the eighties (1980), Y. Koide has proposed a formula connecting the masses of leptons in a "trans-generation" manner. There were two remarkable facts about it: (i) its value: 2/3 and (ii) a prediction for the mass of the tau-electron. The value of the ratio gave rise to numerous speculations, most of them flirting with the numerology and bringing no real progress in the understanding of the formula. Thirty years later, the author was able to get the same ratio but with a different approach connecting it in a better manner with the standard model [08].

Since the mass of the tau is now known and since the value of the ratio can be verified by anyone, this formula no longer holds any particular interest today unless it helps revealing a until now hidden rule of nature.

The first experimental statement is a desacralization of the historical ratio since all other ratios that can be formed and calculated with the formula differ from 2/3 (see annex 01):

	Ko(leptons)	Ko(quarks I)	Ko(quarks II)
Trans- generations	2/3	0.85	0.73
	Ko(**I**)	Ko(**II**)	Ko(**III**)
Intra-generations	0.39	0.47	0.65

Table n°1

If this ratio is really meaning- and useful for physics, this can only entirely be because it would tell us that and how three masses (energies related to what is perceived as particles) can be linked together. This is the intellectual path that will be followed in this speculative document.

Definition: the generic Ko-ratio

Let consider a space vector $V_3 = E(3, K)$; in this document, K will be either $\mathbb{R}$ or $\mathbb{C}$. The triple (b_1, b_2, b_3) in K^3 represents the components of some vector **b** in V_3. The historical ratio can be understood as the prototype for a numerical function depending on three variables which are nothing but the components of **b**:

$$\forall\ \mathbf{b} \in V_3 \xrightarrow{Ko} \mathrm{Ko}(\mathbf{b}) = \mathrm{Ko}(b_1, b_2, b_3) = \frac{(b_1)^2 + (b_2)^2 + (b_3)^2}{(b_1 + b_2 + b_3)^2}$$

This generalized Ko-ratio can be synthetized as:

$$\mathrm{Ko}(\mathbf{b}) = \frac{||\mathbf{b}||^2}{(\mathbf{b}^{\oplus})^2},\ \mathbf{b}^{\oplus} \neq 0$$

... where ||**b**|| denotes the classical Euclidean norm of **b** whilst $\mathbf{b}^{\oplus}$ denotes the sum of its components. This function exists only if the sum of the components of **b** doesn't vanish. The original formula concerning the charged leptons is recovered in writing:

$$b^1 = \sqrt[2]{m_e},\ b^2 = \sqrt[2]{m_\mu},\ b^3 = \sqrt[2]{m_\tau}$$

Unfortunately, generalizing a seemingly intuitive formula explains neither its existence nor its experimental validity. This is the reason why I start this exploration in looking for a well-accepted fact concerning all particles in physics.

Preserving the speed of light

The evolutions of the Poynting-vector in a changing vacuum

The geometry can change, even in the empty regions of the universe. This affirmation is not a scoop, and everybody can verify it in reading, e.g., [09-DE; § 92, pp. 312-319].

But, in this changing environment, the speed of light should not change due to the information implicitly contained in the Morley and Michelson experiments [10].

They are at the origin of a deep revolution in our understanding of space-time. Not only the electromagnetic waves propagate at a given speed (it is not infinite) – see J. C. Maxwell's work [11]- but this speed is invariant for observers situated at the origin of inertial frames. This experimental fact gives rise to the Lorentz-Poincare transformations which indirectly appear in the first version of the Einstein's theory of relativity [12].

The leptons are the source of electromagnetic fields. Therefore, at a technical level, they can be understood as a set of pairs of spatial vectors (${}^{(3)}\mathbf{E}$, ${}^{(3)}\mathbf{B}$). A Poynting vector ${}^{(3)}\mathbf{S}$ can be associated with each state of these fields. Its usual definition is a classical cross product [09; § 31]:

$$\mathbf{E} = -\frac{1}{c}.\, \mathbf{v} \wedge \mathbf{B} \qquad\qquad \mathbf{S} = \frac{c}{4.\pi}.\, \mathbf{E} \wedge \mathbf{B}$$

$$\rho_{EM} = \frac{1}{8.\pi}.\, (<\mathbf{E}, \mathbf{E}>_{Id3} + <\mathbf{B}, \mathbf{B}>_{Id3}) = \frac{1}{4.\pi}.\, <\mathbf{E}, \mathbf{E}>_{Id3} = \frac{1}{4.\pi}.\, <\mathbf{B}, \mathbf{B}>_{Id3}$$

$$\mathbf{S} = \frac{c}{4.\pi}.\, (-\frac{1}{c}.\, \mathbf{v} \wedge \mathbf{B}) \wedge \mathbf{B} = \frac{1}{4.\pi}.\, \mathbf{B} \wedge (\mathbf{v} \wedge \mathbf{B}) = \frac{1}{4.\pi}.\, \{<\mathbf{B}, \mathbf{B}>_{Id3}.\, \mathbf{v} - \{<\mathbf{B}, \mathbf{v}>_{Id3}.\, \mathbf{B}\} = \rho_{EM}.\, \mathbf{v}$$

$$||\, \mathbf{S}\, || = \frac{c}{4.\pi}.\, ||\, \mathbf{E}\, ||.\, ||\, \mathbf{B}\, ||.\, \sin(\mathbf{E}, \mathbf{B})$$

It gives information on the energy carried by them and we can write:

$$\mathbf{E} \wedge \mathbf{B} = \frac{4.\pi.\rho_{EM}}{c}.\, \mathbf{v}$$

As mentioned before, the geometry can change, even in the empty regions of the universe. Within A. Einstein's theory of gravitation these modifications are carried by the components of the metric tensor and their partial derivates (first and second order). The theory[1] studying the deformation and the decomposition of tensor products explains how the classical cross product can also be deformed with the help of (3-3-3) cubes.

[1] The Theory of the (E) Question (TEQ).

Up to now, we consider (i) two three-dimensional frames respectively characterized by the pairs (Id_3, $\blacktriangledown\varepsilon$) and ($^{(3)}[G]$, $\blacktriangledown C$) and the evolution (hypothesis 1):

$$Id_3 \rightarrow [G] \Rightarrow \blacktriangledown\varepsilon \rightarrow \blacktriangledown C$$

This evolution of the geometry and of the context acts on the Poynting vector: $^{(3)}\mathbf{S} \rightarrow {}^{(3)}\mathbf{S}'$; we implicitly suppose that it preserves the global formalism and we write (hypothesis 2):

$$< {}^{(3)}\mathbf{S}, {}^{(3)}\mathbf{S} >_{Id3} = (\frac{4.\pi.\rho_{EM}}{c})^2. < {}^{(3)}\mathbf{v}, {}^{(3)}\mathbf{v} >_{Id3} \rightarrow < {}^{(3)}\mathbf{S}', {}^{(3)}\mathbf{S}' >_{[G]} = (\frac{4.\pi.\rho'_{EM}}{c})^2. < {}^{(3)}\mathbf{v}', {}^{(3)}\mathbf{v}' >_{[G]}$$

The next important question is to know how to define the deformed Poynting vector. We may suppose the existence of the formal transformation:

$$^{(3)}\mathbf{E} \wedge {}^{(3)}\mathbf{B} \rightarrow \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')$$

... with the intuitive prerequisite that the cube C is at least represented by three matrices with a vanishing diagonal (hypothesis 3).

$$\forall\, \alpha, \chi: C_{\alpha\chi\chi} = 0$$

This means that the general formulation of this deformed tensor product is reduced to:

$$\forall\, \alpha = 1, 2, 3:$$

$$\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')$$

$$=$$

$$\sum_\lambda \sum_\mu C_{\alpha\lambda\mu}. E'^\lambda. B'^\mu$$

$$=$$

$$C_{\alpha xx}. E'_x. B'_x + C_{\alpha xy}. E'_x. B'_y + C_{\alpha xz}. E'_x. B'_z$$

$$+ C_{\alpha yx}. E'_y. B'_x + C_{\alpha yy}. E'_y. B'_y + C_{\alpha yz}. E'_y. B'_z$$

$$+ C_{\alpha zx}. E'_z. B'_x + C_{\alpha zy}. E'_z. B'_y + C_{\alpha zz}. E'_z. B'_z$$

$$=$$

$$C_{\alpha xy}. E'_x. B'_y + C_{\alpha xz}. E'_x. B'_z + C_{\alpha yx}. E'_y. B'_x + C_{\alpha yz}. E'_y. B'_z + C_{\alpha zx}. E'_z. B'_x + C_{\alpha zy}. E'_z. B'_y$$

To be able to start calculations concerning these evolutions, we adopt a convention: "All symbols which are followed by a "'" concern what can be measured in a second frame. With prior hypotheses we can now describe the preservation of the speed of light with this new relation:

$$(\frac{4\pi}{c})^2 = \frac{1}{(\rho.\ v)^2}. < {}^{(3)}\mathbf{E} \wedge {}^{(3)}\mathbf{B}, {}^{(3)}\mathbf{E} \wedge {}^{(3)}\mathbf{B} >_{Id3} = \frac{1}{(\rho'.\ v')^2}. < \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}'), \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}') >_{[G]} \neq 0$$

This is the so-called "*fundamental relation*" within this approach.

The left-hand term in the fundamental relation

Some manipulations transform the left-hand term in:

$$\text{l.h.t.} = \frac{1}{(\rho.\ v)^2}.\ (\delta_{\lambda\nu}.\ \delta_{\mu\omega} - \delta_{\mu\nu}.\ \delta_{\lambda\omega}).\ E^{\lambda}.\ B^{\mu}.\ E^{\nu}.\ B^{\omega}$$

The l.h.t. contains a term that can always be understood as a peculiar representation of the object:

$$R_{\lambda\mu\nu\omega} = g_{\lambda\nu}.\ g_{\mu\omega} - g_{\mu\nu}.\ g_{\lambda\omega}$$

Precisely, the one which is obtained when $^{(3)}[G]$ can be identified with the Euclidean geometry:

$$\text{Lim}_{[G] \to Id3}\ R_{\lambda\nu\mu\omega} = \delta_{\lambda\nu}.\ \delta_{\mu\omega} - \delta_{\mu\nu}.\ \delta_{\lambda\omega}$$

As a matter of mathematical facts, the object "R" owns the same properties than the Rieman's curvature tensor; more precisely:

$$R_{\lambda\mu\nu\omega} = -\ R_{\mu\lambda\nu\omega}$$

$$R_{\lambda\mu\nu\omega} = -\ R_{\lambda\mu\omega\mu}$$

$$R_{\lambda\mu\nu\omega} = R_{\nu\omega\lambda\mu}$$

$$R_{\lambda\mu\nu\omega} + R_{\lambda\omega\mu\nu} + R_{\lambda\nu\omega\mu} = 0$$

The right-hand term in the fundamental relation

Up to now, we must start annoying but necessary calculations. For example, we must calculate the product $< \otimes_C(^{(3)}\mathbf{E'}, ^{(3)}\mathbf{B'}), \otimes_C(^{(3)}\mathbf{E'}, ^{(3)}\mathbf{B'}) >_{[G]}$.

A simple observation is enough to understand that we shall have terms in $E'^{\lambda}.\ E'^{\mu}.\ E'^{\nu}.\ E'^{\omega}$, $E'^{\lambda}.\ E'^{\mu}.\ E'^{\nu}.\ B'^{\omega}$, $E'^{\lambda}.\ B'^{\mu}.\ E'^{\nu}.\ B'^{\omega}$, $E'^{\lambda}.\ B'^{\mu}.\ B'^{\nu}.\ B'^{\omega}$ and $B'^{\lambda}.\ B'^{\mu}.\ B'^{\nu}.\ B'^{\omega}$. Let call them respectively T'_0, T'_1, T'_2, T'_3 and T'_4.

A second remark can be done: if the transformations relating the second frame (notation with "'") to the first one (notation without "'") are linear, e.g.: $E'^{\alpha} = f_{\alpha}(E^{\nu}, B'^{\omega})$, $B'^{\alpha} = h_{\alpha}(E^{\nu}, B'^{\omega})$, then the transformed product will be a sum of terms in $E^{\lambda}.\ E^{\mu}.\ E^{\nu}.\ E^{\omega}$, $E^{\lambda}.\ E^{\mu}.\ E^{\nu}.\ B^{\omega}$, $E^{\lambda}.\ B^{\mu}.\ E^{\nu}.\ B^{\omega}$, $E^{\lambda}.\ B^{\mu}.\ B^{\nu}.\ B^{\omega}$ and $B^{\lambda}.\ B^{\mu}.\ B^{\nu}.\ B^{\omega}$. Let call them respectively T_0, T_1, T_2, T_3 and T_4.

Since, at the end of the calculations, the right-hand term of the fundamental relation must be confronted with the left-hand term which only contains terms in T_2, the interesting situations are those for which the transformations are yielding:

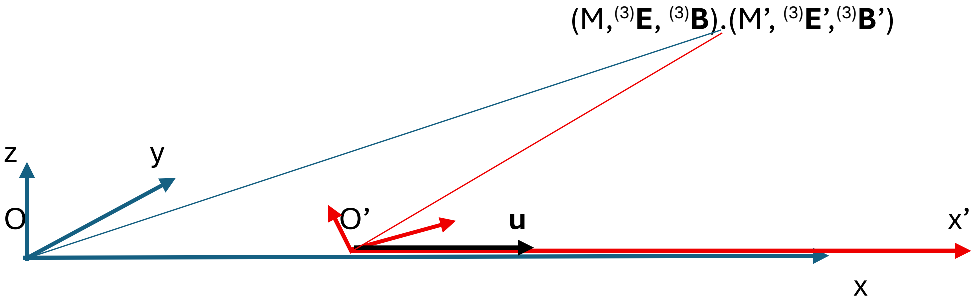
$$T_0 + T_1 + T_3 + T_4 = 0$$

This will be the so-called "*crucial condition*".

A warmup with a simple Lorentz transformation

In general, the Lorentz transformations are the tools allowing to confront the information concerning two inertial frames. As warm up, we start with the simple, usual and reduced Lorentz transformations given in [13 ; § 10.A.2, pp. 354-356. [10.A.6]], [09-DE ; §24, pp. 74-75]. Two physicists are observing the same electromagnetic field when this field is at point (M, t) for the observer in O but at point (M', t') for the observer in O'. It is in fact the same event which is studied with two different perspectives. This event is supposed to be at the boarder separating two physical contexts. The first physicist lives in a classical Euclidean world characterized by the pair (Id_3, ▼ε). The second one lives somewhere else in a different environment where the usual calculations are modified and must be realized with the information contained in the pair ($^{(3)}$[G], ▼C). Therefore, for the observer living in the second frame, the Poynting's vector looks like a deformed tensor product built around a cube ▼C of which the knots respect the hypothesis 3.

Both frames have the same x-axis, the origins of which coincides at t = 0. The second frame moves relatively to the first one with a constant speed u along the x-axis.



The Lorentz transformations describing this situation are characterized by the relations:

$$k = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$E'_x = E_x,\ E'_y = k.\ (E_y - u.\ B_z),\ E'_z = k.\ (E_z + u.\ B_y)$$

$$B'_x = B_x,\ B'_y = k.\ (B_y + \frac{u}{c^2}.\ E_Z),\ B'_z = k.\ (B_z - \frac{u}{c^2}.\ E_y)$$

These transformations are linear. Within this context, the deformed cross product writes now:

$$\{\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')\}_\alpha$$

$$=$$

$$k.\ C_{\alpha xy}.\ E_x.\ (B_y + \frac{u}{c^2}.\ E_Z) + k.\ C_{\alpha xz}.\ E_x.\ (B_z - \frac{u}{c^2}.\ E_y) + k.\ C_{\alpha yx}.\ (E_y - u.\ B_z).\ B_x + k.\ C_{\alpha zx}.\ (E_z + u.\ B_y).\ B_x$$

$$+ k^2.\ C_{\alpha zy}.\ (E_z + u.\ B_y).\ (B_y + \frac{u}{c^2}.\ E_Z) + k^2.\ C_{\alpha yz}.\ (E_y - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ E_y)$$

The compatibility with the description of plane waves

We verify that this expression doesn't systematically vanish when the electromagnetic field describes a wave in the plane (Oy, Oz). Indeed, in that case, although $E_x = B_x = 0$:

$$\{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_\alpha = k^2. \{C_{\alpha zy}. (E_z + u. B_y). (B_y + \frac{u}{c^2}. E_Z) + C_{\alpha yz}. (E_y - u. B_z). (B_z - \frac{u}{c^2}. E_y)\}$$

This mathematical fact tell us that what can be interpreted as a plane wave in the first frame continues to have an existence in the second inertial frame.

The fundamental relation for non-polarized plane waves

We can now calculate the r.h.t in the fundamental relation for *non-polarized plane waves*. Although there is no necessity to impose supplementary constraints than the third hypothesis on the matrices $[_\alpha C]$, it can be pedagogically interesting to work with fully *antisymmetric matrices* ; in that case, see the annex 02.

$$(\frac{4\pi}{c})^2$$

$$=$$

$$\frac{1}{(\rho'.\ v')^2}. g_{\alpha\beta}. \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_\alpha. \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_\beta$$

$$=$$

$$\frac{16,\pi^2}{(\rho'.\ v')^2}. g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz}. \frac{(c^2 - u.v)^2}{(c^2 - u^2)^2}. (v - u)^2. \rho^2_{EM}$$

We state that:

$$(\rho_{EM}'. \frac{v'}{c})^2 = (g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz}). \frac{(c^2 - u.v)^2}{(c^2 - u^2)^2}. (v - u)^2. \rho^2_{EM}$$

Since the relation is equating two squares, we can go a small step further and impose:

$$g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz} \geq 0$$

From which we get[2]:

$$\rho_{EM}'. v' = (g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz})^{1/2}. \rho_{EM}. \frac{\left(1 - \frac{u.v}{c^2}\right)}{\left(1 - \frac{u^2}{c^2}\right)}. |v - u|$$

[2] The c on the l.h.s is probably incorrect. The error comes from the relations in annex 02 describing non-polarized waves. I presume that, here, there should be a v/c instead of a v.

Let analyse the plausibility of this relation.

- When the inertial frames are static:

$$u = 0 \Rightarrow (\rho_{EM}'.\ v') = (g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz})^{1/2}.\ (\rho_{EM}.\ v)$$

We know that the impulse-energy tensor is preserved but not the kinetic momentum (here per unit of volume) alone. In this theory, $(g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz})^{1/2}$ is the factor of proportionality connecting the momentums.

- When the plane wave is propagating at the same speed and in the same direction than the second frame is moving, the observer at the origin of the second inertial frame gets the illusion that this wave is static: v' = 0 (see explanation below):

$$u = v \Rightarrow (\rho_{EM}'.\ v') = 0$$

- When the wave is propagating at c speed in the first frame, we can easily admit that it travels quicker than the origin O' of the second frame (u < c). This situation allows the calculation of:

$$(c - u).\ \frac{\left(1-\frac{u.c}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)} = \frac{1-u.c}{c+u}$$

The consequence is:

$$v = c \Rightarrow (\rho_{EM}'.\ v') = (g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz})^{1/2}.\ \frac{c-u}{c+u}.\ (\rho_{EM}.\ c)$$

This expression contains the ratio (c - u)/(c + u) which is signing the existence of a *Doppler-Fizeau effect* [14].

- Furthermore, we know:
- how to connect the speeds [09 ; § 5, p. 15, (5,3)] when ${}^{(3)}[G] \cong Id_3$:

$$\mathbf{v} - \mathbf{u} = (1 - \frac{1}{c^2}.\ <\mathbf{u}, \mathbf{v}'>_{Id3}).\ \mathbf{v}'$$

Here, the plane wave is propagating along the x-axis which coincides with the x'-axis. Therefore, **v**, **u** and **v**' are aligned and we can project the relation on Ox to get:

$$||\mathbf{v} - \mathbf{u}|| = (1 - \frac{1}{c^2}.\ <\mathbf{u}, \mathbf{v}'>_{Id3}).\ ||\ \mathbf{v}'\ ||$$

Or equivalently:

$$v - u = (1 - \frac{1}{c^2}.\ <\mathbf{u}, \mathbf{v}'>_{Id3}).\ v'$$

This projection partially eliminates the dependence on v':

$$\rho_{EM}' = |1 - \frac{1}{c^2}. < \mathbf{u}, \mathbf{v}' >_{Id3}|. (g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz})^{1/2}. \frac{\left(1-\frac{u.v}{c^2}\right)}{\left(1-\frac{u^2}{c^2}\right)}. \rho_{EM}$$

In general, the density of energy, alone, is not preserved. It is a statement which is easy to understand since the context for the observer at the origin of the second inertial frame may eventually totally differ from the one in the first frame.

- that the polarization (or its absence) is rarely preserved in a change of frame. This experimental fact can be for example translated into the equation through the relation:

$$g_{\alpha\beta}. C_{\alpha yz}. C_{\beta yz} = \sin^2\{d(\mathbf{E}, \mathbf{B})\}$$

... which is only valid if the geometry in the second inertial frame is spheric.

The fundamental relation for any electromagnetic field

Up to now, we examine what happens to any electromagnetic field when the relations linking both inertial frames are the Lorentz transformations given in [13 ; § 10.A.2, pp. 354-356. [10.A.6]], [09 ; §24, pp. 74-75]; see previous paragraph.

In that case, the scalar product remains an invariant, $< \mathbf{E}, \mathbf{B} > = Inv_1$, but the calculations are no longer simple; see annex 03. After complicated manipulations, we get:

$$(\frac{4\pi}{c})^2$$

$$=$$

$$\frac{1}{(\rho'.\ v')^2}. < \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}'), \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}') >_{[G]}$$

$$=$$

$$\frac{1}{(\rho'.\ v')^2}. \{T_0 + T_1 + T_2 + T_3 + T_4\}$$

$$=$$

$$(\frac{4.\pi.\rho_{EM}}{c})^2. < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{V} >_{[G]}$$

$$+$$

$$\frac{1}{(\rho'.\ v')^2}. (\frac{k.u}{c^2})^2.$$

$$\Sigma_\alpha \Sigma_\beta\, g_{\alpha\beta}. \{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x. ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x$$

$$- \rho_{EM}. \{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x. V_\beta + V_\alpha. ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x\}$$

$$+ ({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x. (c.\ {}^{(3)}\mathbf{B} \wedge {}_\beta\psi)_x + (c.\ {}^{(3)}\mathbf{B} \wedge {}_\alpha\psi)_x. ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x$$

$$- \rho_{EM}. \{(c.\ ^{(3)}\mathbf{B} \wedge {}_{\alpha}\psi)_x.\ V_\beta + V_\alpha.\ (c.\ ^{(3)}\mathbf{B} \wedge {}_{\beta}\psi)_x\}$$

$$+ (c.\ \mathbf{B} \wedge {}_{\alpha}\psi)_x.\ (c.\ \mathbf{B} \wedge {}_{\beta}\psi)_x\}$$

The physical interpretation of this relation is a challenge. We may recognize two parts; the first one introduces the x-components of three vectors respectively related to the electrical field and to the magnetic field:

$$\alpha = 1, 2, 3: (^{(3)}\mathbf{E} \wedge {}_{\alpha}\kappa)_x, (c.\ \mathbf{B} \wedge {}_{\alpha}\psi)_x$$

This statement allows the building of two vectors:

$$|\ ^{(3)}\mathbf{P} > = \begin{bmatrix} (^{(3)}\mathbf{E} \wedge {}_1\kappa)_x \\ (^{(3)}\mathbf{E} \wedge {}_2\kappa)_x \\ (^{(3)}\mathbf{E} \wedge {}_3\kappa)_x \end{bmatrix}, |\ ^{(3)}\mathbf{Q} > = c. \begin{bmatrix} (^{(3)}\mathbf{B} \wedge {}_1\psi)_x \\ (^{(3)}\mathbf{B} \wedge {}_2\psi)_x \\ (^{(3)}\mathbf{B} \wedge {}_3\psi)_x \end{bmatrix}$$

These vectors allow to state that:

$$T_0 = (\frac{k.u}{c^2})^2. <\ ^{(3)}\mathbf{P},\ ^{(3)}\mathbf{P} >_{[G]}$$

$$T_4 = (\frac{k.u}{c^2})^2. <\ ^{(3)}\mathbf{Q},\ ^{(3)}\mathbf{Q} >_{[G]}$$

The second one introduces a supplementary actor related to the deforming cube at hand and to the speed **v** (but take care of the order of the components):

$$|\ \mathbf{V} > = [\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}$$

It allows to write:

$$T_1 = (\frac{k.u}{c^2})^2. \{<\ ^{(3)}\mathbf{P},\ ^{(3)}\mathbf{V} >_{[G]} + <\ ^{(3)}\mathbf{V},\ ^{(3)}\mathbf{P} >_{[G]}\}$$

$$T_2 = (\frac{k.u}{c^2})^2. \{<\ ^{(3)}\mathbf{P},\ ^{(3)}\mathbf{Q} >_{[G]} + <\ ^{(3)}\mathbf{Q},\ ^{(3)}\mathbf{P} >_{[G]}\} + (\frac{4.\pi.\rho_{EM}}{c})^2. <\ ^{(3)}\mathbf{V},\ ^{(3)}\mathbf{V} >_{[G]}$$

$$T_3 = (\frac{k.u}{c^2})^2. \{<\ ^{(3)}\mathbf{Q},\ ^{(3)}\mathbf{V} >_{[G]} + <\ ^{(3)}\mathbf{V},\ ^{(3)}\mathbf{Q} >_{[G]}\}$$

The crucial condition

For logical reasons which have already been exposed in prior paragraphs, we shall only consider the situations such that:

$$\forall\ (\mathbf{E}, \mathbf{B}): T_0 + T_1 + T_3 + T_4 = 0$$

More precisely:

$$T_0 + T_1 + T_3 + T_4$$

$$=$$

$$(\frac{k.u}{c^2})^2. \{<\ ^{(3)}\mathbf{P},\ ^{(3)}\mathbf{P} >_{[G]} + <\ ^{(3)}\mathbf{P},\ ^{(3)}\mathbf{V} >_{[G]} + <\ ^{(3)}\mathbf{V},\ ^{(3)}\mathbf{P} >_{[G]}$$

$$+ < {}^{(3)}\mathbf{Q}, {}^{(3)}\mathbf{V} >_{[G]} + < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{Q} >_{[G]} + < {}^{(3)}\mathbf{Q}, {}^{(3)}\mathbf{Q} >_{[G]}\}$$

$$=$$

$$(\frac{k.u}{c^2})^2.\ \{< {}^{(3)}\mathbf{V} + {}^{(3)}\mathbf{P}, {}^{(3)}\mathbf{P} >_{[G]} + \{< {}^{(3)}\mathbf{P} + {}^{(3)}\mathbf{Q}, {}^{(3)}\mathbf{V} >_{[G]} + \{< {}^{(3)}\mathbf{Q} + {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{Q} >_{[G]}\}$$

$$=$$

$$0$$

We recognize a cyclicity. We also state that if:

$${}^{(3)}\mathbf{P} \perp_{[G]} {}^{(3)}\mathbf{V}$$

$${}^{(3)}\mathbf{Q} \perp_{[G]} {}^{(3)}\mathbf{V}$$

Then:

$$T_0 + T_1 + T_3 + T_4 = (\frac{k.u}{c^2})^2.\ \{< {}^{(3)}\mathbf{P}, {}^{(3)}\mathbf{P} >_{[G]} + \{< {}^{(3)}\mathbf{Q}, {}^{(3)}\mathbf{Q} >_{[G]}\} = 0$$

The crucial condition is up to now supposed to be *a priori* true, and we shall focus all our attention on a reduced fundamental relation:

$$g_{\alpha\beta}.\ \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_\alpha.\ \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_\beta$$

$$=$$

$$T_2$$

$$=$$

$$(\frac{4.\pi.\rho_{EM}}{c})^2.\ < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{V} >_{[G]} + (\frac{k.u}{c^2})^2.\ \{< {}^{(3)}\mathbf{P}, {}^{(3)}\mathbf{Q} >_{[G]} + < {}^{(3)}\mathbf{Q}, {}^{(3)}\mathbf{P} >_{[G]}\}$$

$$=$$

$$g_{\alpha\beta}.\ A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega}.\ E^\lambda.\ B^\mu.\ E^\nu.\ B^\omega$$

We must introduce a cube A which certainly differs from the cube C because a part of the terms in the initial and complete sum disappears when the crucial condition is true.

The condition on the pairs (▼A, $^{(3)}$[G]) and its sufficient version

We can go a step further in remarking that:

$$(\mathrm{Lim}_{[G]\to Id3}\ R_{\lambda\nu\mu\omega} + \mathrm{Lim}_{[G]\to Id3}\ R_{\lambda\omega\mu\nu} + \mathrm{Lim}_{[G]\to Id3}\ R_{\lambda\nu\omega\mu})$$

$$=$$

$$\mathrm{Lim}_{[G]\to Id3}\ \{\frac{(\rho.v)^2}{(\rho'.v')^2}.\ g_{\alpha\beta}.\ (A_{\alpha\lambda\mu}.\ A_{\beta\nu\omega} + A_{\alpha\lambda\omega}.\ A_{\beta\omega\nu} + A_{\alpha\lambda\nu}.\ A_{\beta\omega\mu})\}$$

$$=$$

$$0$$

The way of thinking which has been exposed here concludes that the admissible cubes ▼A and the spatial part of the metric tensor, $^{(3)}[G]$, must respect the condition:

$$\Sigma_{\alpha,\beta}\, g_{\alpha\beta}.\ \{[_\alpha A]^{\oplus}.\ [_\beta A]^{\oplus} + \{[_\alpha A].\ \{[_\beta A] + [_\beta A]^t\}\}^{\oplus}\} = 0$$

A "*sufficient condition*" not depending on the metric tensor is given through the relation:

$$\forall\ \alpha,\ \beta:\ [_\alpha A]^{\oplus}.\ [_\beta A]^{\oplus} + \{[_\alpha A].\ \{[_\beta A] + [_\beta A]^t\}\}^{\oplus} = 0$$

See the demonstration in annex 04.

Lemma 01: The antisymmetric matrices and the sufficient condition

It is easy to verify (without calculation) that all anti(skew)-symmetric matrices in M(3, $\mathbb{R}$ or $\mathbb{C}$) respect the sufficient condition. Their anti-symmetry is a sufficient condition, not a necessary one.

The sufficient condition for symmetric deforming matrices

The lemma induces a curiosity: "And what about the symmetric matrices?" When:

$$\forall\ \alpha = 1, 2, 3:\ [_\alpha A]^t = [_\alpha A]$$

The sufficient condition is modified and written as:

$$\forall\ \alpha,\ \beta:\ [_\alpha A]^{\oplus}.\ [_\beta A]^{\oplus} + \{2.\ [_\alpha A].\ [_\beta A]\}^{\oplus} = 0$$

Hence, for a given value of α:

$$\forall\ \alpha = 1, 2, 3:\ \{[_\alpha A]^{\oplus}\}^2 + 2.\ \{[_\alpha A]^2\}^{\oplus} = 0$$

Remark

We must note that, in general, the square of the sum of the entries in a symmetric matrix is not equal to the sum of the entries of the square of that given matrix:

$$\{[_\alpha A]^{\oplus}\}^2 \neq \{[_\alpha A]^2\}^{\oplus}$$

A simple example proves this affirmation; the identity matrix is obviously symmetric:

$$[A] = Id_3,\ [A]^{\oplus} = Id_3^{\oplus} = 3,\ \{[A]^{\oplus}\}^2 = 9,\ [A]^2 = Id_3,\ \{[_\alpha A]^2\}^{\oplus} = Id_3^{\oplus} = 3 \neq 9$$

Conventions and the sufficient condition

Let consider:

$$\forall\ \alpha = 1, 2, 3:\ [_\alpha A] = \begin{bmatrix} _\alpha a_{11} & _\alpha a_{12} & _\alpha a_{13} \\ _\alpha a_{12} & _\alpha a_{22} & _\alpha a_{23} \\ _\alpha a_{13} & _\alpha a_{23} & _\alpha a_{33} \end{bmatrix}$$

We get:

$$[_\alpha A]^{\oplus} = Tr[_\alpha A] + 2.\ (_\alpha a_{12} + {_\alpha a_{23}} + {_\alpha a_{13}})$$

Let consider that $[{}_{\alpha}A]$ contains the components of two vectors:

$${}_{\alpha}\mathbf{a}: ({}_{\alpha}a_{11}, {}_{\alpha}a_{22}, {}_{\alpha}a_{33}),\ {}_{\alpha}\mathbf{b}: ({}_{\alpha}a_{12}, {}_{\alpha}a_{23}, {}_{\alpha}a_{13})$$

We can then write that the sum of the entries of $[{}_{\alpha}A]$ is the sum of the sums of the components of both vectors:

$$[{}_{\alpha}A]^{\oplus} = {}_{\alpha}\mathbf{a}^{\oplus} + {}_{\alpha}\mathbf{b}^{\oplus}$$

Let calculate the square of $[{}_{\alpha}A]$:

$$[{}_{\alpha}A]^2 = [\sum_p {}_{\alpha}a_{mp} . {}_{\alpha}a_{pn}]$$

The sum of the entries of the square is:

$$\{[{}_{\alpha}A]^2\}^{\oplus} = \sum_m \sum_n \sum_p {}_{\alpha}a_{mp} . {}_{\alpha}a_{pn}$$

To simplify the notations, let prefer working with:

$$[S] = \begin{bmatrix} a & w & v \\ w & b & u \\ v & u & c \end{bmatrix}$$

It implicitly contains two vectors:

$$\mathbf{a}: (a, b, c) \text{ and } \mathbf{b}: (u, v, w)$$

Its square is:

$$[S]^2 = \begin{bmatrix} a^2 + w^2 + v^2 & a.w + w.b + v.u & a.v + w.u + v.c \\ w.a + b.w + u.v & w^2 + b^2 + u^2 & w.v + b.u + u.c \\ v.a + u.w + c.v & v.w + u.b + c.u & v^2 + u^2 + c^2 \end{bmatrix}$$

The sum of the entries is:

$$\{[S]^2\}^{\oplus}$$

$$=$$

$$(a^2 + b^2 + c^2) + 2. (u^2 + v^2 + w^2) + a. (u + v + w) - a. u + b. (u + v + w) - b. v + c. (u + v + w) - c. w$$

$$=$$

$$||\mathbf{a}||^2 + 2. ||\mathbf{b}||^2 + \mathbf{a}^{\oplus}. \mathbf{b}^{\oplus} - < \mathbf{a}, \mathbf{b} >_{Id3}$$

Therefore, with these conventions, the sufficient condition for any symmetric matrix is:

$$\{[S]^{\oplus}\}^2 + 2. \{[S]^2\}^{\oplus} = (\mathbf{a}^{\oplus} + \mathbf{b}^{\oplus})^2 + 2. ||\mathbf{a}||^2 + 4. ||\mathbf{b}||^2 + 2. \mathbf{a}^{\oplus}. \mathbf{b}^{\oplus} - 2. < \mathbf{a}, \mathbf{b} >_{Id3} = 0$$

The sufficient condition for symmetric matrices when the Ko ratios exist

Let suppose that there exists a Ko ratio for both vectors:

$$\exists\ Ko(\mathbf{a}) = \frac{||\mathbf{a}||^2}{(\mathbf{a}^{\oplus})^2},\ Ko(\mathbf{b}) = \frac{||\mathbf{b}||^2}{(\mathbf{b}^{\oplus})^2}$$

Let inject the ratios into the sufficient condition with the hope to simplify it:

$$(\mathbf{a}^{\oplus})^2 + (\mathbf{b}^{\oplus})^2 + 2.\ \mathbf{a}^{\oplus}.\ \mathbf{b}^{\oplus} + 2.\ ||\mathbf{a}||^2 + 4.\ ||\mathbf{b}||^2 + 2.\ \mathbf{a}^{\oplus}.\ \mathbf{b}^{\oplus} - 2. < \mathbf{a}, \mathbf{b} >_{Id3} = 0$$

$$(1 + 2.\ Ko(\mathbf{a})).\ (\mathbf{a}^{\oplus})^2 + (1 + 4.\ Ko(\mathbf{b})).\ (\mathbf{b}^{\oplus})^2 + 4.\ \mathbf{a}^{\oplus}.\ \mathbf{b}^{\oplus} - 2. < \mathbf{a}, \mathbf{b} >_{Id3} = 0$$

Let focus attention on symmetric matrices such that **a** = K. **b**. They are characterized by: Ko(**a**) = Ko(**b**), $\mathbf{a}^{\oplus} = K.\ \mathbf{b}^{\oplus}$, and the formalism:

$$[S(K.\ \mathbf{b}, \mathbf{b})] = \begin{bmatrix} K.u & w & v \\ w & K.v & u \\ v & u & K.w \end{bmatrix}$$

For these matrices, the sufficient condition is:

$$(1 + 2.\ Ko(\mathbf{b})).\ K^2.\ (\mathbf{b}^{\oplus})^2 + (1 + 4.\ Ko(\mathbf{b})).\ (\mathbf{b}^{\oplus})^2 + 4.\ K.\ \mathbf{b}^{\oplus}.\ \mathbf{b}^{\oplus} - 2.\ K. < \mathbf{b}, \mathbf{b} >_{Id3} = 0$$

$$\{(1 + 2.\ Ko(\mathbf{b})).\ K^2 + 4.\ K + (1 + 4.\ Ko(\mathbf{b}))\}.\ (\mathbf{b}^{\oplus})^2 - 2.\ K.\ ||\ \mathbf{b}\ ||^2 = 0$$

$$\{(1 + 2.\ Ko(\mathbf{b})).\ K^2 + 4.\ K + (1 + 4.\ Ko(\mathbf{b}))\}.\ (\mathbf{b}^{\oplus})^2 - 2.\ K.\ Ko(\mathbf{b}).\ (\mathbf{b}^{\oplus})^2 = 0$$

$$\{(1 + 2.\ Ko(\mathbf{b})).\ K^2 + 2.\ (2 - Ko(\mathbf{b})).\ K + (1 + 4.\ Ko(\mathbf{b}))\}.\ (\mathbf{b}^{\oplus})^2 = 0$$

Ko-ratios only exist when $(\mathbf{b}^{\oplus})^2 \neq 0$. Therefore, the sufficient condition is a polynomial of degree two:

$$(1 + 2.\ Ko(\mathbf{b})).\ K^2 + 2.\ (2 - Ko(\mathbf{b})).\ K + (1 + 4.\ Ko(\mathbf{b})) = 0$$

The Ko-ratios for particles in the standard model exist (see the table n°1). This fact suggests that we may have a chance to find an ad hoc value for K allowing to say that the matrix [S(K. **b**, **b**)] represents the three particles of which the Ko-ratio is Ko(**b**).

$$\Delta'$$

$$=$$

$$(2 - Ko(\mathbf{b}))^2 - (1 + 2.\ Ko(\mathbf{b})).\ (1 + 4.\ Ko(\mathbf{b}))$$

$$=$$

$$4 - 4.\ Ko(\mathbf{b}) + (Ko(\mathbf{b}))^2 - 1 - 6.\ Ko(\mathbf{b}) - 8.\ (Ko(\mathbf{b}))^2$$

$$=$$

$$3 - 10.\ Ko(\mathbf{b}) - 7.\ (Ko(\mathbf{b}))^2$$

Let calculate the discriminant for the diverse ratios:

Δ'	Leptons	Quarks I	Quarks II
Trans- generations	- 6.7777	- 10.5575	- 8.0303
	(I)	**(II)**	**(III)**
Intra-generations	- 1.9647	- 3.2463	- 6.4575

Table n°2

They all are negative real numbers. There are always two possible values for K in $\mathbb{C}$.

$$K_{\pm} = \{Ko(\mathbf{b}) - 2 \pm i.\ (7.\ (Ko(\mathbf{b}))^2 + 10.\ Ko(\mathbf{b}) - 3)^{1/2}\} / \{1 + 2.\ Ko(\mathbf{b})\}$$

We can always find two symmetric matrices [S($K_{\pm}$. **b**, **b**)] representing the ratio Ko(**b**) and respecting the sufficient condition; example: for Ko(**b**) = 2/3, K_+ = - 0,5714 + i. 1,1157.

The Perian matrices

Definition

Per convention, a Perian matrix is the image of a pair[3] ($_P\mathbf{a}$, $_P\mathbf{b}$) through the function M:

$$\forall\ {}_P\mathbf{a}: (\alpha, \beta, \chi),\ {}_P\mathbf{b} \in V_3 \overset{M}{\rightarrow} M({}_P\mathbf{a}, {}_P\mathbf{b}) = \alpha.\ Id_3 + \beta.\ T_2(\otimes)({}_P\mathbf{b}, {}_P\mathbf{b}) + \chi.\ {}_{[J]}\Phi({}_P\mathbf{b}) \in M(3, K)$$

In this document, K = $\mathbb{R}$ or $\mathbb{C}$.

The sufficient condition for Perian matrices

It can be proved that the sufficient condition for a Perian matrices is (see the demonstration in annex 05):

$$\forall\ [A] = M({}_P\mathbf{a}, {}_P\mathbf{b}) \qquad {}_P\mathbf{a}: (\alpha, \beta, \chi)$$

$$\beta^2.\ ({}_P\mathbf{b}^{\oplus})^4 + (10.\ \alpha.\ \beta + 2.\ \beta^2.\ ||{}_P\mathbf{b}||^2).\ ({}_P\mathbf{b}^{\oplus})^2 + 15.\ \alpha^2 = 0$$

We immediately remark that this condition doesn't depend on the third component of $_P\mathbf{a}$. Therefore, it doesn't depend on the anti-symmetric part of the Perian matrices.

The interplays between symmetric matrices and Perian matrices

The first evidence is that a symmetric matrix is in general not a Perian matrix. But it is a matter of facts that any symmetric matrix [S] can be related to a pair (**a**, **b**); see prior paragraphs. At the same time, nothing forbids the construction of a Perian matrix with this given pair. Let examine if there exist relations linking [S(**a**, **b**)] and M(**a**, **b**).

$$(\mathbf{a}, \mathbf{b}) \rightarrow [S(\mathbf{a}, \mathbf{b})] = \begin{bmatrix} a & w & v \\ w & b & u \\ v & u & c \end{bmatrix}$$

$$\downarrow$$

$$M(\mathbf{a}, \mathbf{b})$$

$$=$$

$$a.\ Id_3 + b.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) + c.\ {}_{[J]}\Phi(\mathbf{b})$$

[3] **Caution**: Here, the pair involved in a Perian matrix is in general totally different than the pair in any symmetric matrix. The vector $_P\mathbf{a}$ in a Perian matrix (subscript P) is α. $\mathbf{n}_0$ + $_P\mathbf{m}$; with $\mathbf{n}_0$: (1, 1, 1), $_P\mathbf{m}$: (($_Pb_1)^2$, $(_Pb_2)^2$, $(_Pb_3)^2$). Hence $_P\mathbf{a}^{\oplus}$ = 3. α + $||_P\mathbf{b}||^2$.

=

$$\begin{bmatrix} a + b.u^2 & b.u.v - c.w & b.u.w + c.v \\ b.v.u + c.w & a + b.v^2 & b.v.w - c.u \\ b.w.u - c.v & b.w.v + c.u & a + b.w^2 \end{bmatrix}$$

The Perian matrix associated with the pair (**a**, **b**) characterizing a symmetric matrix [S] is no more a symmetric matrix, except if the third component of **a** vanishes (c = 0). In that very special case:

$$(\mathbf{a}, \mathbf{b}) \rightarrow [S(\mathbf{a}, \mathbf{b})] = \begin{bmatrix} a & w & v \\ w & b & u \\ v & u & 0 \end{bmatrix}$$

↓

M(**a**, **b**)

=

a. Id$_3$ + b. T$_2$(⊗)(**b**, **b**)

=

$$\begin{bmatrix} a + b.u^2 & b.u.v & b.u.w \\ b.v.u & a + b.v^2 & b.v.w \\ b.w.u & b.w.v & a + b.w^2 \end{bmatrix}$$

=

[S(**a**', **b**')]

The Perian matrix related to [S(**a**, **b**)] is now a new symmetric matrix [S(**a**', **b**')] such that:

a' = a. $\mathbf{n}_0$ + b. **m**

b': b. (v. w, w. u, u. v)

Conversely, we may ask ourselves if a symmetric matrix can sometimes be a Perian matrix. This is the case each time we can find a pair ($_P$**a**, $_P$**b**) such that:

[S(**a**, **b**)]

=

$$\begin{bmatrix} a & w & v \\ w & b & u \\ v & u & c \end{bmatrix}$$

=

M($_P$**a**, $_P$**b**)

=

α. Id$_3$ + β. T$_2$(⊗)($_P$**b**, $_P$**b**)

$$=$$

$$\begin{bmatrix} \alpha + b_1^2 & b_1.b_2 & b_1.b_3 \\ b_2.b_1 & \alpha + b_2^2 & b_2.b_3 \\ b_3.b_1 & b_3.b_2 & \alpha + b_2^2 \end{bmatrix}$$

This exigence is equivalent to the system:

$$a = \alpha + (b_1)^2$$

$$b = \alpha + (b_2)^2$$

$$c = \alpha + (b_3)^2$$

$$u = b_2.\ b_3$$

$$v = b_1.\ b_3$$

$$w = b_1.\ b_2$$

Here, (a, b, c) and (u, v, w) are known whilst (α, b_1, b_2, b_3) are unknown. The resolution of this system is in some way equivalent to the discovery of an element in V_4 when two elements in V_3 are already known.

This statement roughly suggests a way connecting any pair (**E**, **B**) representing an electromagnetic field and a significant physical four-dimensional tool associated with it.

Let try to characterize the domain where solutions may eventually exist. We state that:

$$u.\ v.\ w = (a - \alpha).\ (b - \alpha).\ (c - \alpha)$$

This relation tells us that we shall find three values for α when the discussion is developed with components in $\mathbb{C}$. For each value of α, we shall then discover two values for b_1, two for b_2 and two for b_3. Summa summorum, each pair (**a**, **b**) can be related to six symmetric Perian matrices.

In what follows, we abandon the subscript P to simplify the notation but insist on the important information recalled in footnote 3 and illustrated through the discussion within this paragraph.

A misleading indication and what it tells us

It is a matter of mathematical fact that the polynomials:

$$4.\ (\mathbf{b}^{\oplus})^4 - 16.\ (\mathbf{b}^{\oplus})^2 + 15 = 0$$

... always have two solutions:

$$\forall\ n: (\mathbf{b}^{\oplus})^2 = 3/2 \text{ and } (\mathbf{b}^{\oplus})^2 = 5/2$$

Hence, with the supplementary condition:

$$||\mathbf{b}|| = 1$$

We dispose of a set of vectors denoted **b** such that:

$$\mathrm{Ko}(\mathbf{b}) = 2/3 \text{ and } \mathrm{Ko}(\mathbf{b}) = 2/5$$

We get the historical value 2/3 for the ratio Ko(**b**). "Can we obtain this interesting polynomial in starting from:

$$\beta^2 . (\mathbf{b}^{\oplus})^4 + (10. \alpha . \beta + 2. \beta^2 . ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2 + 15. \alpha^2 = 0$$

... when $||\mathbf{b}|| = 1$?" To get it, we must impose:

$$\alpha^2 = 1, \beta^2 = 4, 10. \alpha . \beta + 2. \beta^2 = - 16$$

Let consider the diverse possibilities:

$$\alpha = 1, \beta = 2 \Rightarrow 10. \alpha . \beta + 2. \beta^2 = 28 \neq -16$$

$$\alpha = - 1, \beta = 2 \Rightarrow 10. \alpha . \beta + 2. \beta^2 = - 12 \neq -16$$

$$\alpha = 1, \beta = - 2 \Rightarrow 10. \alpha . \beta + 2. \beta^2 = - 12 \neq -16$$

$$\alpha = - 1, \beta = - 2 \Rightarrow 10. \alpha . \beta + 2. \beta^2 = 28 \neq -16$$

Conclusion: we cannot recover this promising polynomial in starting from the sufficient condition. This example proves that we must give much attention to the logical progression.

The logical progression structuring this approach

We intuitively believe there is a link between the preservation of $(4.\pi/c)$ and the Ko ratios. Since the Ko ratios exist and since their values are known (see table n°1), we should first rewrite the sufficient condition in taking the Ko(**b**) ratios into account:

$$\beta^2 . (\mathbf{b}^{\oplus})^4 + (10. \alpha . \beta + 2. \beta^2 . \mathrm{Ko}(\mathbf{b}). (\mathbf{b}^{\oplus})^2). (\mathbf{b}^{\oplus})^2 + 15. \alpha^2 = 0$$

$$\downarrow$$

$$(1 + 2. \mathrm{Ko}(\mathbf{b})). \beta^2 . (\mathbf{b}^{\oplus})^4 + 10. \alpha . \beta . (\mathbf{b}^{\oplus})^2 + 15. \alpha^2 = 0$$

All Perian matrices respecting this sufficient condition for a given value of some experimental Ko-ratio can join this conversation. Anticipating a development of the discussion, we want to recall here the sum of the entries of any Perian matrix:

$$[\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus} = 3. \alpha + \beta . (\mathbf{b}^{\oplus})^2$$

As consequence, we can also rework the sufficient condition in eliminating α. Let start with:

$$(1 + 2. \mathrm{Ko}(\mathbf{b})). \beta^2 . (\mathbf{b}^{\oplus})^4 + 10/3. \{[\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus} - \beta . (\mathbf{b}^{\oplus})^2\}. \beta . (\mathbf{b}^{\oplus})^2 + 5/3. \{[\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus} - \beta . (\mathbf{b}^{\oplus})^2\}^2 = 0$$

And get after some simple manipulations:

$$(2. \mathrm{Ko}(\mathbf{b}) - 2/3). \beta^2 . (\mathbf{b}^{\oplus})^4 + 5/3. \{[\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus}\}^2 = 0$$

For all Ko-ratios related to particles in the standard model (see table n°1 again):

$$2.\ \mathrm{Ko}(\mathbf{b}) - 2/3 > 0$$

2. Ko(**b**) - 2/3	Leptons	Quarks I	Quarks II
Trans- generations	2/3	1.0334	0.7934
	(I)	**(II)**	**(III)**
Intra-generations	0.1134	0.2734	0.6334

Table n°3

Therefore, the sufficient condition is the vanishing sum of two positive real numbers, and it is equivalent to two simultaneous sub-conditions:

$$(2.\ \mathrm{Ko}(\mathbf{b}) - 2/3).\ \beta^2.\ (\mathbf{b}^{\oplus})^4 = 5/3.\ \{[\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus}\}^2 = 0$$

Since the Ko-ratios doesn't exist when the sum of the components of **b** is null, the first sub-condition is realized when either Ko(**b**) = 1/3 or $\beta = 0$ whilst the second sub-condition is exclusively realized when the sum of the entries of the Perian matrix at hand is null. There are two sets of allowed situations:

$$\mathrm{Ko}(\mathbf{b}) = 1/3,\ [\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus} = 0 = 3.\ \alpha + \beta.\ (\mathbf{b}^{\oplus})^2 \Rightarrow [\mathrm{M}(\mathbf{a}, \mathbf{b})] = \alpha.\ \mathrm{Id}_3 - \frac{3.\alpha}{(\mathbf{b}^{\oplus})^2}.\ \mathrm{T}_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi.\ {}_{[J]}\Phi(\mathbf{b})$$

$$\forall\ \mathrm{Ko}(\mathbf{b}),\ \beta = 0,\ [\mathrm{M}(\mathbf{a}, \mathbf{b})]^{\oplus} = 0 \Rightarrow \alpha = 0,\ [\mathrm{M}(\mathbf{a}, \mathbf{b})] = \chi.\ {}_{[J]}\Phi(\mathbf{b})$$

This result is interesting because it indicates the formalism of Perian matrices for which the sufficient condition is true. But it is incomplete because we must discover pairs of deforming matrices respecting the sufficient condition:

$$\forall\ \lambda, \mu: [{}_{\lambda}A]^{\oplus}.\ [{}_{\mu}A]^{\oplus} + \{[{}_{\lambda}A].\ \{[{}_{\mu}A] + [{}_{\mu}A]^t\}\}^{\oplus} = 0$$

There is no reason forcing to believe that both arguments in a pair ($[{}_{\lambda}A]$, $[{}_{\mu}A]$) belong to the set of Perian matrices. But if they must or do, then we have at least to manage an embarrassing interrogation because the Perian matrices allowing the existence of a Ko(**b**)-ratio equal to 2/3 and respecting the sufficient condition are only represented through axial rotations. This questioning will reappear a little bit later in this document because we shall prove the existence of Perian matrices representing involutions and related to a Ko(**b**)-ratio equal to 2/3. Fortunately for the coherence of this approach, we shall also prove that none of them can respect the sufficient condition; see below the lemma 02.

For now, let only consider pairs of deforming matrices such that:

$$\forall\ \lambda, \mu: [{}_{\lambda}A]^{\oplus} = [{}_{\mu}A]^{\oplus} = 0$$

For these pairs, the sufficient condition has a simplified formulation:

$$\forall\ \lambda, \mu: \{[{}_{\lambda}A].\ \{[{}_{\mu}A] + [{}_{\mu}A]^t\}\}^{\oplus} = 0$$

The lemma 01 gives the insurance that this condition is automatically true when $[_\mu A]$ is a Perian matrix related to the second set of allowed situations, whatever the Ko($\mathbf{b}_\mu$)-ratio is. When $[_\mu A]$ is a Perian matrix related to the first set of allowed situations, the Ko($\mathbf{b}_\mu$)-ratio is equal to 1/3 and:

$$[M(\mathbf{a}_\mu, \mathbf{b}_\mu)] + [M(\mathbf{a}_\mu, \mathbf{b}_\mu)]^t = 2.\ \alpha_\mu.\ \{Id_3 - \frac{3}{\left(\mathbf{b}_\mu{}^{\oplus}\right)^2}.\ T_2(\otimes)(\mathbf{b}_\mu, \mathbf{b}_\mu)\}$$

We remark at this stage that if we would not know that this symmetric matrix is representing a precise Ko-ratio, we could also write it:

$$[M(\mathbf{a}_\mu, \mathbf{b}_\mu)] + [M(\mathbf{a}_\mu, \mathbf{b}_\mu)]^t = 2.\ \alpha_\mu.\ \{Id_3 - \frac{3.Ko(\mathbf{b}_\mu)}{(||\mathbf{b}_\mu||)^2}.\ T_2(\otimes)(\mathbf{b}_\mu, \mathbf{b}_\mu)\}$$

And if the ratio at hand would be the historical one (equal to 2/3), then we would write this symmetric Perian matrix:

$$[M(\mathbf{a}_\mu, \mathbf{b}_\mu)] + [M(\mathbf{a}_\mu, \mathbf{b}_\mu)]^t = 2.\ \alpha_\mu.\ \{Id_3 - \frac{2}{(||\mathbf{b}_\mu||)^2}.\ T_2(\otimes)(\mathbf{b}_\mu, \mathbf{b}_\mu)\}$$

This formalism will reappear later when we shall study the representation of involution through Perian matrices.

Perian matrices representing an involution

The previous results induce a question: "Is it possible to adjust the Euclidean norm $||\mathbf{b}||$ in a manner ensuring that the admissible Perian matrices are involutions, i.e.: elements in Invol(3)?" An answer to this legitimate question can only be obtained in precising the formalism of Perian matrices in Invol(3). For that purpose, recall that:

$$[M(\mathbf{a}, \mathbf{b})]^2$$

$$=$$

$$\{\alpha.\ Id_3 + \beta.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi.\ {}_{[J]}\Phi(\mathbf{b})\}.\ \{\alpha.\ Id_3 + \beta.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi.\ {}_{[J]}\Phi(\mathbf{b})\}$$

$$=$$

$$(\alpha^2 - \chi^2).\ Id_3 + (2.\ \alpha.\ \beta + \beta^2.\ ||\mathbf{b}||^2 + \chi^2).\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) + 2.\ \alpha.\ \chi.\ {}_{[J]}\Phi(\mathbf{b})$$

There are only a small number of possibilities to get:

$$[M(\mathbf{a}, \mathbf{b})]^2 = Id_3$$

We state that (annex 07):

- Either $\alpha = 0$ and we get the Perian matrices:

$$[M(\mathbf{a}, \mathbf{b})] = \frac{1}{||\mathbf{b}||}.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) + i.\ {}_{[J]}\Phi(\mathbf{b})$$

$$[M(\mathbf{a}, \mathbf{b})] = -\frac{1}{||\mathbf{b}||}.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) - i.\ {}_{[J]}\Phi(\mathbf{b})$$

$$[M(\mathbf{a}, \mathbf{b})] = \frac{1}{||\mathbf{b}||}. T_2(\otimes)(\mathbf{b}, \mathbf{b}) - i. {}_{[J]}\Phi(\mathbf{b})$$

$$[M(\mathbf{a}, \mathbf{b})] = - \frac{1}{||\mathbf{b}||}. T_2(\otimes)(\mathbf{b}, \mathbf{b}) + i. {}_{[J]}\Phi(\mathbf{b})$$

For the details see [c; annex 5.1] and [4]; here, $|M(\mathbf{a}, \mathbf{b})| = \beta. \chi^2. ||\mathbf{b}||^4$. More precisely, since $\beta = \pm 1/||\mathbf{b}||$, $\chi^2 = -1$, the determinants of these matrices are:

$$\left| \frac{1}{||\mathbf{b}||}. T_2(\otimes)(\mathbf{b}, \mathbf{b}) \pm i. {}_{[J]}\Phi(\mathbf{b})\right| = - ||\mathbf{b}||^3$$

$$\left| - \frac{1}{||\mathbf{b}||}. T_2(\otimes)(\mathbf{b}, \mathbf{b}) \pm i. {}_{[J]}\Phi(\mathbf{b})\right| = ||\mathbf{b}||^3$$

- Or $\chi = 0$ and we get the Perian matrices:

$$[M(\mathbf{a}, \mathbf{b})] = \pm Id_3$$

$$[M(\mathbf{a}, \mathbf{b})] = Id_3 - \frac{2}{||\mathbf{b}||^2}. T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

$$[M(\mathbf{a}, \mathbf{b})] = - Id_3 + \frac{2}{||\mathbf{b}||^2}. T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

For the details see [c; annex 5.1] and [5]; here, $|M(\mathbf{a}, \mathbf{b})| = \alpha^2. (\alpha + \beta. ||\mathbf{b}||^2)$. More precisely, since $\alpha = \pm 1$ and $\beta = \pm 2/||\mathbf{b}||^2$, the determinants of these matrices are:

$$|\pm Id_3| = \pm 1$$

$$\left|Id_3 - \frac{2}{||\mathbf{b}||^2}. T_2(\otimes)(\mathbf{b}, \mathbf{b})\right| = - 1$$

$$\left|- Id_3 + \frac{2}{||\mathbf{b}||^2}. T_2(\otimes)(\mathbf{b}, \mathbf{b})\right| = 1$$

- Or $(\alpha^2 - \chi^2) = 1$ and all eventualities have already been treated just before.

Characteristics of Perian matrices in Invol(3)

There are only two specific subsets of Perian matrices in Invol(3); we call them $Invol(3)_1$ and $Invol(3)_2$. Of course, they all respect the definition:

$$Invol(3) = \{[M(\mathbf{a}, \mathbf{b})]: \forall\, \mathbf{b} \in E(3, K = \mathbb{R} \text{ or } \mathbb{C}), [M(\mathbf{a}, \mathbf{b})] = [M(\mathbf{a}, \mathbf{b})]^{-1}\}$$

A) In both subsets:

- the first argument in the pairs (**a**, **b**), precisely the vector **a**, is not free.
- Each subset contains two families of matrices; more precisely: if [M(**a**, **b**)] denotes the generic formalism of an element in the first family, then an element in the second family has the formalism of minus [M(**a**, **b**)]. Furthermore, for each given **b**, each family contains only two elements. Hence, for each given **b**, each subset

[4] In general, $|M(\mathbf{a}, \mathbf{b})| = (\alpha + \beta. ||\mathbf{b}||^2). (\alpha^2 + \chi^2. ||\mathbf{b}||^2)$.
[5] In general, $|M(\mathbf{a}, \mathbf{b})| = (\alpha + \beta. ||\mathbf{b}||^2). (\alpha^2 + \chi^2. ||\mathbf{b}||^2)$.

contains only four elements and for each given **b** again, there are eight elements in Invol(3).

B) The elements in $Invol(3)_1$:

- are the sum of (i) a symmetric Pythagorean matrix built on the classical tensor product and (ii) an anti-symmetric matrix related to an axial rotation around **b**.
- the matrices $\{- T_2(\otimes)(\mathbf{b}, \mathbf{b}) \pm i. {}_{[J]}\Phi(\mathbf{b})\}$ when $||\mathbf{b}|| = \pm 1$ are in SO(3), respectively in SU(3).

C) The elements in $Invol(3)_2$, are symmetric matrices. Therefore:

- they verify the supplementary property:

$$Invol(3)_2 = \{[M(\mathbf{a}, \mathbf{b})]: \forall\ \mathbf{b}, [M(\mathbf{a}, \mathbf{b})] = [M(\mathbf{a}, \mathbf{b})]^{-1} = [M(\mathbf{a}, \mathbf{b})]^t\}$$

- The identity matrix Id_3 and the matrices $- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})$ are in SO(3), respectively in SU(3).

The Perian matrices in SU(3) and the Koide ratio for leptons

Proposition: It is a strange matter of mathematical facts that when (i) the vector **b** doesn't vanish ($\mathbf{b} \neq \mathbf{0}$), (ii) the sum of the entries of the matrices $- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})$ is null, then Ko(**b**) is equal to 2/3.

Proof - (easy): $\forall\ \mathbf{b} \neq \mathbf{0}$: $\{- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})\}^{\oplus} = - 3 + 2. (\mathbf{b}^{\oplus})^2/||\mathbf{b}||^2$. Hence, when $\{- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})\}^{\oplus} = 0$, it is evident that: $||\mathbf{b}||^2/(\mathbf{b}^{\oplus})^2 = Ko(\mathbf{b}) = 2/3$. □

Comments: When the vector **b** is not null whilst its components are real and positive numbers (elements in $\mathbb{R}^+$), it may eventually represent three masses. In that case, the Perian matrices representing an involution in SO(3) are a subset in O(3). Prior short demonstration proves that the matrices $- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})$ can represent a transgenerational set of three leptons. They can be involved in nuclear collisions and reactions when the three generations appears simultaneously. We shall see a little bit later that these matrices have supplementary interesting mathematical characteristics because they also are the representations of Euler-Rodrigues parametrisations.

If the discussion is extended to vectors representing trios of energies (like it may be the case within the quantum chromodynamics (QCD), then these matrices form also a subset in U(3).

Perian matrices in SU(3) and Gell-Mann matrices

The set SU(3) plays also an important role in quantum physics, especially in QCD. This sub-result allows at least a formal link with the Gell-Mann matrices and with the gluons. Recall that the gluons are transferring momentum and colours between quarks. This fact suggests testing the structure of Perian matrices in SU(3). Let label them for convenience:

$$M_1 = Id_3$$

$$M_2 = - Id_3 + 2.\ T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

$$M_{3+} = - T_2(\otimes)(\mathbf{b}, \mathbf{b}) + i.\ {}_{[J]}\Phi(\mathbf{b})$$

$$M_{3-} = - T_2(\otimes)(\mathbf{b}, \mathbf{b}) - i.\ {}_{[J]}\Phi(\mathbf{b})$$

It is obvious that:

$$M_1 + M_2 + M_{3+} + M_{3+} = [0]$$

Hence, if an addition in the set of Perian matrices in SU(3) represents the interaction between trios of energies (particles), then this formula describes the first "*nuclear reaction*" of this theory. This is a beautiful idea, but it would be better to relate this relation to something which is known concerning the gluons or the quarks.

For this purpose, which will not be achieved in this document, it is necessary to reproduce the Gell-Mann matrices and the formula connecting them to the gluons. I loan the convention to reference [19; Table 1]:

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Starting with these matrices, I state that:

$$T_2(\otimes)(\mathbf{b}, \mathbf{b}) = \begin{bmatrix} (b^1)^2 & 0 & 0 \\ 0 & (b^2)^2 & 0 \\ 0 & 0 & (b^3)^2 \end{bmatrix} + b^1.\ b^2.\ \lambda^1 + b^1.\ b^3.\ \lambda^4 + b^2.\ b^3.\ \lambda^6$$

$${}_{[J]}\Phi(\mathbf{b}) = i.\ \{b^1.\ \lambda^7 - b^2.\ \lambda^5 + b^3.\ \lambda^2\}$$

These relations allow to write:

$$M_{3+} = - \{b^1.\ \lambda^7 - b^2.\ \lambda^5 + b^3.\ \lambda^2 + \begin{bmatrix} (b^1)^2 & 0 & 0 \\ 0 & (b^2)^2 & 0 \\ 0 & 0 & (b^3)^2 \end{bmatrix} + b^1.\ b^2.\ \lambda^1 + b^1.\ b^3.\ \lambda^4 + b^2.\ b^3.\ \lambda^6\}$$

$$M_{3-} = \{b^1.\ \lambda^7 - b^2.\ \lambda^5 + b^3.\ \lambda^2\} - \begin{bmatrix} (b^1)^2 & 0 & 0 \\ 0 & (b^2)^2 & 0 \\ 0 & 0 & (b^3)^2 \end{bmatrix} - b^1.\ b^2.\ \lambda^1 + b^1.\ b^3.\ \lambda^4 + b^2.\ b^3.\ \lambda^6$$

Concerning the identity matrix, it is useful to recall that (when $||\mathbf{b}|| = \pm 1$):

$$\{{}_{[J]}\Phi(\mathbf{b})\}^2 = T_2(\otimes)(\mathbf{b}, \mathbf{b}) - Id_3 \Leftrightarrow M_1 = T_2(\otimes)(\mathbf{b}, \mathbf{b}) - \{{}_{[J]}\Phi(\mathbf{b})\}^2$$

Since:

$$½.\ \{M_{3+} - M_{3-}\} = i.\ {}_{[J]}\Phi(\mathbf{b}),\ ½.\ \{M_{3+} + M_{3-}\} = - T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

It is obvious that:

$$M_1 = -\tfrac{1}{2}.\ \{M_{3+} + M_{3-}\} + \tfrac{1}{4}.\ \{M_{3+} - M_{3-}\}^2 = Id_3$$

This paragraph essentially introduces three independent matrices: $\{M_2, M_{3+}, M_{3-}\}$. It remains to connect them with known particles.

Complements: the Perian matrices and the fundamental relation

This paragraph examines which Perian matrices in Invol(3) may eventually be a part of a deforming cube ▼A respecting the sufficient condition; synonym: preserving the speed of light whatever the metric tensor is.

The Perian matrices in Invol(3) and the sufficient condition

Proposition: There are Perian matrices in Invol(3) for which the sufficient condition is true (recall):

$$\beta^2.\ (\mathbf{b}^{\oplus})^4 + (10.\ \alpha.\ \beta + 2.\ \beta^2.\ ||\mathbf{b}||^2).\ (\mathbf{b}^{\oplus})^2 + 15.\ \alpha^2 = 0$$

Proof - Let suppose that the ratio Ko(**b**) exists and introduce it into the sufficient condition:

$$\beta^2.\ ||\mathbf{b}||^4/Ko^2(\mathbf{b}) + (10.\ \alpha.\ \beta + 2.\ \beta^2.\ ||\mathbf{b}||^2).\ ||\mathbf{b}||^2/Ko(\mathbf{b}) + 15.\ \alpha^2 = 0$$

First subset

Let consider the matrices:

$$[M(\mathbf{a}, \mathbf{b})] = -1/||\mathbf{b}||^2.\ T_2(\otimes)(\mathbf{b}, \mathbf{b}) \pm i.\ {}_{[J]}\Phi(\mathbf{b})$$

They are characterized by a first argument **a**: $(0, -1/||\mathbf{b}||^2, \pm i)$. Let inject these values in the sufficient condition (recall):

$$\forall\ \mathbf{b}:\ \beta^2.\ (\mathbf{b}^{\oplus})^4 + (10.\ \alpha.\ \beta + 2.\ \beta^2.\ ||\mathbf{b}||^2).\ (\mathbf{b}^{\oplus})^2 + 15.\ \alpha^2 = 0$$

This polynomial is now:

$$\forall\ \mathbf{b}:\ (\mathbf{b}^{\oplus})^4/||\mathbf{b}||^4 + 2.\ (\mathbf{b}^{\oplus})^2/||\mathbf{b}||^2 = 0$$

Let reduce the discussion to vectors **b** for which the Ko ratio exists:

$$\forall\ \mathbf{b}:\ \mathbf{b}^{\oplus} \neq 0 \Rightarrow \exists\ Ko(\mathbf{b}) = \frac{||\mathbf{b}||^2}{(\mathbf{b}^{\oplus})^2}$$

The polynomial of interest writes now:

$$(1/Ko(\mathbf{b})).\ (1/Ko(\mathbf{b}) + 2) = 0$$

The solution is evident: $Ko(\mathbf{b}) = -\tfrac{1}{2}$.

Second subset - first family

Let consider the matrices:

$$\forall\ \mathbf{b}: [M(\mathbf{a}, \mathbf{b})] = \pm\ Id_3$$

They are characterized by a first argument **a**: (± 1, 0, 0); hence the sufficient condition is equal to 15:

$$\forall\ \mathbf{b}: \beta^2. (\mathbf{b}^{\oplus})^4 + (10. \alpha. \beta + 2. \beta^2. ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2 + 15. \alpha^2 = 15$$

The identity matrix is a Perian matrix in Invol(3) but it cannot respect the sufficient condition (the same negative result applies to minus the identity matrix).

Second subset - second family

Let consider the matrices:

$$\forall\ \mathbf{b}: [M(\mathbf{a}, \mathbf{b})] = \pm\ \{Id_3 - 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})\}$$

They are characterized by a first argument **a**: (-1, $2/||\mathbf{b}||^2$, 0) or by **a**: (1, - $2/||\mathbf{b}||^2$, 0). Let inject these values in the sufficient condition:

$$\forall\ \mathbf{b}: \beta^2. (\mathbf{b}^{\oplus})^4 + (10. \alpha. \beta + 2. \beta^2. ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2 + 15. \alpha^2 = 0$$

The generic polynomial is now:

$$\forall\ \mathbf{b}: 4. (\mathbf{b}^{\oplus})^4/||\mathbf{b}||^4 - 12. (\mathbf{b}^{\oplus})^2/||\mathbf{b}||^2 + 15 = 0$$

Let restrict the discussion to a domain in which the vectors have a Ko ratio:

$$\forall\ \mathbf{b}: \mathbf{b}^{\oplus} \neq 0 \Rightarrow \exists\ Ko(\mathbf{b}) = \frac{||\mathbf{b}||^2}{(\mathbf{b}^{\oplus})^2}$$

The polynomial of interest can be rewritten as:

$$\forall\ \mathbf{b}: \mathbf{b}^{\oplus} \neq 0,\ 4/Ko^2(\mathbf{b}) - 12/Ko(\mathbf{b}) + 15 = 0$$

Unfortunately, the solutions are not real (because $\Delta' = - 24$). The sufficient condition can therefore not account for the values of Ko(**b**) which have been calculated for the leptons and for the quarks; see the table n°1. Amazingly, the historical ratio 2/3 can be recovered in imposing the vanishing of the entries of these matrices. This fact exhibits the strong disconnection between diverse logical affirmations: (i) to be related to a real effective ratios; (ii) to respect the reduced sufficient condition; (iii) to be a Perian matrix.

Lemma 02: Perian matrices, Invol(3), the sufficient condition and the ratios

For each vector **b**, there are eight ordinary Perian matrices in Invol(3) (recall). The elements in $Invol(3)_1$ are eventually associated with a ratio equal to - ½ if they respect the sufficient condition. Two elements in $Invol(3)_2$, precisely $\pm\ Id_3$, does not respect the sufficient condition at all. The two remaining elements in $Invol(3)_2$ may eventually be related to ratios in $\mathbb{C}$ if they respect the sufficient condition. Beside this evidence, we have proved that the two remaining elements in $Invol(3)_2$ are related to the historical Ko(**b**) ratio characterizing the leptons when the sum of their entries is null.

Conclusion: this specific family can be associated with the leptons without respecting the sufficient condition. This fact means that these matrices may eventually respect the fundamental relation within an ad hoc metric which remains to be discovered.

This statement justifies having interest for another type of Perian matrices: those which are representations of Euler-Rodrigues parametrisations.

Perian matrices and Euler parametrisations

Any four-dimensional sphere with radius 1 can be represented with a (3-3) matrix, thanks to the Euler-Rodrigues parametrisation.

$$(b_0)^2 + (b_1)^2 + (b_2)^2 + (b_3)^2 = 1$$

$$\downarrow$$

$$[\text{E-R}({}^{(4)}\mathbf{b})]$$

$$=$$

$$\begin{bmatrix} (b_0)^2 + (b_1)^2 - (b_2)^2 - (b_3)^2 & 2.(b_2.b_1 - b_0.b_3) & 2.(b_3.b_1 + b_0.b_2) \\ 2.(b_1.b_2 + b_0.b_3) & (b_0)^2 - (b_1)^2 + (b_2)^2 - (b_3)^2 & 2.(b_3.b_2 - b_0.b_1) \\ 2.(b_1.b_3 - b_0.b_2) & 2.(b_2.b_3 + b_0.b_1) & (b_0)^2 - (b_1)^2 - (b_2)^2 + (b_3)^2 \end{bmatrix}$$

$$=$$

$$\{2.(b_0)^2 - 1\}.\,Id_3 + 2.\,T_2(\otimes)({}^{(3)}\mathbf{b}, {}^{(3)}\mathbf{b}) + 2.\,b_0.\,{}_{[J]}\Phi({}^{(3)}\mathbf{b})$$

$$=$$

$$[M({}^{(3)}\mathbf{a}, {}^{(3)}\mathbf{b})]$$

This matrix is obviously a special type of Perian matrices characterized by the arguments:

$${}^{(3)}\mathbf{a}: (\alpha, \beta, \chi) = (2.(b_0)^2 - 1, 2, 2.b_0) = ((b_0)^2 - ||{}^{(3)}\mathbf{b}||^2, 2, 2.b_0) = (<{}^{(4)}\mathbf{b}, {}^{(4)}\mathbf{b}>_{[\eta(+---)]}, 2, 2.b_0)$$

$${}^{(3)}\mathbf{b}: (b_1, b_2, b_3)$$

Lemma 03: Perian matrices and Euler-Rodrigues parametrisation

Any Euler-Rodrigues parametrisation (ERP) of a four-dimensional unit sphere is represented by a Perian matrix. The inverse is false: any Perian matrix doesn't obligatorily represent a Euler-Rodrigues parametrisation.

Remark: Working with a Euler-Rodrigues parametrisation doesn't really introduce a new variable into the discussion, precisely b_0, because there is the condition $(b_0)^2 + ||{}^{(3)}\mathbf{b}||^2 = 1$.

Euler parametrisation and the historical Ko ratio

It is a matter of facts that the Perian matrices which can be associated with the historical Ko ratio also are a specific subset of the matrices representing ERP, precisely those for which $b^0 = 0$ and $||\mathbf{b}||^2 = 1$.

The sufficient condition for matrices representing a Euler parametrisation

Let consider the sufficient condition for any ordinary Perian matrix (OPM):

$$\forall\, \mathbf{b}: \beta^2.\, (\mathbf{b}^{\oplus})^4 + (10.\, \alpha.\, \beta + 2.\, \beta^2.\, ||\mathbf{b}||^2).\, (\mathbf{b}^{\oplus})^2 + 15.\, \alpha^2 = 0$$

Let inject the relation specifying that the OPM is the representation of some ERP:

$$\forall\, {}^{(3)}\mathbf{b}: 4.\, ({}^{(3)}\mathbf{b}^{\oplus})^4 + (20.\, ((b^0)^2 - ||{}^{(3)}\mathbf{b}||^2) + 8.\, ||{}^{(3)}\mathbf{b}||^2).\, ({}^{(3)}\mathbf{b}^{\oplus})^2 + 15.\, ((b^0)^2 - ||{}^{(3)}\mathbf{b}||^2) = 0$$

Let state that when, exceptionally, $b_0 = 0$, then:

$$\forall\, {}^{(3)}\mathbf{b}: 4.\, ({}^{(3)}\mathbf{b}^{\oplus})^4 - 12.\, ||{}^{(3)}\mathbf{b}||^2.\, ({}^{(3)}\mathbf{b}^{\oplus})^2 - 15.\, ||{}^{(3)}\mathbf{b}||^2 = 0$$

Let restrict the domain of investigation to the part in which the vector ${}^{(3)}\mathbf{b}$ can be associated with a ratio $Ko({}^{(3)}\mathbf{b})$:

$$\forall\, \mathbf{b}: \mathbf{b}^{\oplus} \neq 0 \Rightarrow \exists\, Ko({}^{(3)}\mathbf{b}) = \frac{||\mathbf{b}||^2}{(\mathbf{b}^{\oplus})^2}$$

Within this domain, the sufficient condition is:

$$\forall\, \mathbf{b}: \mathbf{b}^{\oplus} \neq 0:$$

$$4.\, ||{}^{(3)}\mathbf{b}||^4/Ko^2({}^{(3)}\mathbf{b}) + (20.\, (b^0)^2 - 12.\, ||{}^{(3)}\mathbf{b}||^2).\, ||{}^{(3)}\mathbf{b}||^2/Ko({}^{(3)}\mathbf{b}) + 15.\, ((b^0)^2 - ||{}^{(3)}\mathbf{b}||^2) = 0$$

Let state that when, exceptionally, $b_0 = 0$, then:

$$4.\, ||{}^{(3)}\mathbf{b}||^4/Ko^2({}^{(3)}\mathbf{b}) - 12.\, ||{}^{(3)}\mathbf{b}||^4/Ko({}^{(3)}\mathbf{b}) - 15.\, ||{}^{(3)}\mathbf{b}||^2 = 0$$

If, in peculiar, $Ko({}^{(3)}\mathbf{b}) = 2/3$, then the sufficient condition writes:

$$9.\, ||{}^{(3)}\mathbf{b}||^4 - 18.\, ||{}^{(3)}\mathbf{b}||^4 - 15.\, ||{}^{(3)}\mathbf{b}||^2 = -\, 9.\, ||{}^{(3)}\mathbf{b}||^4 - 15.\, ||{}^{(3)}\mathbf{b}||^2 = -\, 3.\, ||{}^{(3)}\mathbf{b}||^2.\, (3.\, ||{}^{(3)}\mathbf{b}||^2 + 5) = 0$$

The mathematical solutions are $||\mathbf{b}|| = \pm\, i.\, \sqrt{5/3}$ which, obviously, are not real and not coherent with the expected result $||\mathbf{b}||^2 = 1$. Again (see lemma 02), this result proves that the matrices $\{-\, Id_3 + 2.\, T_2(\otimes)(\mathbf{b}, \mathbf{b})\}$ such that $||\mathbf{b}||^2 = 1$ can sometimes be associated with the historical $Ko(\mathbf{b}) = 2/3$ ratio although they never respect the sufficient condition. The consequence is that the fundamental relation can only hold true for a limited set of metrics.

In general, the sufficient condition concerning the ERP can be analysed in diverse manners.

- Let consider that the ratio $Ko({}^{(3)}\mathbf{b})$ is the (unknown) variable whilst $||\mathbf{b}||$ is known through experiments:

$$15.\ ((b^0)^2 - ||^{(3)}\mathbf{b}||^2).\ Ko^2(^{(3)}\mathbf{b}) + 2.\ (10.\ (b^0)^2 - 6.\ ||^{(3)}\mathbf{b}||^2).\ ||^{(3)}\mathbf{b}||^2.\ Ko(^{(3)}\mathbf{b}) + 4.\ ||^{(3)}\mathbf{b}||^4 = 0$$

The sufficient condition is now a polynomial form of degree two and its discriminant is:

$$\Delta'$$

$$= (10.\ (b^0)^2 - 6.\ ||^{(3)}\mathbf{b}||^2)^2.\ ||^{(3)}\mathbf{b}||^4 - 60.\ ((b^0)^2 - ||^{(3)}\mathbf{b}||^2).\ ||^{(3)}\mathbf{b}||^4$$

$$= \{100.\ (b^0)^4 - 120.\ (b^0)^2 + 36.\ ||^{(3)}\mathbf{b}||^4 - 60.\ (b^0)^2 + 60.\ ||^{(3)}\mathbf{b}||^2\}.\ ||^{(3)}\mathbf{b}||^4$$

$$= \{100.\ (b^0)^4 - 180.\ (b^0)^2 + 96.\ ||^{(3)}\mathbf{b}||^4\}.\ ||^{(3)}\mathbf{b}||^4$$

But here, the ERP imposes the constraint:

$$(b^0)^2 + ||^{(3)}\mathbf{b}||^2 = 1$$

We therefore can focus attention on $||^{(3)}\mathbf{b}||^2$ and write:

$$(b^0)^4 = 1 - 2.\ ||^{(3)}\mathbf{b}||^2 + ||^{(3)}\mathbf{b}||^4$$

$$\Delta'$$

$$=$$

$$\{100.\ (1 - 2.\ ||^{(3)}\mathbf{b}||^2 + ||^{(3)}\mathbf{b}||^4) - 180.\ (1 - ||^{(3)}\mathbf{b}||^2) + 96.\ ||^{(3)}\mathbf{b}||^4\}.\ ||^{(3)}\mathbf{b}||^4$$

$$=$$

$$\{100 - 200.\ ||^{(3)}\mathbf{b}||^2 + 100.\ ||^{(3)}\mathbf{b}||^4 - 180 + 180.\ ||^{(3)}\mathbf{b}||^2 + 96.\ ||^{(3)}\mathbf{b}||^4\}.\ ||^{(3)}\mathbf{b}||^4$$

$$=$$

$$(196.\ ||^{(3)}\mathbf{b}||^4 - 20.\ ||^{(3)}\mathbf{b}||^2 - 80).\ ||^{(3)}\mathbf{b}||^4$$

$$=$$

$$4.\ (49.\ ||^{(3)}\mathbf{b}||^4 - 5.\ ||^{(3)}\mathbf{b}||^2 - 20).\ ||^{(3)}\mathbf{b}||^4$$

Hence, since:

$$(10.\ (b^0)^2 - 6.\ ||^{(3)}\mathbf{b}||^2).\ ||^{(3)}\mathbf{b}||^2 = \{10.\ (1 - ||^{(3)}\mathbf{b}||^2) - 6.\ ||^{(3)}\mathbf{b}||^2\}.\ ||^{(3)}\mathbf{b}||^2 = (10 - 16.\ ||^{(3)}\mathbf{b}||^2).\ ||^{(3)}\mathbf{b}||^2$$

$$15.\ ((b^0)^2 - ||^{(3)}\mathbf{b}||^2) = 15.\ (1 - 2.\ ||^{(3)}\mathbf{b}||^2)$$

At the end we get the solutions:

$$Ko(^{(3)}\mathbf{b}) = \frac{-\left(10-1\ .\ ||\mathbf{b}||^2\right) \pm 2.\ (49.\ ||\mathbf{b}||^4 - 5.\ ||\mathbf{b}||^2 - 20)^{1/2}}{15.\ (1-2.\ ||\mathbf{b}||^2)}.\ ||^{(3)}\mathbf{b}||^2$$

We state that the ratios which are associated with a Euler-Rodrigues parametrisation can have any value in $\mathbb{C}$. The challenge will be the discovery of criterion isolating the real ratios.

- Let now consider that $||\mathbf{b}||^2$ is the unknown variable whilst $Ko(^{(3)}\mathbf{b})$ is known through experimental measurements (see the table n°1). The sufficient condition (recall):

$$\forall\ \mathbf{b}: \mathbf{b}^{\oplus} \neq 0:$$

$$4.\ ||^{(3)}\mathbf{b}||^4/Ko^2(^{(3)}\mathbf{b}) + (20.\ (b^0)^2 - 12.\ ||^{(3)}\mathbf{b}||^2).\ ||^{(3)}\mathbf{b}||^2/Ko(^{(3)}\mathbf{b}) + 15.\ ((b^0)^2 - ||^{(3)}\mathbf{b}||^2) = 0$$

... can be reformulated as a polynomial form of degree two depending on $||\mathbf{b}||^2$:

$$\{4/\ Ko^2(^{(3)}\mathbf{b}) - 12/\ Ko(^{(3)}\mathbf{b})\}.\ ||^{(3)}\mathbf{b}||^4 + \{20.\ (b^0)^2/Ko(^{(3)}\mathbf{b}) - 15\}.\ ||^{(3)}\mathbf{b}||^2 + 15.\ (b^0)^2 = 0$$

But this formulation must be modified when the sufficient condition concerns an ERP:

$$\{4/\ Ko^2(^{(3)}\mathbf{b}) - 12/\ Ko(^{(3)}\mathbf{b})\}.\ ||^{(3)}\mathbf{b}||^4 + \{20.\ (1 - ||^{(3)}\mathbf{b}||^2)/Ko(^{(3)}\mathbf{b}) - 15\}.\ ||^{(3)}\mathbf{b}||^2 + 15.\ (1 - ||^{(3)}\mathbf{b}||^2) = 0$$

A reorganisation allows the writing:

$$\{4/Ko^2(^{(3)}\mathbf{b}) - 32/Ko(^{(3)}\mathbf{b})\}.\ ||^{(3)}\mathbf{b}||^4 + 20/Ko(^{(3)}\mathbf{b}).\ ||^{(3)}\mathbf{b}||^2 + 15 = 0$$

Its discriminant is:

$$\Delta'$$

$$= 100/Ko^2(^{(3)}\mathbf{b}) - 15.\ \{4/Ko^2(^{(3)}\mathbf{b}) - 32/Ko(^{(3)}\mathbf{b})\}$$

$$= 40/Ko^2(^{(3)}\mathbf{b}) + 480/Ko(^{(3)}\mathbf{b})$$

$$= 40/Ko(^{(3)}\mathbf{b}).\ \{1/Ko(\mathbf{b}) + 12\}$$

The solutions are:

$$||^{(3)}\mathbf{b}||^2 = \frac{-\frac{20}{K} \pm \sqrt{\frac{40}{K}.\ (\frac{1}{K} + 12)}}{\frac{4}{K^2} - \frac{32}{K}} = \frac{-5 \pm 5\sqrt{0{,}1 + 1{,}2.\ K}}{1 - 8.\ K}.\ K$$

Since this part of the work concerns the ERP, the solutions must be $||^{(3)}\mathbf{b}||^2 = 1$. This is imposing supplementary conditions concerning the ratios.

$$(2{,}5 + 30.\ K).\ K^2 = (1 - 8.\ K)^2 + 25.\ K^2 + 10.\ (1 - 8.\ K).\ K$$

$$2{,}5.\ K^2 + 30.\ K^3 = 1 - 16.\ K + 64.\ K^2 + 25.\ K^2 + 10.\ K - 80.\ K^2$$

$$30.\ K^3 - 6{,}5.\ K^2 + 6.\ K - 1 = 0$$

The unique real ratio is:

$$Ko(\mathbf{b}) = 0{,}17319$$

It is not in the table n°1 but it would be interesting to look for three particles in the standard model zoo which are related to this value. There are also two roots in $\mathbb{C}$ $(0{,}02174 \pm 0{,}43817.\ i)$.

The non-Perian matrices

The non-Perian matrices respecting the sufficient condition

Let come back to the sufficient condition (recall):

$$\forall\ n = 1, 2, 3: \{[{}_nA]^{\oplus}\}^2 + \{[{}_nA]^2 + [{}_nA].\ [{}_nA]^t\}^{\oplus} = 0$$

There is no obligation to work with Perian matrices. For example, any matrix [A(**a**, **b**)] with the generic formalism:

$$[A(\mathbf{a}, \mathbf{b})] = \begin{bmatrix} a^1 & -b^3 & b^2 \\ b^3 & a^2 & -b^1 \\ -b^2 & b^1 & a^3 \end{bmatrix}$$

... is not a Perian matrix (because of the formalism of its diagonal) but can respect the sufficient condition if a convenient constraint is imposed to the pair (**a**, **b**). See proof in annex 06.

Another strategy to discover matrices respecting the sufficient condition

The discovery of matrices respecting the sufficient condition can be managed with diverse strategies. The easiest one searches for sets of matrices of which the sum of the entries (components) is null in adding the constraint:

$$\{[{}_nA]^{\oplus}\}^2 = \{[{}_nA]^2\}^{\oplus}$$

Applying for example this supplementary constraint to the non-Perian matrices which have been introduces in former paragraph, we get:

$$\{[A]^{\oplus}\}^2 = (\mathbf{a}^{\oplus})^2$$

$$\{[A(\mathbf{a}, \mathbf{b})]^2\}^{\oplus} = ||\mathbf{a}||^2 + (\mathbf{b}^{\oplus})^2 - 3.\ ||\mathbf{b}||^2$$

The constraint $\{[{}_nA]^{\oplus}\}^2 = \{[{}_nA]^2\}^{\oplus}$ is not systematically true. It is true when:

$$(\mathbf{a}^{\oplus})^2 = ||\mathbf{a}||^2 + (\mathbf{b}^{\oplus})^2 - 3.\ ||\mathbf{b}||^2$$

Discussion

The statement and the claim

A strange formula connecting the masses of three types of electrons has been established in the eighties. To my best but modest knowledge, it is the unique tool relating their masses with exactitude. This statement is the starting point for the exploration which is presented in this document. Its main claim is the discovery of rationalistic causes for the existence of three generations of electrons and quarks. The historical formulation of the ratio connecting the masses of electrons is interpreted as a peculiar manifestation of a more fundamental rule which we try to reveal all along the quest.

Experimental evidence

Most of all volumes which can be defined in our universe are empty. Sometimes, something happens that we usually label with the word "particle".

All fermions (leptons and quarks), except their associated neutrinos have an electrical charge. This fact justifies that the behaviour of these particles can be described within the theory of electromagnetism.

The speed of light in vacuum, "c" (or should I prefer to say: in the empty regions of the universe) is a universal constant for all observers working at the origin of an inertial frame.

But the following facts must be noted: (i) the nearest to the nucleus the quickest is an electron and its speed can reach 1/10 c for heavy atoms; (ii) the speed of quarks in nucleus depends on the weight of this nucleus. For light atoms, the speed is close to c whilst it is no more the case for heavier atoms like those of iron. This fact is called the European Muon Collaboration (EMC) Effect and is known since 1983 [15].

Trying to extract a global image from these diverse experimental events, we gain the intuition of the existence of an interplay between the presence of nuclear particles in a relatively small volume and the fact that their speed becomes smaller than c. As if a deceleration of the energy would be synonym of materialization of a particle and modification of the geometry.

The main topic of this quest implies or presupposes a mandatory coupling between trios of particles. And it naturally legitimates the questions: "Does such coupling exist? When? Why?" To the best of my very limited experimental knowledge, the phenomenon called "entanglement" [16] and "pairing" [17; chapter 3, pp. 115-196] of particles exist in nuclei or between atoms. These phenomenon are realities justifying to not immediately reject a way of thinking looking for collective behaviours in nuclear physics.

Mathematical tools

Since there is absolutely no need to reinvent the Lorentz-Poincare transformations, another manner to translate the invariance of the speed of light with the help of mathematical equations is needed. This is exactly the place where the Poynting's vector and the theory studying the deformations of classical cross products through cubes enter the discussion. At a technical level, when the crucial condition is realized, this approach introduces a fundamental relation and its direct consequence: the generic sufficient condition (recall):

$$\Sigma_{\alpha,\beta}\, g_{\alpha\beta}.\, \{[{}_{\alpha}A]^{\oplus}.\, [{}_{\beta}A]^{\oplus} + \{[{}_{\alpha}A].\, \{[{}_{\beta}A] + [{}_{\beta}A]^{t}\}\}^{\oplus}\} = 0$$

Or its reduced version not depending on the spatial geometry:

$$\forall\, [G],\ \forall\, \alpha, \beta:\ [{}_{\alpha}A]^{\oplus}.\, [{}_{\beta}A]^{\oplus} + \{[{}_{\alpha}A].\, \{[{}_{\beta}A] + [{}_{\beta}A]^{t}\}\}^{\oplus} = 0$$

This logical path unfortunately gives no significant hint allowing a precise interpretation of the deforming matrices. This fact imposes a systematic analyse of all types of matrices.

Main results

Antisymmetric matrices systematically validate the reduced sufficient condition (RSC). They represent axial rotations. A specific set of symmetric matrices validates the RSC and can represent the Ko-ratios (anyone). The ordinary Perian matrices are also reintroduced into this discussion. The RSC for these matrices is written and the analyse yields two subsets respecting it: (i) the axial rotations which can be associated with any value of a Ko-ratio; (ii) specific symmetric Perian matrices exclusively related to the ratio 1/3. A part of the Perian matrices representing an involution can be related to the historical ratio 2/3 but the latter is not a solution for the RSC applied to this type of matrices.

We may also add that:

- All antisymmetric matrices representing an axial rotations are ordinary Perian matrices respecting the reduced version of the sufficient condition. They can be associated with any value of the Ko-ratio.

- There exist non-Perian matrices respecting the sufficient condition.

- There is a subset of all ordinary Perian matrices representing involutions.
  - A part of them are elements in SU(3).
  - More specifically, the matrices $\{- Id_3 + 2/||\mathbf{b}||^2. T_2(\otimes)(\mathbf{b}, \mathbf{b})\}$ such that (i) $\mathbf{b} \neq \mathbf{0}$ and (ii) the sum of the entries vanishes are naturally associated with the historical ratio concerning the masses of the electrons: Ko(**b**) = 2/3.
  - A part of them cannot respect the reduced version of the sufficient condition. The part of them which is respecting it is associated with Ko-ratios either equal to - ½ (negative ratio) or in $\mathbb{C}$. Hence, they don't represent the particles of the standard model (table n°1).

- All matrices representing the Euler-Rodrigues parametrisation of some 4D-unit sphere are ordinary Perian matrices.
  - The converse is false.
  - The matrices $\{- Id_3 + 2. T_2(\otimes)(\mathbf{b}, \mathbf{b})\}$ are ordinary Perian matrices representing Euler-Rodrigues parametrisation of some 4D-unit sphere for vectors **b** such that $||\mathbf{b}||^2 = 1$.

The discussion itself

Previous results need a deep analyse to be understood and interpreted. Since the formula giving the historical ratio can be generalized to any trio of masses (or of energies), diverse Ko-ratios can be calculated.

The ratios related to the standard model are strongly "*type of particles-dependent*". This generalization doesn't explain the origin of the formula and the formula itself has per se until now no rational justification.

But if it is a priori believed that the formalism of this formula contains information about how trios of masses interact, the discussion gradually shifts to another topic. The goal is no longer to explain how this formalism has been discovered or to prove the existence of the ratios, they exist anyway. The goal is the discovery of mathematical tools representing trios of particles in a manner which is compatible with well-admitted physical rules.

The preservation of $(4.\pi/c)^2$ when the Poynting's vector is deformed gives the opportunity to introduce sets of (3-3) deforming matrices into the discussion. The quest continues with the unsaid hypothesis betting that particles or trios of particles can be represented by these matrices.

The deforming matrices [A] must be cautiously chosen and have a formalism allowing the existence of the experimental trans- and inner- generations ratios (table n°1). The important actors are no longer the ratios but the matrices compatible with these ratios… and the fact that these matrices carry crucial information about the collective behaviour of the trios.

As I have proved it in diverse documents, the Perian matrices can sometimes be interpreted as fermionic operators [a], [b], [c]. This is the reason why they have been reintroduced here. A part of them can account for the historical ratio (2/3) but they don't validate the reduced version of the sufficient condition. This is perhaps not a problem if it can be proved that they can validate the complete version of this condition (with dependence to the local spatial geometry).

The remaining problem accompanying this exploration is related to the fact that, at this stage, we don't know with certitude how the Perian matrices must be/are linked to the particles. Do they represent only one pair (**E**, **B**) and its associated particle? Do they intrinsically represent three particles? This is the crucial question that must yet be answered to bring clarity and coherence into this work.

Conclusion

This work is a pioneer approach attempting to describe the behaviour of any trio of particles in a nuclei or around it with the help of (3-3) matrices which are not the usual density matrices. There is certainly yet a long and difficult way between the intuitive idea which has been suggested here and its achievement inside a complete and coherent theory. For now, just take it as a curiosity because fermions are well described with two-components spinors [18] and the Standard Model may not have said its last word [23].

Personal works

[a] Les matrices périennes; recherche d'un lien avec les opérateurs quantiques ; vixra.org: 2509.0014, 33 pages.

[b] Algebraic Dynamics, the first stones; vixra.org: 2509.0108, 49 pages.

[c] Les matrices périennes ordinaires ; lien avec le problème des trois générations ; vixra.org: 2510.0013, 62 pages.

Declaration of competing interest

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Annexes

Annex 01 - The Ko-ratios

Trans-generations

The leptons:

m_e = 0,511 MeV/c^2, m_μ = 105,66 MeV/c^2 [05] and m_τ = 1777,09 ± 0,14 MeV/c^2 [06]

$$\text{Ko(leptons)} = \frac{0{,}511 + 105{,}66 + 1777{,}09}{(0{,}7148 + 10{,}2791 + 42{,}1555)^2} = \frac{1883{,}261}{2824{,}8587} = 0{,}66667 \sim 2/3$$

For the quarks:

Up : 2,2 MeV/c^2, Charm : 1 280 MeV/c^2 et Top : 173 100 MeV/c^2

Ko(quarks I) = 0,8486

For the quarks:

Down 4,67 MeV/c^2, Strange : 93,4 MeV/c^2 et Bottom : 4180 MeV/c^2

Ko(quarks II) = 0,7314.

Intra-generations

The first generation:

$m_{e^-} = 0{,}511$ MeV/c², $m_{down} = 4{,}7$ MeV/c², $m_{up} = 2{,}2$ MeV/c²

$\sqrt{0{,}511} = 0{,}71484264$, $\sqrt{4{,}7} = 2{,}1679483389$, $\sqrt{2{,}2} = 1{,}4832396974$

$(m_{e^-} + m_{down} + m_{up}) = 7{,}411$, $(\sqrt{0{,}511} + \sqrt{4{,}7} + \sqrt{2{,}2})^2 = 19{,}0622238664$

$$Ko(\mathbf{I}) = \frac{7{,}411}{19{,}06222386664} = 0{,}3887794022$$

The second generation:

$m_\mu = 105{,}66$ MeV/c² [20], $m_{strange} = 93{,}5$ MeV/c², $m_{charm} = 1273$ MeV/c²

$\sqrt{105{,}66} = 10{,}279$, $\sqrt{93{,}5} = 9{,}669$, $\sqrt{1273} = 35{,}679$

$(m_\mu + m_{strange} + m_{charm}) = 1472{,}16$, $(\sqrt{105{,}66} + \sqrt{93{,}5} + \sqrt{1273})^2 = 3094{,}363$

$$Ko(\mathbf{II}) = \frac{1472{,}16}{3094{,}363} = 0{,}475$$

The third generation:

$m_\tau = 1776{,}93$ MeV/c² [20], $m_{bottom} = 4183$ MeV/c², $m_{top} = 172\,570$ MeV/c²

$\sqrt{1776{,}93} = 42{,}15$, $\sqrt{4183} = 64{,}676$, $\sqrt{172570} = 415{,}415$

$(m_\tau + m_{bottom} + m_{top}) = 178532{,}96$, $(42{,}15 + 64{,}676 + 415{,}415)^2 = 272735{,}662$

$$Ko(\mathbf{III}) = \frac{178532{,}96}{272735{,}662} = 0{,}6546$$

The results are summarized in table n°1.

Annex 02 - The fundamental relation for plane waves

Let calculate:

$$g_{\alpha\beta}.\ \{\otimes_C({}^{(3)}\mathbf{E'},{}^{(3)}\mathbf{B'})\}_\alpha.\ \{\otimes_C({}^{(3)}\mathbf{E'},{}^{(3)}\mathbf{B'})\}_\beta$$

$$=$$

$$k^4.\ \{C_{\alpha zy}.\ (E_z + u.\ B_y).\ (B_y + \frac{u}{c^2}.\ E_Z) + C_{\alpha yz}.\ (E_y - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ E_y)\}$$

$$.\ \{C_{\beta zy}.\ (E_z + u.\ B_y).\ (B_y + \frac{u}{c^2}.\ E_Z) + C_{\beta yz}.\ (E_y - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ E_y)\}$$

The non polarized plane waves

For these waves especially:

$$E_x = B_x = 0,\ E_z = -\ v.\ B_y \text{ et } E_y = v.\ B_z,\ <\mathbf{E},\mathbf{B}>_{Id3} = 0$$

As consequences:

- If all matrices $[{}_{\alpha}C]$ are such that $C_{\alpha yz} + C_{\alpha zy} = 0$:

$$\{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\alpha}$$

$$=$$

$$k^2.\ \{C_{\alpha zy}.\ (-\ v.\ B_y + u.\ B_y).\ (B_y - \frac{u.v}{c^2}.\ B_y) + C_{\alpha yz}.\ (v.\ B_z - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ v.\ B_z)\}$$

$$=$$

$$k^2.\ \{-\ C_{\alpha yz}.\ (-\ v + u).\ (1 - \frac{u.v}{c^2}).\ (B_y)^2 + C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ (B_z)^2\}$$

$$=$$

$$k^2.\ C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ \{(B_y)^2 + (B_z)^2\}$$

$$=$$

$$k^2.\ C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ <\mathbf{B}, \mathbf{B}>_{Id3}$$

$$=$$

$$4.\ \pi.\ k^2.\ C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ \rho_{EM}$$

Hence:

$$g_{\alpha\beta}.\ \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\alpha}.\ \{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\beta}$$

$$=$$

$$16.\ \pi^2.\ k^4.\ (g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz}).\ (v - u)^2.\ (1 - \frac{u.v}{c^2})^2.\ \rho^2_{EM}$$

$$=$$

$$16.\ \pi^2.\ (g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz}).\ \frac{(c^2 - u.v)^2}{(c^2 - u^2)^2}.\ (v - u)^2.\ \rho^2_{EM}$$

- If all matrices $[{}_{\alpha}C]$ are such that $C_{\alpha yz} = C_{\alpha zy}$:

$$\{\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\alpha}$$

$$=$$

$$k^2.\ \{C_{\alpha zy}.\ (-\ v.\ B_y + u.\ B_y).\ (B_y - \frac{u.v}{c^2}.\ B_y) + C_{\alpha yz}.\ (v.\ B_z - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ v.\ B_z)\}$$

$$=$$

$$k^2.\ \{C_{\alpha yz}.\ (-\ v + u).\ (1 - \frac{u.v}{c^2}).\ (B_y)^2 + C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ (B_z)^2\}$$

=

$$k^2.\ C_{\alpha yz}.\ (v - u).\ (1 - \frac{u.v}{c^2}).\ \{(B_z)^2 - (B_y)^2\}$$

Annex 03 - Calculations for any electromagnetic field

Starting with:

$$\{\otimes_C({}^{(3)}\mathbf{E'},{}^{(3)}\mathbf{B'})\}_\alpha$$

=

$$k.\ C_{\alpha xy}.\ E_x.\ (B_y + \frac{u}{c^2}.\ E_Z) + k.\ C_{\alpha xz}.\ E_x.\ (B_z - \frac{u}{c^2}.\ E_y)$$

$$+ k.\ C_{\alpha yx}.\ (E_y - u.\ B_z).\ B_x + k^2.\ C_{\alpha yz}.\ (E_y - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ E_y)$$

$$+ k.\ C_{\alpha zx}.\ (E_z + u.\ B_y).\ B_x + k^2.\ C_{\alpha zy}.\ (E_z + u.\ B_y).\ (B_y + \frac{u}{c^2}.\ E_Z)$$

We make our calculations with totally antisymmetric matrices $[_\alpha C]$, but a complete exploration should envisage the same calculations with symmetric matrices and with any matrices.

For antisymmetric $[_\alpha C]$ matrices

In that case:

$$\{\otimes_C({}^{(3)}\mathbf{E'},{}^{(3)}\mathbf{B'})\}_\alpha$$

=

$$k.\ C_{\alpha xy}.\ \{E_x.\ (B_y + \frac{u}{c^2}.\ E_z) - (E_y - u.\ B_z).\ B_x\} + k.\ C_{\alpha xz}.\ \{E_x.\ (B_z - \frac{u}{c^2}.\ E_y) - (E_z + u.\ B_y).\ B_x\}$$

$$+ k^2.\ C_{\alpha yz}.\ \{(E_y - u.\ B_z).\ (B_z - \frac{u}{c^2}.\ E_y) - (E_z + u.\ B_y).\ (B_y + \frac{u}{c^2}.\ E_Z)\}$$

=

$$k.\ \{C_{\alpha xy}.\ (E_x.\ B_y - E_y.\ B_x) + C_{\alpha xz}.\ (E_x.\ B_z - E_z.\ B_x)\}$$

$$+ k.\ \{C_{\alpha xy}.\ (\frac{u}{c^2}.\ E_x.\ E_z + u.\ B_x.\ B_z) - C_{\alpha xz}.\ (\frac{u}{c^2}.\ E_x.\ E_y + u.\ B_x.\ B_y)\}$$

$$+ k^2.\ C_{\alpha yz}.\ \{(E_y.\ B_z - E_z.\ B_y) - \frac{u}{c^2}.\ \{(E_y)^2 + (E_z)^2\} - u.\ \{(B_y)^2 + (B_z)^2\} + \frac{u^2}{c^2}.\ (E_y.\ B_z - E_z.\ B_y)\}$$

Since in the first frame (recall):

$$\mathbf{S} = \frac{c}{4.\pi}.\ \mathbf{E} \wedge \mathbf{B} = \rho_{EM}.\ \mathbf{v}$$

The next step is:

$$\{\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')\}_\alpha$$

$$=$$

$$k.\frac{4.\pi}{c}.\{C_{\alpha xy}.\,v_z - C_{\alpha xz}.\,v_y\}.\,\rho_{EM} + k.\frac{u}{c^2}.\{E_x.\,(C_{\alpha xy}.\,E_z - C_{\alpha xz}.\,E_y) + c^2.\,B_x.\,(C_{\alpha xy}.\,B_z - C_{\alpha xz}.\,B_y)\}$$

$$+\,k^2.\,C_{\alpha yz}.\{\frac{4.\pi}{c}.\,(1+\frac{u^2}{c^2}).\,v_x.\,\rho_{EM} - \frac{u}{c^2}.\{||\mathbf{E}||^2 - (E_x)^2\} - u.\{||\mathbf{B}||^2 - (B_x)^2\}\}$$

At this stage, it is not really evident to go further. We can at least remark that each antisymmetric matrix $[_\alpha C]$ is automatically equivalent to a pseudo-vector $_\alpha\mathbf{C}$ with components $(-C_{\alpha yz}, C_{\alpha xz}, -C_{\alpha xy}) = ({}_\alpha C_x, {}_\alpha C_y, {}_\alpha C_z)$. This pseudo-vector appears explicitly in prior expression. We unfortunately don't know how to interpret it. Any plane wave ($E_x = B_x = 0$) reduces this expression drastically:

$$\{\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')\}_\alpha$$

$$=$$

$$k.\frac{4.\pi}{c}.\,\rho_{EM}.\{C_{\alpha xy}.\,v_z - C_{\alpha xz}.\,v_y\} + k^2.\,C_{\alpha yz}.\{\frac{4.\pi}{c}.\,(1+\frac{u^2}{c^2}).\,\rho_{EM}.\,v_x - \frac{u}{c^2}.\,(||\mathbf{E}||^2 + c^2.\,||\mathbf{B}||^2\}$$

... without giving any immediate hint on its interpretation. We only guess that $||\mathbf{E}||^2 + c^2.\,||\mathbf{B}||^2$ is related to ρ_{EM} in some ad hoc unit's system. If we understand the deformed tensor $\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')$ as a generalized Poynting's vector and if we accept to believe that the formalism of physical rules is preserved in all inertial frames, then it is meaningful to write:

$$\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}') = \rho'_{EM}.\,{}^{(3)}\mathbf{v}'$$

Each component of this tensor product can be seen as the sum of two sub-expressions.

- The first one is:

$$\forall\,\alpha = 1, 2, 3:\ \rho'_{EM}.\,v'_\alpha = k.\frac{4.\pi}{c}.\,\rho_{EM}.\{C_{\alpha xy}.\,v_z - C_{\alpha xz}.\,v_y + k.\,C_{\alpha yz}.\,(1+\frac{u^2}{c^2}).\,v_x\}$$

  In details:

$$\rho'_{EM}.\,v'_1 = k.\frac{4.\pi}{c}.\,\rho_{EM}.\{C_{1xy}.\,v_z - C_{1xz}.\,v_y + k.\,C_{1yz}.\,(1+\frac{u^2}{c^2}).\,v_x\}$$

$$\rho'_{EM}.\,v'_2 = k.\frac{4.\pi}{c}.\,\rho_{EM}.\{C_{2xy}.\,v_z - C_{2xz}.\,v_y + k.\,C_{2yz}.\,(1+\frac{u^2}{c^2}).\,v_x\}$$

$$\rho'_{EM}.\,v'_3 = k.\frac{4.\pi}{c}.\,\rho_{EM}.\{C_{3xy}.\,v_z - C_{3xz}.\,v_y + k.\,C_{3yz}.\,(1+\frac{u^2}{c^2}).\,v_x\}$$

These relations can be synthetized as:

$$| \rho'_{EM}.\, {}^{(3)}\mathbf{v}' > = \rho'_{EM}. \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \frac{4.\pi.k}{c}.\, \rho_{EM}. \begin{bmatrix} C_{1xy} & -C_{1xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{1yz} \\ C_{2xy} & -C_{2xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{2yz} \\ C_{3xy} & -C_{3xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{2yz} \end{bmatrix}. \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Let speculate a little bit. Suppose that the field occupies a small cubic volume $\tau = r^3$ in the first inertial frame. This cube contains the electromagnetic energy:

$$E_{EM} = \rho_{EM}.\, r^3$$

The observer at the centre of the second inertial frame perceives this volume as:

$$\tau' = r^3. \sqrt{1-\frac{u^2}{c^2}} = \frac{r^3}{k},\ k \neq 0$$

It contains the electromagnetic energy:

$$E'_{EM} = \rho'_{EM}.\, r^3. \frac{r^3}{k}$$

Due to Einstein's theory of relativity:

$$(E'_{EM})^2 - < {}^{(3)}\mathbf{p}', {}^{(3)}\mathbf{p}' >_{[G]}.\, c^2 = (m_0)^2.\, c^4 = (E_{EM})^2 - < {}^{(3)}\mathbf{p}, {}^{(3)}\mathbf{p} >_{[Id3]}.\, c^2$$

In the context at hand, this invariance can be written as:

$$\frac{1}{8.\pi}.\, (< \mathbf{E}', \mathbf{E}' >_{[G]} + < \mathbf{B}', \mathbf{B}' >_{[G]}). \frac{r^3}{k} - < {}^{(3)}\mathbf{p}', {}^{(3)}\mathbf{p}' >_{[G]}.\, c^2$$

$$=$$

$$\frac{1}{8.\pi}.\, (< \mathbf{E}, \mathbf{E} >_{[Id3]} + < \mathbf{B}, \mathbf{B} >_{[Id3]}).\, r^3 - < {}^{(3)}\mathbf{p}, {}^{(3)}\mathbf{p} >_{[Id3]}.\, c^2$$

If the volume which is occupied by the field is equivalent to a particle with a mass, then we can write:

$${}^{(3)}\mathbf{p} = m.\, {}^{(3)}\mathbf{v} \qquad {}^{(3)}\mathbf{p}' = m'.\, {}^{(3)}\mathbf{v}'$$

We know how to relate the masses:

$$m' = k.\, m$$

Recall that:

$$k = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

$$E'_x = E_x,\ E'_y = k.\,(E_y - u.\, B_z),\ E'_z = k.\,(E_z + u.\, B_y)$$

$$B'_x = B_x,\ B'_y = k.\,(B_y + \frac{u}{c^2}.\, E_z),\ B'_z = k.\,(B_z - \frac{u}{c^2}.\, E_y)$$

We can formally write the density of electromagnetic energy in the second frame. But since we don't know how to relate the geometry in the second frame to the geometry in the first one, concrete calculations can only be done when the new geometry remains quasi-Euclidean. In this case ($^{(3)}[G] \cong Id_3$):

$$(< \mathbf{E'}, \mathbf{E'} >_{[G]} + < \mathbf{B'}, \mathbf{B'} >_{[G]})$$

$$\cong$$

$$(< \mathbf{E'}, \mathbf{E'} >_{[Id3]} + < \mathbf{B'}, \mathbf{B'} >_{[Id3]})$$

$$=$$

$$(E_x)^2 + k^2.\,(E_y - u.\,B_z)^2 + k^2.\,(E_z + u.\,B_y)^2 + (B_x)^2 + k^2.\,(B_y + \frac{u}{c^2}.\,E_z)^2 + k^2.\,(B_z - \frac{u}{c^2}.\,E_y)^2$$

$$=$$

$$(E_x)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_y)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_z)^2 + (B_x)^2$$

$$+ k^2.\,(1 + u^2).\,(B_y)^2 + k^2.\,(1 + u^2).\,(B_z)^2$$

$$-\ 2.\,k^2.\,u.\,(1 + \frac{1}{c^2}).\,E_y.\,B_z + 2.\,k^2.\,u.\,(1 + \frac{1}{c^2}).\,E_z.\,B_y$$

$$=$$

$$(E_x)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_y)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_z)^2 + (B_x)^2$$

$$+ k^2.\,(1 + u^2).\,(B_y)^2 + k^2.\,(1 + u^2).\,(B_z)^2$$

$$-\ 8.\,\pi.\,k^2.\,(u/c).\,(1 + \frac{1}{c^2}).\,\rho_{EM}.\,v_x$$

And:

$$< {}^{(3)}\mathbf{p'}, {}^{(3)}\mathbf{p'} >_{[G]}.\,c^2 \cong < m'.{}^{(3)}\mathbf{v'}, m'.{}^{(3)}\mathbf{v'} >_{[Id3]}.\,c^2 = k^2.\,m^2.\,< {}^{(3)}\mathbf{v'}, {}^{(3)}\mathbf{v'} >_{[Id3]}.\,c^2$$

We know how to relate the speeds:

$$v'_x = \frac{v_x - u}{1 - \frac{v_x.u}{c^2}} \qquad v'_y = \frac{1}{k}.\,\frac{v_y}{1 - \frac{v_x.u}{c^2}} \quad v'_z = \frac{1}{k}.\,\frac{v_z}{1 - \frac{v_x.u}{c^2}}$$

As consequence:

$$< {}^{(3)}\mathbf{v'}, {}^{(3)}\mathbf{v'} >_{[Id3]} = \frac{1}{k^2}.\,\{k^2.\,(v_x - u)^2 + (v_y)^2 + (v_z)^2\}/(1 - \frac{v_x.u}{c^2})^2$$

Hence, the invariant quantity is:

$$\frac{1}{8.\pi}.\,(< \mathbf{E}, \mathbf{E} >_{[Id3]} + < \mathbf{B}, \mathbf{B} >_{[Id3]}).\,r^3 - < {}^{(3)}\mathbf{p}, {}^{(3)}\mathbf{p} >_{[Id3]}.\,c^2$$

$$\cong$$

$$\frac{1}{8.\pi}.\,\{(E_x)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_y)^2 + k^2.\,(1 + \frac{u^2}{c^4}).\,(E_z)^2 + (B_x)^2$$

$$+ k^2 . (1 + u^2). (B_y)^2 + k^2 . (1 + u^2). (B_z)^2$$

$$- 8. \pi. k^2 . (u/c). (1 + \frac{1}{c^2}). \rho_{EM} . v_x\}. \frac{r^3}{k}$$

$$- k^2 . m^2 . c^2 . \frac{1}{k^2} . \{k^2 . (v_x - u)^2 + (v_y)^2 + (v_z)^2\}/(1 - \frac{v_x . u}{c^2})^2$$

When the second frame is static (u = 0, k = 1), this fundamental relation is a tautology (synonym: trivially true). In all other circumstances, we get a relation insuring the coherence of this exercise with the theory of relativity.

$$\frac{1}{8.\pi} . \{\{k. (1 + \frac{u^2}{c^4}) - 1\}. (E_y)^2 + \{k. (1 + \frac{u^2}{c^4}) - 1\}. (E_z)^2$$

$$+ \{k. (1 + u^2) - 1\}. (B_y)^2 + \{k. (1 + u^2) - 1\}. (B_z)^2$$

$$- 8. \pi. k. (u/c). (1 + \frac{1}{c^2}). \rho_{EM} . v_x\}. r^3$$

$$+ m^2 . c^2 . \{||\mathbf{v}||^2 - \{k^2 . (v_x - u)^2 + (v_y)^2 + (v_z)^2\}/(1 - \frac{v_x . u}{c^2})^2\}$$

$$=$$

$$0$$

Let write for convenience:

$$[\Xi] = \frac{4.\pi.k}{c} . \begin{bmatrix} C_{1xy} & -C_{1xz} & k.\left(1 + \frac{u^2}{c^2}\right).C_{1yz} \\ C_{2xy} & -C_{2xz} & k.\left(1 + \frac{u^2}{c^2}\right).C_{2yz} \\ C_{3xy} & -C_{3xz} & k.\left(1 + \frac{u^2}{c^2}\right).C_{3yz} \end{bmatrix}$$

The sub-system can be condensed as:

$$| \rho'_{EM} . {}^{(3)}\mathbf{v}' > = [\Xi]. | \rho_{EM} . {}^{(3)}\mathbf{v} >$$

This is effectively a linear system telling how to calculate the density of kinetic momentum per unit of volume in the second frame when this density is already known in the first frame. When the second frame is static (u = 0, k = 1), then:

$$[\Xi(u = 0)] = \frac{4.\pi}{c} . \begin{bmatrix} C_{1xy} & -C_{1xz} & C_{1yz} \\ C_{2xy} & -C_{2xz} & C_{2yz} \\ C_{3xy} & -C_{3xz} & C_{3yz} \end{bmatrix} = - \frac{4.\pi}{c} . \begin{bmatrix} < {}_1\mathbf{C}| \\ < {}_2\mathbf{C} \\ < {}_3\mathbf{C} \end{bmatrix}$$

- Each second sub-expression is proportional to the x-component of the pseudo-vector ${}_\alpha\mathbf{C}$:

$$- k^2 . \frac{u}{c^2} . (||\mathbf{E}||^2 + c^2 . ||\mathbf{B}||^2\}. C_{\alpha yz}$$

Therefore, the second part of $\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')$ is proportional to

$$- k^2. \frac{u}{c^2}. (||\mathbf{E}||^2 + c^2. ||\mathbf{B}||^2\}. \begin{bmatrix} C_{1yz} \\ C_{2yz} \\ C_{3yz} \end{bmatrix}$$

$$= - k^2. u. \mu_0. (\varepsilon_0. ||\mathbf{E}||^2 + \frac{1}{\mu_0}. ||\mathbf{B}||^2\}. \begin{bmatrix} {}_1C_x \\ {}_2C_x \\ {}_3C_x \end{bmatrix}$$

$$= - 2. \mu_0. \frac{u}{1-\frac{u^2}{c^2}}. \rho_{EM}. \begin{bmatrix} {}_1C_x \\ {}_2C_x \\ {}_3C_x \end{bmatrix}$$

We have no interpretation for this vector. We can only state that it vanishes when the second frame is static (u = 0).

In that case, the information concerning the physical phenomenon acting on the cross product is entirely contained in the deforming cube C for which a physical interpretation is also absent:

$$| \rho'_{EM}. {}^{(3)}\mathbf{v}' > = [\Xi(u = 0)]. | \rho_{EM}. {}^{(3)}\mathbf{v} > = - \frac{4.\pi}{c}. \begin{bmatrix} < {}_1\mathbf{C}| \\ < {}_2\mathbf{C} \\ < {}_3\mathbf{C} \end{bmatrix}. | \rho_{EM}. {}^{(3)}\mathbf{v} >$$

It tells how to connect the densities of kinetic momentum per unit of volume of two different frames.

In general (for any u), we can write:

$$|\otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}') >$$

$$=$$

$$- \frac{4.\pi.k}{c}. \begin{bmatrix} {}_1C_z & {}_1C_y & k.\left(1 + \frac{u^2}{c^2}\right).\ {}_1C_x \\ {}_2C_z & {}_2C_y & k.\left(1 + \frac{u^2}{c^2}\right).\ {}_2C_x \\ {}_3C_z & {}_3C_y & k.\left(1 + \frac{u^2}{c^2}\right).\ {}_3C_x \end{bmatrix}. | \rho_{EM}. {}^{(3)}\mathbf{v} > - 2. \mu_0. \frac{u}{1-\frac{u^2}{c^2}}. \rho_{EM}. \begin{bmatrix} {}_1C_x \\ {}_2C_x \\ {}_3C_x \end{bmatrix}$$

This formulation looks like a decomposition of the deformed tensor product at hand; with different words: it is perhaps an illustration of the TEQ. The underlying thought is to ask ourselves if the manner to manage the deformed tensor products built on antisymmetric cubes can be related to the main part of prior decomposition.

Until now, the theory considers that the cube C can be reduced to an element [C] in M(3, K) such that[6]:

$$\blacktriangledown C \in \square(3, K)^{-} \Rightarrow |\otimes_{C}({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')> \rightarrow |\otimes_{[C]}({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')> = \{[C]^{t}. [J]\}. |{}^{(3)}\mathbf{E}' \wedge {}^{(3)}\mathbf{B}'>$$

And that:

$$|{}^{(3)}\mathbf{E}' \wedge {}^{(3)}\mathbf{B}'> = \{½. [\mathrm{Hess}_{E'}\, \Lambda({}^{(3)}\mathbf{E}') + {}_{[J]}\Phi(_{\wedge}\mathbf{s})\}. |{}^{(3)}\mathbf{B}'> + |\,\textbf{residual part}'>$$

Hence, if we look for a link between both approaches, we must first find a link between ${}^{(3)}\mathbf{v}$ and ${}^{(3)}\mathbf{B}'$. This questioning is an open debate.

For symmetric [$_{\alpha}$C] matrices

In this case:

$$\{\otimes_{C}({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\alpha}$$

$$=$$

$$k.\, C_{\alpha xy}. \{E_x. (B_y + \frac{u}{c^2}. E_z) + (E_y - u.\, B_z). B_x\} + k.\, C_{\alpha xz}. \{(E_z + u.\, B_y). B_x + E_x. (B_z - \frac{u}{c^2}. E_y)\}$$

$$+ k^2. C_{\alpha yz}. \{(E_y - u.\, B_z). (B_z - \frac{u}{c^2}. E_y) + (E_z + u.\, B_y). (B_y + \frac{u}{c^2}. E_z)\}$$

$$=$$

$$k.\, \{C_{\alpha xy}. (E_x. B_y + E_y. B_x) + C_{\alpha xz}. (E_x. B_z + E_z. B_x)\}$$

$$+ k.\, u.\, \{C_{\alpha xy}. (\frac{1}{c^2}. E_x. E_z - B_x. B_z) + C_{\alpha xz}. (-\frac{1}{c^2}. E_x. E_y + B_x. B_y)\}$$

$$+ k^2. C_{\alpha yz}. \{(1 + \frac{u^2}{c^2}). (E_y. B_z + E_z. B_y) + \frac{u}{c^2}. \{(E_z)^2 - (E_y)^2\} + u.\, \{(B_y)^2 - (B_z)^2\}\}$$

We cannot go further on this path.

- For a plane wave propagating along the Ox axis ($E_x = B_x = 0$), this expression is drastically simplified:

$$\{\otimes_{C}({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}')\}_{\alpha} = k^2. C_{\alpha yz}. \{(1 + \frac{u^2}{c^2}). (E_y. B_z + E_z. B_y) + \frac{u}{c^2}. \{(E_z)^2 - (E_y)^2\} + u.\, \{(B_y)^2 - (B_z)^2\}\}$$

---

[6] But it is far to be the unique manner to analyse this situation. The cube C is the superposition of three antisymmetric matrices. Each of them contains at most three different elements in K (= R or C); each of them appears twice, once with a positive sign and once with a negative sign. Hence, we may consider that each antisymmetric matrix is represented by three elements in R^+, each of them being the module of an element. They can be regrouped inside an element in M(3, R^+):

$$[_{\alpha}C] \rightarrow \{\pm z_{\alpha 1}, \pm z_{\alpha 2}, \pm z_{\alpha 3}\} \rightarrow \{|z_{\alpha 1}|, |z_{\alpha 2}|, |z_{\alpha 3}|\} \rightarrow \begin{bmatrix} |z_{11}| & |z_{12}| & |z_{13}| \\ |z_{21}| & |z_{22}| & |z_{23}| \\ |z_{31}| & |z_{32}| & |z_{33}| \end{bmatrix} = [M] \in M(3, R^+)$$

- If the frames are static (u = 0):

$$\{\otimes_C({}^{(3)}\mathbf{E'},{}^{(3)}\mathbf{B'})\}_\alpha$$

$$=$$

$$k.\{C_{\alpha xy}.(E_x.B_y + E_y.B_x) + C_{\alpha xz}.(E_x.B_z + E_z.B_x)\} + k^2.C_{\alpha yz}.(E_y.B_z + E_z.B_y)\}$$

The fundamental relation

Let try to calculate the fundamental relation; its general formulation is:

$$g_{\alpha\beta}.(C_{\alpha\lambda\mu}.E'^{\lambda}.B'^{\mu}).(C_{\beta\nu\omega}.E'^{\nu}.B'^{\omega})$$

$$=$$

$$g_{\alpha\beta}.\{k.C_{\alpha xy}.E_x.(B_y + \frac{u}{c^2}.E_z) + k.C_{\alpha xz}.E_x.(B_z - \frac{u}{c^2}.E_y)$$

$$+ k.C_{\alpha yx}.(E_y - u.B_z).B_x + k^2.C_{\alpha yz}.(E_y - u.B_z).(B_z - \frac{u}{c^2}.E_y)$$

$$+ k.C_{\alpha zx}.(E_z + u.B_y).B_x + k^2.C_{\alpha zy}.(E_z + u.B_y).(B_y + \frac{u}{c^2}.E_z)\}$$

$$.\{k.C_{\beta xy}.E_x.(B_y + \frac{u}{c^2}.E_z) + k.C_{\beta xz}.E_x.(B_z - \frac{u}{c^2}.E_y)$$

$$+ k.C_{\beta yx}.(E_y - u.B_z).B_x + k^2.C_{\beta yz}.(E_y - u.B_z).(B_z - \frac{u}{c^2}.E_y)$$

$$+ k.C_{\beta zx}.(E_z + u.B_y).B_x + k^2.C_{\beta zy}.(E_z + u.B_y).(B_y + \frac{u}{c^2}.E_z)\}$$

We are primarily interested by the general formulation and by the crucial condition.

The T_0 term.

$$\frac{k.u}{c^2}.C_{\alpha xy}.E_x.E_z.\{\frac{k.u}{c^2}.C_{\beta xy}.E_x.E_z - \frac{k.u}{c^2}.C_{\beta xz}.E_x.E_y - \frac{k^2.u}{c^2}.C_{\beta yz}.(E_y)^2 + \frac{k^2.u}{c^2}.C_{\beta zy}.(E_z)^2\}$$

$$- \frac{k.u}{c^2}.C_{\alpha xz}.E_x.E_y.\{\frac{k.u}{c^2}.C_{\beta xy}.E_x.E_z - \frac{k.u}{c^2}.C_{\beta xz}.E_x.E_y - \frac{k^2.u}{c^2}.C_{\beta yz}.(E_y)^2 + \frac{k^2.u}{c^2}.C_{\beta zy}.(E_z)^2\}$$

$$- \frac{k^2.u}{c^2}.C_{\alpha yz}.(E_y)^2.\{\frac{k.u}{c^2}.C_{\beta xy}.E_x.E_z - \frac{k.u}{c^2}.C_{\beta xz}.E_x.E_y - \frac{k^2.u}{c^2}.C_{\beta yz}.(E_y)^2 + \frac{k^2.u}{c^2}.C_{\beta zy}.(E_z)^2\}$$

$$+ \frac{k^2.u}{c^2}.C_{\alpha zy}.(E_z)^2.\{\frac{k.u}{c^2}.C_{\beta xy}.E_x.E_z - \frac{k.u}{c^2}.C_{\beta xz}.E_x.E_y - \frac{k^2.u}{c^2}.C_{\beta yz}.(E_y)^2 + \frac{k^2.u}{c^2}.C_{\beta zy}.(E_z)^2\}$$

- It is null when the frames are static: u = 0.
- For waves in the (Oy, Oz) plane, T_0 is reduced to:

$$T_0 = (\frac{k^2.u}{c^2})^2.g_{\alpha\beta}.\{C_{\alpha zy}.(E_z)^2 - C_{\alpha yz}.(E_y)^2\}.\{C_{\beta zy}.(E_z)^2 - C_{\beta yz}.(E_y)^2\}$$

It vanishes whatever the metric tensor is when :

$$\forall\ \alpha: C_{\alpha zy}.(E_z)^2 - C_{\alpha yz}.(E_y)^2 = 0$$

We may remark that if the matrices $[_{\alpha}C]$ are antisymmetric, then this condition is here equivalent to:

$$< {}^{(3)}\mathbf{E}, {}^{(3)}\mathbf{E} > = 0$$

... because $E_x = 0$. These circumstances typically represent non-vanishing electrical fields in vacuum; or more precisely: such fields are isotropic vectors (see E: Cartan's work on this topic [--]) which have components in $\mathbb{C}$. Although we don't know the deep reason justifying this fact, the intervening of complex numbers in electricity is not a scoop. We guess that the T_4 term will give similar relations for the magnetic field.

- For any EM-field:

$$\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2$$

$$=$$

$$\frac{k.u}{c^2}.\ \{E_z.\ (C_{\beta xy}.\ E_x + k.\ C_{\beta zy}.\ E_z) - E_y.\ (C_{\beta xz}.\ E_x + k.\ C_{\beta yz}.\ E_y)\}$$

And:

$$\frac{k.u}{c^2}.\ C_{\alpha xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\alpha xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\alpha yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\alpha zy}.\ (E_z)^2$$

$$=$$

$$\frac{k.u}{c^2}.\ \{E_z.\ (C_{\alpha xy}.\ E_x + k.\ C_{\alpha zy}.\ E_z) - E_y.\ (C_{\alpha xz}.\ E_x + k.\ C_{\alpha yz}.\ E_y)\}$$

If we introduce:

$$\forall\ \alpha:\ {}_{\alpha}\kappa_y = C_{\alpha xy}.\ E_x + k.\ C_{\alpha zy}.\ E_z$$

$$\forall\ \alpha:\ {}_{\alpha}\kappa_z = C_{\alpha xz}.\ E_x + k.\ C_{\alpha yz}.\ E_y$$

We can condense each factor and get:

$$-\frac{k.u}{c^2}.\ (\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa})_x$$

With these conventions, the T_0 term is:

$$T_0 = \left(\frac{k.u}{c^2}\right)^2.\ g_{\alpha\beta}.\ ({}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa})_x.\ ({}^{(3)}\mathbf{E} \wedge {}_{\beta}\boldsymbol{\kappa})_x$$

It vanishes whatever the metric tensor is when the electrical field is aligned with each vector ${}_{\alpha}\boldsymbol{\kappa}$ or when the x components of the cross products $({}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa})$ are null.

To complete this first part, we may also remark that each vector ${}_{\alpha}\boldsymbol{\kappa}$ is a linear combination of the components of **E** when k = 1:

$$u = 0 \Rightarrow k = 1 \Rightarrow \forall\ \alpha:\ {}_{\alpha}\kappa_m = \sum_n C_{\alpha nm}.\ E_n$$

The T_1 term.

The first half:

$$\frac{k.u}{c^2}.\ C_{\alpha xy}.\ E_x.\ E_z.\ \{k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z$$

$$+ k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta zy}.\ E_z.\ B_y\}$$

$$- \frac{k.u}{c^2}.\ C_{\alpha xz}.\ E_x.\ E_y.\ \{k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z$$

$$+ k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta zy}.\ E_z.\ B_y\}$$

$$- \frac{k^2.u}{c^2}.\ C_{\alpha yz}.\ (E_y)^2.\ \{k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z$$

$$+ k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta zy}.\ E_z.\ B_y\}$$

$$+ \frac{k^2.u}{c^2}.\ C_{\alpha zy}.\ (E_z)^2.\ \{k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z$$

$$+ k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta zy}.\ E_z.\ B_y\}$$

- It is null when $u = 0$.
- For waves in the (Oy, Oz) plane, T_1 is reduced to:

$$T_{11} = \frac{k^2.u}{c^2}.\ k^2.\ (1 + \frac{u^2}{c^2}).\ g_{\alpha\beta}.\ \{C_{\alpha zy}.\ (E_z)^2 - C_{\alpha yz}.\ (E_y)^2\}.\ \{C_{\beta yz}.\ E_y.\ B_z + C_{\beta zy}.\ E_z.\ B_y\}$$

  If the matrices $[_\alpha C]$ are antisymmetric:

$$T_{11} = \frac{4.\ \pi.\ k^2.\ u}{c^3}.\ k^2.\ (1 + \frac{u^2}{c^2}).\ (g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz}).\ < {}^{(3)}\mathbf{E},\ {}^{(3)}\mathbf{E} >_{Id3}.\ \rho_{EM}.\ v_x$$

  It vanishes when:

  - $g_{\alpha\beta}.\ C_{\alpha yz}.\ C_{\beta yz} = 0$
  - $< {}^{(3)}\mathbf{E},\ {}^{(3)}\mathbf{E} >_{Id3} = 0$

- For any EM-field when the matrices $[_\alpha C]$ are antisymmetric:

$$k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z$$

$$+ k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta zy}.\ E_z.\ B_y$$

$$=$$

$$\frac{4.\pi.\rho_{EM}}{c}.\ \{k.\ (C_{\beta xy}.\ v_z - C_{\beta xz}.\ v_y) + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ v_x\}$$

Hence, in these circumstances:

$$T_{11} = - 4.\ \pi.\ \frac{k.u}{c^3}.\ \rho_{EM}.\ g_{\alpha\beta}.\ ({}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa})_x.\ \{k.\ C_{\beta xy}.\ v_z + k.\ C_{\beta zx}.\ v_y + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ v_x\}$$

The term T_1 vanishes whatever the metric tensor is when:

- The x-component of ${}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa}$ is null.
- The cross product ${}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa}$ is null.
- The second factor is null:

$$C_{\beta xy}.\ v_z + C_{\beta zx}.\ v_y + k.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ v_x = 0$$

The second half of T_1:

$$k.\ C_{\alpha xy}.\ E_x.\ B_y.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

$$+ k.\ C_{\alpha xz}.\ E_x.\ B_z.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

$$+ k.\ C_{\alpha yx}.\ E_y.\ B_x.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

$$+ k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\alpha yz}.\ E_y.\ B_z.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

$$+ k.\ C_{\alpha zx}.\ E_z.\ B_x.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

$$+ k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\alpha zy}.\ E_z.\ B_y.\ \{\frac{k.u}{c^2}.\ C_{\beta xy}.\ E_x.\ E_z - \frac{k.u}{c^2}.\ C_{\beta xz}.\ E_x.\ E_y - \frac{k^2.u}{c^2}.\ C_{\beta yz}.\ (E_y)^2 + \frac{k^2.u}{c^2}.\ C_{\beta zy}.\ (E_z)^2\}$$

When the matrices $[{}_{\alpha}C]$ are antisymmetric, it is now easier to understand that we face the expression:

$$T_{12} = - 4.\ \pi.\ \frac{k.u}{c^3}.\ \rho_{EM}.\ g_{\alpha\beta}.\ \{k.\ C_{\alpha xy}.\ v_z + k.\ C_{\alpha zx}.\ v_y + k^2.\ (1 + \frac{u^2}{c^2}).\ C_{\alpha yz}.\ v_x\}.\ ({}^{(3)}\mathbf{E} \wedge {}_{\beta}\boldsymbol{\kappa})_x$$

At the end of this paragraph, we get:

$$T_1$$

$$=$$

$$T_{11} + T_{12}$$

$$=$$

$$- 4.\ \pi.\ \frac{k^2.u}{c^3}.\ \rho_{EM}.\ g_{\alpha\beta}$$

$$.\ \{({}^{(3)}\mathbf{E} \wedge {}_{\alpha}\boldsymbol{\kappa})_x.\ \{C_{\beta xy}.\ v_z + C_{\beta zx}.\ v_y + k.\ (1 + \frac{u^2}{c^2}).\ C_{\beta yz}.\ v_x\}$$

$$+ \{C_{\alpha xy}.\ v_z + C_{\alpha zx}.\ v_y + k.\ (1 + \frac{u^2}{c^2}).\ C_{\alpha yz}.\ v_x\}.\ ({}^{(3)}\mathbf{E} \wedge {}_{\beta}\boldsymbol{\kappa})_x\}$$

Let recall the existence of:

$$[\Xi] = \frac{4.\pi.k}{c}.\begin{bmatrix} C_{1xy} & -C_{1xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{1yz} \\ C_{2xy} & -C_{2xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{2yz} \\ C_{3xy} & -C_{3xz} & k.\left(1+\frac{u^2}{c^2}\right).C_{3yz} \end{bmatrix}$$

State that:

$$T_1 = -\frac{k.u}{c^2}.\ \rho_{EM}.\ \sum_\alpha \sum_\beta g_{\alpha\beta}.\ \{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x.\ \{[\Xi].\begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\beta + \{[\Xi].\begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\alpha.\ ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x\}$$

The T_2 term.

It is the most important term for the development of this exploration. It contains three parts that we can roughly describe as a sum of terms successively proportional to (i) $E^\lambda.\ E^\mu.\ B^\nu.\ B^\omega$, (ii) $B^\lambda.\ B^\mu.\ E^\nu.\ E^\omega$ and (iii) $E^\lambda.\ B^\mu.\ E^\nu.\ B^\omega$.

$$T_2 = \left(\frac{k.u}{c^2}\right)^2.\ \sum_\alpha \sum_\beta g_{\alpha\beta}.\ \{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x.\ ({}^{(3)}\mathbf{B} \wedge {}_\beta\psi)_x + ({}^{(3)}\mathbf{B} \wedge {}_\alpha\psi)_x.\ ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x\} + \text{(iii)}$$

The term (iii) is:

$$\text{(iii)} = g_{\alpha\beta}.\ \{k.\ C_{\alpha xy}.\ E_x.\ B_y + k.\ C_{\alpha xz}.\ E_x.\ B_z + k.\ C_{\alpha yx}.\ E_y.\ B_x + k^2.\ (1+\frac{u^2}{c^2}).\ C_{\alpha yz}.\ E_y.\ B_z + k.\ C_{\alpha zx}.\ E_z.\ B_x + k^2.\ C_{\alpha zy}.\ (1+\frac{u^2}{c^2}).\ E_z.\ B_y\}$$

$$.\{k.\ C_{\beta xy}.\ E_x.\ B_y + k.\ C_{\beta xz}.\ E_x.\ B_z + k.\ C_{\beta yx}.\ E_y.\ B_x + k^2.\ (1+\frac{u^2}{c^2}).\ C_{\beta yz}.\ E_y.\ B_z + k.\ C_{\beta zx}.\ E_z.\ B_x + k^2.\ C_{\beta zy}.\ (1+\frac{u^2}{c^2}).\ E_z.\ B_y\}$$

Let consider one factor:

$$f_\alpha = C_{\alpha xy}.\ E_x.\ B_y + C_{\alpha xz}.\ E_x.\ B_z + C_{\alpha yx}.\ E_y.\ B_x + k.\ (1+\frac{u^2}{c^2}).\ C_{\alpha yz}.\ E_y.\ B_z + C_{\alpha zx}.\ E_z.\ B_x + k.\ C_{\alpha zy}.\ (1+\frac{u^2}{c^2}).\ E_z.\ B_y$$

When the matrices $[{}_\alpha C]$ are antisymmetric:

$$f_\alpha = C_{\alpha xy}.\ (E_x.\ B_y - E_y.\ B_x) + k.\ (1+\frac{u^2}{c^2}).\ C_{\alpha yz}.\ (E_y.\ B_z - E_z.\ B_y) + C_{\alpha zx}.\ (E_z.\ B_x - E_x.\ B_z)$$

=

$$\frac{4.\pi.\rho_{EM}}{c}. \{C_{\alpha xy}. v_z + k. (1 + \frac{u^2}{c^2}). C_{\alpha yz}. v_x + C_{\alpha zx}. v_y\}$$

=

$$\frac{4.\pi.\rho_{EM}}{c}. \{[\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\alpha$$

The supplementary term is:

$$\text{(iii)} = g_{\alpha\beta}. f_\alpha. f_\beta = (\frac{4.\pi.\rho_{EM}}{c})^2. g_{\alpha\beta}. \{[\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\alpha. \{[\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\beta = (\frac{4.\pi.\rho_{EM}}{c})^2. < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{V} >_{[G]}$$

It doesn't vanish when $u = 0$, $k = 1$.

The T_3 term.

This term is the sum of two parts proportional to (i) $(E^\lambda. B^\mu). (B^\nu. B^\omega)$ or $(B^\lambda. E^\mu). (B^\nu. B^\omega)$, and (ii) $(B^\lambda. B^\mu). (B^\nu. E^\omega)$ or $(B^\lambda. B^\mu). (E^\nu. B^\omega)$.

T_3

=

$$- 4. \pi. \frac{k^2.u}{c^3}. \rho_{EM}.$$

$$g_{\alpha\beta}. \{\{C_{\alpha xy}. v_z + C_{\alpha zx}. v_y + k. (1 + \frac{u^2}{c^2}). C_{\alpha yz}. v_x\}. ({}^{(3)}\mathbf{B} \wedge {}_\beta\psi)_x$$

$$+ ({}^{(3)}\mathbf{B} \wedge {}_\alpha\psi)_x. \{C_{\beta xy}. v_z + C_{\beta zx}. v_y + k. (1 + \frac{u^2}{c^2}). C_{\beta yz}. v_x\}\}$$

=

$$- \frac{k.u}{c^2}. \rho_{EM}. \sum_\alpha \sum_\beta g_{\alpha\beta}. \{\{[\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\alpha. ({}^{(3)}\mathbf{B} \wedge {}_\beta\psi)_x + \{({}^{(3)}\mathbf{B} \wedge {}_\alpha\psi)_x. \{[\Xi]. \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_\beta\}$$

The T_4 term.

Let introduce:

$$\forall\ \alpha: {}_\alpha\psi_y = C_{\alpha yx}. B_x + k. C_{\alpha yz}. B_z$$

$$\forall\ \alpha: {}_\alpha\psi_z = C_{\alpha zx}. B_x + k. C_{\alpha zy}. B_y$$

With these conventions, the T_4 term is:

T_4

=

$$g_{\alpha\beta}. \{- k. u. C_{\alpha yx}. B_z. B_x - k^2. u. C_{\alpha yz}. (B_z)^2 + k. u. C_{\alpha zx}. B_y. B_x + k^2. u. C_{\alpha zy}. (B_y)^2\}$$

$$. \{- k.\ u.\ C_{\beta yx}.\ B_z.\ B_x - k^2.\ u.\ C_{\beta yz}.\ (B_z)^2 + k.\ u.\ C_{\beta zx}.\ B_y.\ B_x + k^2.\ u.\ C_{\beta zy}.\ (B_y)^2\}$$

=

$$g_{\alpha\beta}.\ \{k.\ u.\ \{B_y.\ (C_{\alpha zx}.\ B_x + k.\ C_{\alpha zy}.\ B_y) - B_z.\ (C_{\alpha yx}.\ B_x + k.\ C_{\alpha yz}.\ B_z)\}$$

$$. k.\ u.\ \{B_y.\ (C_{\beta zx}.\ B_x + k.\ C_{\beta yx}.\ B_y) - B_z.\ (C_{\beta yx}.\ B_x + k.\ C_{\beta yz}.\ B_z)\}\}$$

=

$$g_{\alpha\beta}.\ k^2.\ u^2.\ (B_y.\ {}_\alpha\psi_z - B_z.\ {}_\alpha\psi_y).\ (B_y.\ {}_\beta\psi_z - B_z.\ {}_\beta\psi_y)$$

=

$$g_{\alpha\beta}.\ k^2.\ u^2.\ (\mathbf{B} \wedge {}_\alpha\psi)_x.\ (\mathbf{B} \wedge {}_\beta\psi)_x$$

This term vanishes whatever the metric tensor is when:

- The frames are static: $u = 0$, $k = 1$.
- The x-component of $(\mathbf{B} \wedge {}_\alpha\psi)$ vanishes.
- The cross product $(\mathbf{B} \wedge {}_\alpha\psi)$ is null.

Summa Summorum, we get the complete visage of:

$$(\frac{4\pi}{c})^2$$

=

$$\frac{1}{(\rho'.\ v')^2}.\ g_{\alpha\beta}.\ \{\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')\}_\alpha.\ \{\otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}')\}_\beta$$

=

$$\frac{1}{(\rho'.\ v')^2}.\ < \otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}'),\ \otimes_C({}^{(3)}\mathbf{E}',{}^{(3)}\mathbf{B}') >_{[G]}$$

=

$$\frac{1}{(\rho'.\ v')^2}.\ (T_0 + T_1 + T_2 + T_3 + T_4)$$

=

$$\frac{1}{(\rho'.\ v')^2}.\ (\frac{4.\pi.\rho_{EM}}{c})^2.\ < {}^{(3)}\mathbf{V},\ {}^{(3)}\mathbf{V} >_{[G]}$$

+

$$\frac{1}{(\rho'.\ v')^2}.\ (\frac{k.u}{c^2})^2.\ \sum_\alpha \sum_\beta g_{\alpha\beta}.$$

$$\{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x.\ ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x$$

$$- \rho_{EM}.\ \{({}^{(3)}\mathbf{E} \wedge {}_\alpha\boldsymbol{\kappa})_x.\ \{[\Xi].\ \begin{bmatrix} V_z \\ V_y \\ V_x \end{bmatrix}\}_\beta + \{[\Xi].\ \begin{bmatrix} V_z \\ V_y \\ V_x \end{bmatrix}\}_\alpha.\ ({}^{(3)}\mathbf{E} \wedge {}_\beta\boldsymbol{\kappa})_x\}$$

$$+ ({}^{(3)}\mathbf{E} \wedge {}_{\alpha}\kappa)_x . (c.\ {}^{(3)}\mathbf{B} \wedge {}_{\beta}\psi)_x + (c.\ {}^{(3)}\mathbf{B} \wedge {}_{\alpha}\psi)_x . ({}^{(3)}\mathbf{E} \wedge {}_{\beta}\kappa)_x$$

$$- \rho_{EM} . \{(c.\ {}^{(3)}\mathbf{B} \wedge {}_{\alpha}\psi)_x . \{[\Xi] . \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_{\beta} + \{[\Xi] . \begin{bmatrix} v_z \\ v_y \\ v_x \end{bmatrix}\}_{\alpha} . (c.\ {}^{(3)}\mathbf{B} \wedge {}_{\beta}\psi)_x\}$$

$$+ (c.\ \mathbf{B} \wedge {}_{\alpha}\psi)_x . (c.\ \mathbf{B} \wedge {}_{\beta}\psi)_x\}$$

When the frames are static (u = 0 and k = 1):

$$(\frac{4\pi}{c})^2 = \frac{1}{(\rho'.\ v')^2} . < \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}'), \otimes_C({}^{(3)}\mathbf{E}', {}^{(3)}\mathbf{B}') >_{[G]} = \frac{1}{(\rho'.\ v')^2} . (\frac{4.\pi.\rho_{EM}}{c})^2 . < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{V} >_{[G]}$$

With:

$$|\ {}^{(3)}\mathbf{V} > = [\Xi(u = 0)] .\ |\ {}^{(3)}\mathbf{v} >$$

These calculations imply that, in this case (${}^{(3)}\mathbf{v}' = {}^{(3)}\mathbf{v}$) the correspondence between both static frames is:

$$(\rho'_{EM})^2 . < {}^{(3)}\mathbf{v}', {}^{(3)}\mathbf{v}' >_{[G]} = (\rho'_{EM})^2 . < {}^{(3)}\mathbf{v}, {}^{(3)}\mathbf{v} >_{[G]} = (\rho_{EM})^2 . < {}^{(3)}\mathbf{V}, {}^{(3)}\mathbf{V} >_{[G]}$$

It suggests the new relation connecting the measurements:

$$\{\rho_{EM} . [\Xi(u = 0)] - \rho'_{EM} . Id_3\} .\ |\ {}^{(3)}\mathbf{v} > = |\ {}^{(3)}\mathbf{0} >$$

For static frames, the ratios ρ'_{EM}/ρ_{EM} are the eigenvalues of the matrix $[\Xi(u = 0)]$, the entries of which contain the information on how the propagation of electromagnetic energies is modified in the second frame (static). It evidently depends on the physical characteristics of this second frame (vacuum, air, water, glass, etc.).

Annex 04 - The condition on admissible deforming matrices

We start with:

$$S(\alpha, \beta) = \Sigma_{\lambda, \mu, \nu, \omega} (A_{\alpha\lambda\mu} . A_{\beta\nu\omega} + A_{\alpha\lambda\omega} . A_{\beta\mu\nu} + A_{\alpha\lambda\nu} . A_{\beta\omega\mu})$$

$$\Sigma_{\alpha, \beta}\, g_{\alpha\beta} . S(\alpha, \beta) = 0$$

In general, the $S(\alpha, \beta)$ must not vanish but, there are situations not depending on the metric such that the condition holds if:

$$\forall\ \alpha, \beta: S(\alpha, \beta) = \Sigma_{\lambda, \mu, \nu, \omega} (A_{\alpha\lambda\mu} . A_{\beta\nu\omega} + A_{\alpha\lambda\omega} . A_{\beta\mu\nu} + A_{\alpha\lambda\nu} . A_{\beta\omega\mu}) = 0$$

Let consider a given pair (λ, μ):

$$S_{(\lambda, \mu)}$$

$$=$$

$$A_{\alpha\lambda\mu} . A_{\beta 11} + A_{\alpha\lambda 1} . A_{\beta\mu 1} + A_{\alpha\lambda 1} . A_{\beta 1\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta12} + A_{\alpha\lambda2}.\ A_{\beta\mu1} + A_{\alpha\lambda1}.\ A_{\beta2\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta13} + A_{\alpha\lambda3}.\ A_{\beta\mu1} + A_{\alpha\lambda1}.\ A_{\beta3\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta21} + A_{\alpha\lambda1}.\ A_{\beta\mu2} + A_{\alpha\lambda2}.\ A_{\beta1\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta22} + A_{\alpha\lambda2}.\ A_{\beta\mu2} + A_{\alpha\lambda2}.\ A_{\beta2\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta23} + A_{\alpha\lambda3}.\ A_{\beta\mu2} + A_{\alpha\lambda2}.\ A_{\beta3\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta31} + A_{\alpha\lambda1}.\ A_{\beta\mu3} + A_{\alpha\lambda3}.\ A_{\beta1\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta32} + A_{\alpha\lambda2}.\ A_{\beta\mu3} + A_{\alpha\lambda3}.\ A_{\beta3\mu}$$

$$+ A_{\alpha\lambda\mu}.\ A_{\beta33} + A_{\alpha\lambda3}.\ A_{\beta\mu3} + A_{\alpha\lambda3}.\ A_{\beta3\mu}$$

$$=$$

$$A_{\alpha\lambda\mu}.\ \sum_{\nu,\omega} A_{\beta\nu\omega} + (A_{\alpha\lambda1} + A_{\alpha\lambda2} + A_{\alpha\lambda3}).\ \{(A_{\beta1\mu} + A_{\beta2\mu} + A_{\beta3\mu}) + (A_{\beta\mu1} + A_{\beta\mu2} + A_{\beta\mu3})\}$$

Each cube (3-3-3) is the superposition of three (3-3) matrices:

$$\blacktriangledown A$$

$$=$$

$$\{[{}_1A], [{}_2A], [{}_3A]\}$$

$$=$$

$$\left\{\begin{bmatrix} A^1_{11} & A^1_{12} & A^1_{13} \\ A^1_{21} & A^1_{22} & A^1_{23} \\ A^1_{31} & A^1_{32} & A^1_{33} \end{bmatrix}, \begin{bmatrix} A^2_{11} & A^2_{12} & A^2_{13} \\ A^2_{21} & A^2_{22} & A^2_{23} \\ A^2_{31} & A^2_{32} & A^2_{33} \end{bmatrix}, \begin{bmatrix} A^3_{11} & A^3_{12} & A^3_{13} \\ A^3_{21} & A^3_{22} & A^3_{23} \\ A^3_{31} & A^3_{32} & A^3_{33} \end{bmatrix}\right\}$$

With these conventions, we get:

$$\forall (\lambda, \mu): S_{(\lambda, \mu)} = A_{\alpha\lambda\mu}.\ [{}_\beta A]^\oplus + \{[{}_\alpha A].\ \{[{}_\beta A] + [{}_\beta A]^t\}\}_{\lambda\mu}$$

At the end:

$$\forall \alpha, \beta: S(\alpha, \beta) = \Sigma_{\lambda, \mu} S_{(\lambda, \mu)} = [{}_\alpha A]^\oplus.\ [{}_\beta A]^\oplus + \{[{}_\alpha A].\ \{[{}_\beta A] + [{}_\beta A]^t\}\}^\oplus$$

The condition can be written as:

$$\Sigma_{\alpha, \beta}\, g_{\alpha\beta}.\ \{[{}_\alpha A]^\oplus.\ [{}_\beta A]^\oplus + \{[{}_\alpha A].\ \{[{}_\beta A] + [{}_\beta A]^t\}\}^\oplus\} = 0$$

A sufficient condition is:

$$\forall \alpha, \beta: [{}_\alpha A]^\oplus.\ [{}_\beta A]^\oplus + \{[{}_\alpha A].\ \{[{}_\beta A] + [{}_\beta A]^t\}\}^\oplus = 0$$

Annex 05 - The sufficient condition for the Perian matrices

When [A] = [M(**a**, **b**)], some Perian matrix, the sum of its entries is:

$$[M(\mathbf{a}, \mathbf{b})]^\oplus = 3.\ \alpha + \beta.\ (\mathbf{b}^\oplus)^2$$

The square of this sum is:

$$\{[M(\mathbf{a}, \mathbf{b})]^{\oplus}\}^2 = \beta^2. (\mathbf{b}^{\oplus})^4 + 6. \alpha. \beta. (\mathbf{b}^{\oplus})^2 + 9. \alpha^2$$

The square of any Perian matrix is another Perian matrix:

$$[M(\mathbf{a}, \mathbf{b})]^2 = (\alpha^2 - \chi^2). Id_3 + (2. \alpha. \beta + \beta^2. ||\mathbf{b}||^2 + \chi^2). T_2(\otimes)(\mathbf{b}, \mathbf{b}) + 2. \alpha. \chi. {}_{[J]}\Phi(\mathbf{b})$$

The product between a Perian matrix and its transposed is:

$$[M(\mathbf{a}, \mathbf{b})]. [M(\mathbf{a}, \mathbf{b})]^t$$

$$=$$

$$\{\alpha. Id_3 + \beta. T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \chi. {}_{[J]}\Phi(\mathbf{b})\}. \{\alpha. Id_3 + \beta. T_2(\otimes)(\mathbf{b}, \mathbf{b}) - \chi. {}_{[J]}\Phi(\mathbf{b})\}$$

$$=$$

$$(\alpha^2 + \chi^2). Id_3 + (2. \alpha. \beta + \beta^2. ||\mathbf{b}||^2 - \chi^2). T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

Therefore:

$$[M(\mathbf{a}, \mathbf{b})]^2 + [M(\mathbf{a}, \mathbf{b})]. [M(\mathbf{a}, \mathbf{b})]^t$$

$$=$$

$$2. \{\alpha^2. Id_3 + (2. \alpha. \beta + \beta^2. ||\mathbf{b}||^2). T_2(\otimes)(\mathbf{b}, \mathbf{b}) + \alpha. \chi. {}_{[J]}\Phi(\mathbf{b})\}$$

This allows to calculate:

$$\{[M(\mathbf{a}, \mathbf{b})]^2 + [M(\mathbf{a}, \mathbf{b})]. [M(\mathbf{a}, \mathbf{b})]^t\}^{\oplus} = 2. \{3. \alpha^2 + (2. \alpha. \beta + \beta^2. ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2\}$$

Therefore, the sufficient condition for any Perian matrix is:

$$\{[M(\mathbf{a}, \mathbf{b})]^{\oplus}\}^2 + \{[M(\mathbf{a}, \mathbf{b})]^2 + [M(\mathbf{a}, \mathbf{b})]. [M(\mathbf{a}, \mathbf{b})]^t\}^{\oplus}$$

$$=$$

$$\beta^2. (\mathbf{b}^{\oplus})^4 + 6. \alpha. \beta. (\mathbf{b}^{\oplus})^2 + 9. \alpha^2 + 6. \alpha^2 + 2. (2. \alpha. \beta + \beta^2. ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2$$

$$=$$

$$\beta^2. (\mathbf{b}^{\oplus})^4 + (10. \alpha. \beta + 2. \beta^2. ||\mathbf{b}||^2). (\mathbf{b}^{\oplus})^2 + 15. \alpha^2$$

$$=$$

$$0$$

The result doesn't depend on the component χ.

For the completeness of this approach, we should also note that the sufficient condition can be realized through diverse scenarios. Either in considering the previous formulation as a whole or in looking for situations validating each part of it. For example, for all Perian matrices of which the sum of the entries is null, we must simultaneously verify:

$$[M(\mathbf{a}, \mathbf{b})]^{\oplus} = 3.\ \alpha + \beta.\ (\mathbf{b}^{\oplus})^2 = 0$$

$$3.\ \alpha^2 + (2.\ \alpha.\ \beta + \beta^2.\ ||\mathbf{b}||^2).\ (\mathbf{b}^{\oplus})^2 = 0$$

Therefore, the sufficient condition is true when:

$$\alpha = -\frac{\beta}{3}.\ (\mathbf{b}^{\oplus})^2$$

$$\Downarrow$$

$$3.\ \alpha^2 = 3.\ \frac{\beta^2}{9}.\ (\mathbf{b}^{\oplus})^4 = 3.\ \beta^2.\ (\mathbf{b}^{\oplus})^4$$

$$2.\ \alpha.\ \beta\ = -\frac{2}{3}.\ \beta^2.\ (\mathbf{b}^{\oplus})^2$$

$$\beta^2.\ \{3.\ (\mathbf{b}^{\oplus})^4 + (-\frac{2}{3}.\ (\mathbf{b}^{\oplus})^2 + ||\mathbf{b}||^2).\ (\mathbf{b}^{\oplus})^2\} = 0$$

If a Ko(**b**)-ratio exists for this matrix, the sufficient condition is equivalent to:

$$\alpha = -\frac{\beta}{3}.\ (\mathbf{b}^{\oplus})^2$$

$$\beta^2.\ \{3 + (-\frac{2}{3} + Ko(\mathbf{b})\}.\ (\mathbf{b}^{\oplus})^4 = 0$$

But if a Ko(**b**)-ratio exists for this matrix, then the sum of the components of **b** cannot/must not vanish; hence, the sufficient condition is realized when simultaneously:

$$\alpha = -\frac{\beta}{3}.\ (\mathbf{b}^{\oplus})^2,\ \beta^2.\ \{3 + (-\frac{2}{3} + Ko(\mathbf{b})\} = 0$$

This is the case when either:

$$\alpha = \beta = 0 \Rightarrow [M(\mathbf{a}, \mathbf{b})] = \chi.\ {}_{[J]}\Phi(\mathbf{b})$$

Or:

$$\alpha = -\frac{\beta}{3}.\ (\mathbf{b}^{\oplus})^2,\ Ko(\mathbf{b}) = 7/3$$

Annex 06 - Non-Perian matrices respecting the sufficient condition

Proof: Let consider this generic matrix and state that:

$$[A(\mathbf{a}, \mathbf{b})] = \begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix} + {}_{[J]}\Phi({}^{(3)}\mathbf{b}), \qquad [A(\mathbf{a}, \mathbf{b})]^t = \begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix} - {}_{[J]}\Phi({}^{(3)}\mathbf{b})$$

$$[A(\mathbf{a}, \mathbf{b})]^2$$

$$=$$

$$- ||\mathbf{b}||^2. Id_3 + \begin{bmatrix} (a^1)^2 & 0 & 0 \\ 0 & (a^2)^2 & 0 \\ 0 & 0 & (a^3)^2 \end{bmatrix} + \begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix} . \begin{bmatrix} 0 & -b^3 & b^2 \\ b^3 & 0 & -b^1 \\ -b^2 & b^1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -b^3 & b^2 \\ b^3 & 0 & -b^1 \\ -b^2 & b^1 & 0 \end{bmatrix} . \begin{bmatrix} a^1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^3 \end{bmatrix} + T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

$$=$$

$$- ||\mathbf{b}||^2. Id_3 + \begin{bmatrix} (a^1)^2 & 0 & 0 \\ 0 & (a^2)^2 & 0 \\ 0 & 0 & (a^3)^2 \end{bmatrix} + \begin{bmatrix} 0 & -a^1.b^3 & a^1.b^2 \\ a^2.b^3 & 0 & -a^2.b^1 \\ -a^3.b^2 & a^3.b^1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -a^2.b^3 & a^3.b^2 \\ a^1.b^3 & 0 & -a^3.b^1 \\ -a^1.b^2 & a^2.b^1 & 0 \end{bmatrix} + T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

On the same track:

$$[A(\mathbf{a}, \mathbf{b})]. [A(\mathbf{a}, \mathbf{b})]^t$$

$$=$$

$$||\mathbf{b}||^2. Id_3 + \begin{bmatrix} (a^1)^2 & 0 & 0 \\ 0 & (a^2)^2 & 0 \\ 0 & 0 & (a^3)^2 \end{bmatrix} + \begin{bmatrix} 0 & -a^1.b^3 & a^1.b^2 \\ a^2.b^3 & 0 & -a^2.b^1 \\ -a^3.b^2 & a^3.b^1 & 0 \end{bmatrix}$$

$$- \begin{bmatrix} 0 & -a^2.b^3 & a^3.b^2 \\ a^1.b^3 & 0 & -a^3.b^1 \\ -a^1.b^2 & a^2.b^1 & 0 \end{bmatrix} - T_2(\otimes)(\mathbf{b}, \mathbf{b})$$

Therefore:

$$[A(\mathbf{a}, \mathbf{b})]^2 + [A(\mathbf{a}, \mathbf{b})]. [A(\mathbf{a}, \mathbf{b})]^t$$

$$=$$

$$2. \{ \begin{bmatrix} (a^1)^2 & 0 & 0 \\ 0 & (a^2)^2 & 0 \\ 0 & 0 & (a^3)^2 \end{bmatrix} + \begin{bmatrix} 0 & -a^1.b^3 & a^1.b^2 \\ a^2.b^3 & 0 & -a^2.b^1 \\ -a^3.b^2 & a^3.b^1 & 0 \end{bmatrix} \}$$

The sums of the components are:

$$\{[A]^{\oplus}\}^2 = (\mathbf{a}^{\oplus})^2$$

$$\{[A(\mathbf{a}, \mathbf{b})]^2 + [A(\mathbf{a}, \mathbf{b})]. [A(\mathbf{a}, \mathbf{b})]^t\}^{\oplus} = 2. \{||\mathbf{a}||^2 + a^1. (b^2 - b^3) + a^2. (b^3 - b^1) + a^3. (b^1 - b^2)\}$$

Hence, a sufficient condition on (**a**, **b**) ensuring that [A(**a**, **b**)] is a solution is:

$$(\mathbf{a}^{\oplus})^2 + 2. \{||\mathbf{a}||^2 + a^1. (b^2 - b^3) + a^2. (b^3 - b^1) + a^3. (b^1 - b^2)\} = 0$$

□

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