On quantization of a scalar gravity field
with Feynman's path integral quantization method

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Abstract: Quantization of a scalar field is a standard text book example of Feynman's path integral quantization. As my findings on the Relativity Theory show that gravitation must be a scalar field, not a tensor field, it is natural to try this quantization method on Nordström's and Newton's scalar gravity. It turns out that Feynman's method has many serious errors. The reader doubting it may check the first error very easily. A literature result in equation (11) claims to give a Green function $G(x,x')$ to the Klein-Gordon operator $\Box + m^2$. If so, $(\Box + m^2)G = \delta(x-x')$ and if $(\Box + m^2)y(x) = h(x)$, then $y = \int dx' h(x)G(x,x')dx$. We see that when integrating over x' the delta peak picks up the value of $h(x)$ because $\delta(x-x') \neq 0$ in a single point $x' = 0$. But in (11) there is a delta peak $\delta(x^2)$ where $x^2 = |t-t'|^2 - |\vec{r} - \vec{r}'|^2$ is not zero in a single point, it is zero in a subspace. Other errors in Feynman's method are equally clear and real errors. As expected, quantizing gravitation by this method in Section 6 of this article produces a result that does not look correct.

Keywords: Feynman's path integral, Green function, Feynman's propagator.

1. Introduction

In the old quantum theory 1900-1925 quantization meant imposing rules that allowed energy levels and spin and orbital angular momentums to have only discrete values. Matrix operator view by Werner Heisenberg, Max Born and Pascual Jordan in 1925 and Schrödinger's equation from 1926 formulated quantization as a procedure of replacing energy and momentum by differential operators. This method is called canonizal quantization. Allowing more particles in states was named second quantization, but it is not a quantization method. Today the meaning of quantization seems to be quite restriced and only two procedures seem to be accepted as quantization. It might be more reasonable to allow any discretization of continuous values to be considered as quantization, but this is not the case any more. Canonical quantization is the first method and Feynman's path integral quantization is the second accepted quantization approach.

Feynman's path integral quantization replaces momentum by a partial differential operator and it gives quantum corrections, for instance in the scalar quantum field theory the method adds two small terms to the mass of a scalar particle. There is a problem with the scalar field: the mass correction term includes the Feynman propagator $\Delta(x-x')$ and this function is evaluated at zero. At zero the Feynman propagator diverges. This is solved in the theory by renormalization.

As gravitation must be a scalar field according to my studies on the Relativity Theory, a quantum scalar field is of special interest. This is why I saw it necessary to check Feynman's path integral quantization of a scalar field. It involves finding a Green function to the Klein-Gordon operator. According to this theory, the Green function is the Feynman propagator.

The article shows that the calculation of the Green function is incorrect and points out to many problems in Feynman's path integral method. As Dirac's equation is used as the spinor field for spin $\frac{1}{2}$ particles in quantum field theories, the article also comments on the Dirac equation. The final section before conclusions includes the effort to quantize scalar gravity. The result is not convincing.

2. The Green function of the Klein-Gordon operator

The Klein-Gordon operator is

$$\Box + m^2 = \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_1^2} \right) + m^2. \tag{1}$$

This operator comes from the relativistic mass formula

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{2}$$

by making the substitutions

$$p_i \to -\hbar \frac{\partial}{\partial x_i} \tag{3}$$

in the right side and adding the wavefunction ϕ

$$\left(\frac{E}{\hbar c} \right)^2 \phi = \left(\Box + \left(\frac{m_0 c}{\hbar} \right)^2 \right) \phi. \tag{4}$$

Substitution of energy by time derivation in the left side is not needed in this calculation.

A Green function for a linear operator D_x of variables x is a two-variable function $G(x', x)$ that satisfies

$$D_x G(x', x) = \delta(x' - x) \tag{5}$$

where $\delta(x' - x)$ is Dirac's delta function. The inhomogeneous differential equation

$$D_x \phi(x) = h(x) \tag{6}$$

is solved as

$$\phi(x) = \int_{-\infty}^{+\infty} h(x') G(x', x) dx' \tag{7}$$

as is seen by taking the differential operator inside the definite integral

$$D_x \phi(x) = \int_{-\infty}^{+\infty} h(x') D_x G(x', x) dx' = \int_{-\infty}^{+\infty} h(x') \delta(x' - x) dx' = h(x). \tag{8}$$

In quantum field theory the integral is taken over the four-dimensional Minkowski space where $x_0 = ct$, $x = (x_0, x_1, x_2, x_3)$ and $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$

$$\phi(x) = \int d^4x' h(x') G(x', x) \tag{9}$$

and the operator times the Green function gives a four-dimensional delta function

$$D_x G(x', x) = \delta^4(x' - x). \tag{10}$$

Searching for the Green function for the Klein-Gordon equation in the Internet finds many very similar treatments of the problem. They all propose the following Green function:

$$G(x' - x)_{ret} = \theta(t - t') \left(\delta(x^2) - \frac{m}{2\pi|x - x'|} J_1(m|x - x'|) \right). \tag{11}$$

Here $|x - x'| = \sqrt{(x_0 - x_0')^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2 - (x_3 - x_3')^2}$, often seen in the form $|x - x'| = \sqrt{(t - t')^2 - |\vec{x} - \vec{x}'|^2}$ where c is set to one. The function J_1 is one of the Bessel functions and ret in the subindex of the Green function means that this function is the "retarded" branch of the solution. The function $\theta(t)$ is the Heavyside function, $\theta(t) = 1$ if $t \geq 0$, $\theta(t) = 0$ if $t < 0$.

We can check if this is the Green function. A simple calculation does give the Bessel function J_1. Let us write $y = |x - x'|$. We try to find a solution $f(y)$ to the homogeneous Klein-Gordon equation

$$\left(\Box + m^2\right) f(y) = 0. \tag{12}$$

Since for $i = 1, 2, 3$

$$\frac{\partial}{\partial x_i} f(y) = \frac{\partial y}{\partial x_i} f'(y) = -\frac{x_i}{y} f' \tag{13}$$

$$\frac{\partial^2}{\partial x_i^2} f(y) = -\frac{1}{y} f' - \frac{x_i^2}{y^3} f' + \frac{x_i^2}{y^2} f'' \tag{14}$$

and for $i = 0$

$$\frac{\partial}{\partial x_0} f(y) = \frac{\partial y}{\partial x_0} f'(y) = \frac{x_0}{y} f' \tag{15}$$

$$\frac{\partial^2}{\partial x_0^2} f(y) = \frac{1}{y} f' - \frac{x_0^2}{y^3} f' + \frac{x_0^2}{y^2} f'' \tag{16}$$

Summing the terms to get $\Box + m^2$ gives

$$\left(\Box + m^2\right) f = \frac{1}{y} f' - \frac{x_0^2}{y^3} f' + \frac{x_0^2}{y^2} f'' \tag{17}$$

$$+ 3\frac{1}{y} f' - \frac{-x_1^2 - x_2^2 - x_3^2}{y^3} f' + \frac{-x_1^2 - x_2^2 - x_2^2}{y^2} f'' \tag{18}$$

$$\left(\Box + m^2\right) f = \frac{4}{y} f' - \frac{1}{y} f' + f'' + m^2 f = \frac{3}{y} f' + f'' + m^2 f = 0. \tag{19}$$

This equation is solved by a Taylor series

$$f(y) = \sum_{n=0}^{2} a_n y^n \tag{20}$$

and the recursion equation for y^n is

$$3(n+2)a_{n+2} + (n+2)(n+1)a_{n+2} + m^2 a_n = 0 \tag{21}$$

showing that we must set $a_n = 0$ for odd n and writing $n = 2i$ the recursion gives

$$a_{2i+2} = -\frac{m^2}{(2i+2)(3+2i+1)} a_{2i} \tag{22}$$

$$a_{2i} = (-1)^i \left(\frac{m}{2}\right)^{2i} \frac{1}{(i+1)(i!)^2} a_0 = (-1)^i \left(\frac{m}{2}\right)^{2i} \frac{1}{\Gamma(i+2)(i!)^2} a_0. \tag{23}$$

The Bessel function J_1 has the expression

$$J_1(x) = \sum_{i=0}^{\infty} (-1)^i \left(\frac{x}{2}\right)^{2i+1} \frac{1}{\Gamma(i+2)(i!)^2}. \tag{24}$$

Setting $a_0 = m^2/2\pi$ gives the expression with J_1 in (11)

$$f(y) = a_0 \frac{m}{my} J_1(my) = \frac{m}{2\pi|x-x'|} J_1(m|x-x'|). \tag{25}$$

That is, $f(y)$ as in (25) is a solution to the homogeneous Klein-Gordon equation(12). It is not a Green function for the operator (11): a Green function should give delta, not zero as in (12). The effort to make this solution for a homogeneous equation into a Green function in (11) is to add the delta function and the Heavyside function. This is where the calculation fails.

In this particular case one obvious error is that $|x - x'|$ is not zero at one point because the Minkowski metric is not positive definite. It is zero in a subspace. We can let $|t-t'|$ be positive in an area of interest so that the Heavyside function is one and the solution (11) is

$$G(x'-x)_{ret} = \delta(x^2) - \frac{m}{2\pi|x-x'|} J_1(m|x-x'|). \tag{26}$$

Operating this function with (1) gives differentials of the delta peak, but the main problem is that the delta function in (26) is infinite in a subspace when $x = 0$ and that subspace is not one point.

The error is not only caused by the Minkovski pseudometric. A Green function is not obtained by taking a solution for the homogeneous equation and adding a delta peak to it. A Green function must give the delta peak from the continuous solution, which in this case of $f(y)$, but $f(y)$ is an analytic function and does not give the peak. Feynman may have thought of the Green function for ∇^2 in the Gauss equation

$$\nabla^2 f(r) = \delta(r) \tag{27}$$

where $f(r) = 1/r$. Here we see that the delta peak comes from the pole the function $1/r$ has at $r = 0$. Such an approach cannot work for the operator in (1) because there is the m^2 term. Expanding $f(x)$ as a series in negative powers of x, where x can be an expression $x = \sum k_i x_i^a$ for any a and k_i, and applying the Klein-Gordon operator leaves the smallest negative power of $f(x)$ uncancelled by any other terms. This smallest negative power comes from $m^2 f(x)$, the part $\Box f(x)$ contains only larger negative powers of x.

However, notice that the fiend equation in Nordström's and Newton's gravity does not have the m^2 term. Newton's theory has a Green function $1/r$, but Nordström's theory has the operator $\Box$ and it has the same problem as (11), the argument of the delta peak is zero in a subspace, not in a point. This shows that a Minkowski space is not a correct setting for gravity.

3. The Green function of a one-dimensional Klein-Gordon-type operator

Let us look at the one-dimensional partial differential equation

$$\left(\frac{\partial^2}{\partial x_i^2} - \mu^2 \right) \phi(x) = h(x) \tag{28}$$

where $x = (x_0, x_1, x_2, x_3)$. As μ is not mass in the applications in quantum field theory, it is not denoted by m in this section as it was in the previous section. For $i = 1, 2, 3$ the space coordinates in $\Box$ have a negative sign and therefore μ^2 is negative.

The inhomogeneous second order differential equation of one variable

$$\frac{d^2\phi}{dx^2} - \mu^2 \phi = h \tag{29}$$

can be solved e.g. by inserting $f = d\phi/dx + \mu\phi$ and solving two first-order differential equations. The general solution of the homogeneous equation is

$$\phi = C_1 e^{kx} + C_2 e^{-kx} \tag{30}$$

where C_1 and C_2 are complex constants and $\mu^2 = k^2$. If $-\mu^2$ is negative, k is real and if $-\mu^2$ is positive, k is imaginary. A special solution for the inhomogeneous equation is

$$\phi = \int^x \int^{x'} h(x'') e^{-kx''} dx'' e^{2kx'} dx' e^{-kx}. \tag{31}$$

Let's check that it is a solution:

$$\frac{d\phi}{dx} = \int^x h(x'') e^{-kx''} dx'' e^{kx} - k\phi \tag{32}$$

$$\frac{d^2\phi}{dx^2} = h(x) + k^2\phi. \tag{33}$$

After some manipulation the solution can be expressed in a form

$$\phi = \int_x^\infty h(x')\frac{e^{k(x'-x)}}{2k}dx' + \int_{-\infty}^x h(x')\frac{e^{-k(x'-x)}}{2k}dx' \tag{34}$$

Then

$$\frac{d\phi}{dx} = -k\int_x^\infty h(x')\frac{e^{k(x'-x)}}{2k}dx' + k\int_{-\infty}^x h(x')\frac{e^{-k(x'-x)}}{2k}dx' \tag{35}$$

$$\frac{d^2\phi}{dx^2} = k\frac{h(x)}{2k} + k^2\int_x^\infty h(x')\frac{e^{k(x'-x)}}{2k}dx' \tag{36}$$

$$+ k\frac{h(x)}{2k} + k^2\int_{-\infty}^x h(x')\frac{e^{-k(x'-x)}}{2k}dx' = h(x) + k^2\phi. \tag{37}$$

This is a Green function solution. We should get a Green function solution also by Fourier transform $x \to k$:

$$(k^2 - \mu^2)\Psi(k) = H(k) \tag{38}$$

$$\psi(x) = \mathcal{F}^{-1}[H(k)/(k^2 - \mu^2)]. \tag{39}$$

In order to get the Green function, we transform

$$\mathcal{F}^{-1}[1/(k^2 - \mu^2)] = \frac{1}{2k}\mathcal{F}^{-1}[1/(k-\mu)] + \frac{1}{2k}\mathcal{F}^{-1}[1/(k+\mu)]. \tag{40}$$

Literature gives some alternative forms for $\mathcal{F}^{-1}[1/(k \pm \mu)]$, usually they are $\mathcal{F}^{-1}[1/(k-\mu)] = ie^{i\mu x}\theta(-x)$ and $\mathcal{F}^{-1}[1/(k+\mu)] = -ie^{i\mu x}\theta(-x)$. Inserting these functions yields

$$\mathcal{F}^{-1}[1/(k^2 - \mu^2)] = \frac{1}{2k}ie^{i\mu x}\theta(-x) - \frac{1}{2k}ie^{i\mu x}\theta(-x). \tag{41}$$

In the Green function there is $x - x'$ instead of x

$$G(x - x') = \frac{1}{2k}ie^{i\mu(x-x')}\theta(-(x-x')) - \frac{1}{2k}ie^{i\mu(x-x')}\theta(-(x-x')). \tag{42}$$

Calculating ϕ from the Green function gives

$$\phi(x) = \int_{-\infty}^\infty h(x')\frac{ie^{i\mu(x-x')}}{2k}\theta(-(x-x'))dx' - \int_{-\infty}^\infty h(x')\frac{ie^{i\mu(x-x')}}{2k}\theta(-(x-x'))dx'. \tag{43}$$

$$\phi(x) = i\int_x^\infty h(x')\frac{e^{i\mu(x-x')}}{2k}dx' - i\int_x^\infty h(x')\frac{e^{i\mu(x-x')}}{2k}dx'. \tag{44}$$

Equations (34) and (44) should be the same, but (43) has an incorrect multiplier i and other differences. This is to say that the Fourier transform and inverse

Fourier transform typically cannot be calculated directly as the integral oscillates. They are obtained from tables that have been produced with the idea that if $\mathcal{F}^{-1}[F(k)] = f(x)$ then $\mathcal{F}[f(x)] = F(k)$, and vice versa. There are situations when the correct transform is different than the one given by this logic. In the case of (43), the correct transform is as in (34), (34) does give ϕ. Inverse Fourier transforms yielding (34) are $\mathcal{F}^{-1}[1/(k-\mu)] = e^{i\mu x}\theta(-x)$ and $\mathcal{F}^{-1}[1/(k+\mu)] = e^{-i\mu x}\theta(x)$.

One might imagine that it is possible to extend (34) to four dimensions, at least the literature gives inverse Fourier transforms to different dimensions, but I could not find any other extension of (34) to four dimensions than adding one dimensional Green functions as additive terms where each k_i is μ

$$\phi = -\frac{1}{4}\sum_{i=1}^{3}\int_{x_i}^{\infty} h(x_0,..,x_i',..,x_3)\frac{e^{\mu(x_i'-x_i)}}{2\mu}dx_i' \tag{45}$$

$$-\frac{1}{4}\sum_{i=1}^{3}\int_{-\infty}^{x_i} h(x_0,..,x_i',...x_3)\frac{e^{-\mu(x_i'-x_i)}}{2\mu}dx_i' \tag{46}$$

$$+\frac{1}{4}\int_{x_0}^{\infty} h(x_0',x_1,x_2,x_3)\frac{e^{i\mu(x_0'-x_0)}}{2\mu}dx_0' \tag{47}$$

$$+\frac{1}{4}\int_{-\infty}^{x_0} h(x_0',x_1,x_2,x_3)\frac{-e^{i\mu(x_0'-x_0)}}{2\mu}dx_0' \tag{48}$$

$$= \frac{1}{4}\left(3(h(x)+\mu^2\phi) + (h(x)+\mu^2\phi)\right). \tag{49}$$

Multiply these kind of terms does not give ϕ. In a similar way, the solution (31) can be extended to a 4-dimentional solution as

$$\phi(x) = -\frac{1}{4}\sum_{i=0}^{3}\int^{x_i}\int^{x_i'} h(x_0,..,x_i',...x_3)e^{-\mu x_i''}dx_i''e^{2\mu x_i'}dx_i'e^{-\mu x_i} \tag{50}$$

$$+\frac{1}{4}\int^{x_i}\int^{x_0'} h(x_0',x_1,x_2,x_3)e^{-i\mu x_i''}dx''_0e^{2i\mu x_0'}dx_i'e^{-i\mu x_0} \tag{51}$$

$$= \frac{1}{4}\left(3(h(x)+\mu^2\phi) + (h(x)+\mu^2\phi)\right). \tag{52}$$

Multidimensional integrations of any such terms do not seem to give ϕ.

The homogeneous equation has the solutions

$$\phi = Ce^{k\cdot x} \tag{53}$$

where $x = (x_0, x_1, x_2, x_3)$, $k = (k_0, k_1, k_2, k_3)$, C is a complex constant and $\mu^2 = \sum_{i=0}^{3} k^2$, but notice that unlike in (30) there is no general solution that has two functions. For every choice of k_i satisfying $\mu^2 = \sum_{i=0}^{3} k^2$ the solution is linearly independent of other similar solutions. There is also the solution with the Bessel function J_1. Partial differential equations do not have a small number of basis functions in the general solution of the homogeneous equation.

As a conclusion from this section I mention that there should be a good demonstration that a 4-dimensional Green function as in (9) exists for this operator. Literature gives (11), but that certainly is not correct. If there is no such 4-dimensional valid Green function, as seems to be the case, then diverge of Green functions in the perturbation series and correction of this divergence by renormalization should be reconsidered.

4. Fundamental problems in the two quantization method

Canonical quantization has several issues, the following items describe some issues.

Problem 1 The substition of p by the partial derivative of space is not justified. The momentum p in the Hamiltonian is not $p = \hbar k$, the momentum from de Broglie's formula, see [1] for my arguments. The substitution works in the Schrödinger equation because in that equation the space is one dimensional and calculating with the space derivative world as an alternative way of calculating the time derivative $p = mv = mdx/dt$ when the function on which this derivative operates is a wave $\exp i\vec{k} \cdot \vec{x}$.

Problem 2 Using a 4-dimensional momentum is not justified. In the relativistic mass formula the momentum is 3-dimensional. The relativistic kinetic energy formula is incorrect, but there is apparent mass and apparent momentum, see the apparent momentum in Compton scattering in [2], it is 3-dimensional, not 4-dimensional.

Problem 3 Momentum substitution must be 1-dimensional, not multi-dimensional. In the Schrödinger equation the momentum substitution is one-dimensional. If the mass moves in the constant direction $\vec{r}$, then the substitution gives

$$p^2 \quad \rightarrow \quad -\hbar^2 \frac{\partial^2}{\partial r^2} \tag{54}$$

but the substitution of

$$p_i^2 \quad \rightarrow \quad -\hbar^2 \frac{\partial^2}{\partial x_i^2} \tag{55}$$

gives in 3-dimensional space coordinates

$$p^2 \quad \rightarrow \quad -\hbar^2 \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{56}$$

Notice the term r^2 between the two partial derivatives. It corresponds to a sphere of area $4\pi r^2$ that the force lines must mass. This differential operator comes from a force field radiating from a point mass or a point charge. The term naturally appears in a field equation, i.e., an equation that gives the solution to the field. It is difficult to understand why it should appear in a dynamic equation, i.e., an equation that determines the movement of a test mass or a test charge. The Schrödinger equation is a quantized form of a dynamic equation: though it is derived from the energy equation (Hamiltonian), the substitution of the momentum have the same effect to a wave $\exp(-i\vec{k}\cdot\vec{x})$ as the second time derivative.

Issue 1 Wave equations in canonical quantization (Schrödinger's, Dirac's and Klein-Gordon's equations) are derived by making substitutions to Hamiltonian, i.e., the energy equation. Apparent mass and apparent momentum in apparent kinetic energy corresponds to energy that has to be put into the system. Therefore it is correct to use apparent momentum in the energy equation, but the substition of momentum by a second order derivative actually turns the energy equation into an alternative form of a dynamic equation. A dynamic equation is easier to write when considered from the side of the test mass or test charge, then the mass used is real, not apparent, mass and forces are reduced. This is not an error, but requires attention.

There are also several issues in Feynman's path integral quantization. The path integral is "derived" in a standard text book [3] which was used on a lecture course I once took in the following way. The Gaussian integral

$$\int_{-\infty}^{\infty} dye^{-\frac{1}{2}ay^2} = \sqrt{2\pi}a^{-\frac{1}{2}} \tag{57}$$

can be generalized to an integral of a $n \times n$ diagonal matrix A. The matrix has diagonal entries a_i, $i = 1, ..., n$, and $Y = (y_1, ..., y_n)$ is a vector. The generalization is

$$\int_{-\infty}^{\infty} dy_1 dy_2 ..., dy_n e^{-\frac{1}{2}Y^T AY} = (2\pi)^{\frac{n}{2}} \prod_{i=1}^{n} a_i^{-\frac{1}{2}} = (2\pi)^{\frac{n}{2}} e^{-\frac{1}{2}Tr \ln A}. \tag{58}$$

Logarithm of matrix A means taking the logarithm of every element and trace Tr is the sum of diagonal elements.

Writing $y_i = \phi(x_i)$ and letting n grow to infinity, integrating over each y_i can be seen as integrating over all paths $\phi(x)$ from x_0 to x_n. So, the integral can be understood as a path integral where the path $\phi(x)$ varies over all paths with the fixed end points:

$$\int_{-\infty}^{\infty} dy_1 dy_2 ..., dy_n \quad \rightarrow \quad \int \mathcal{D}\phi(x). \tag{59}$$

The exponent of exp turns into an integral

$$-\frac{1}{2}Y^T AY \quad \rightarrow \quad -\frac{1}{2}\int dx' \int dx \phi(x')^* A(x', x)\phi(x) \tag{60}$$

where the diagonal matrix A has been generalized into a diagonal operator $A(x', x')$.

Subtracting a term $\rho^T Y$ from the exponent $-\frac{1}{2}Y^T AY$ and completing into a square the authors of [3] derive

$$Y^T AY - 2\rho^T Y = (Y^T - A^{-1}\rho)A(Y - A^{-1}Y) + \rho^T A^{-1}\rho \tag{61}$$

which gives Feynman's path integral

$$\int \mathcal{D}\phi(x) \exp\left(-\frac{1}{2}\int dx' \int dx\phi(x')^* A(x',x)\phi(x) + \int dx\rho(x)\phi(x)\right) \tag{62}$$

$$= \exp\left(-\frac{1}{2}Tr \ln A\right) \exp\left(\frac{1}{2}\int dx' \int dx\rho(x)^* A^{-1}(x',x)\rho(x)\right). \tag{63}$$

Later the integration over x' and x is changed to 4-dimensional integration over space or momentum.

This derivation may initially appear to have some sense, but there are certain problems in it, for instance:

Issue 2 Four-dimensional Fourier transform of the Klein-Gordon operator does not have the inverse transform as Feynman's theory assumes. The authors of [3] use as an example the operator

$$A(x',x) = \left(\frac{\partial}{\partial x'}\frac{\partial}{\partial x} + r\right)\delta(x' - x) \tag{64}$$

which is very closely related to the Kein-Gordon operator that is used in the quantization of a scalar field theory, explained in Chapter 4 of [3]. In order to take a trace and logarithm of this operator the authors make a Fourier transform. In the first example it is a 1-dimensional Fourier transform, but in the scalar field theory it is a 4-dimensional Fourier transform. In [1] it is shown by an example that Schrödinger's momentum substitution is not the same as taking a Fourier transform and the earlier sections of the presented article show that the inverse Fourier transform of 4-dimensional $1/(p^2 + m^2)$ is not what quantum field theory claims.

Issue 3 Under equation (53) it is mentioned that partial differential equations, especially the one coming from (1), have uncountably many independent basis functions. Therefore following the appoach of turning a diagonal matrix into an operator and finding the diagonal elements fails. If the operator remains as an operator, it acts on the function $\phi(x)$ on the right side. Therefore we should express the wave functions with eigenvectors of the operator, then we would get a number, an eigenvalue. The diagonal elements could be sums of eigenvalues, but only if the operator had only few eigenvalues. This is not the case.

Issue 4 There is no logical way to make the step from the discrete case to the path integral. In the discrete case all noncontinuous paths are counted while in the path integral paths are continuous. How does limiting to continuous paths in (62) reflect to the right side (63), the right side of (58) has also noncontinuous paths?

The path integral is used to define the generating functional. Let us not go into the logic how the integration over x' and x is changed into integration over time and three dimensional space. The authors of [3] give in equation (4.4) the generating functional as

$$W[J] = N \int \mathcal{D}\phi \int \mathcal{D}\pi \exp i\hbar^{-1} \int dt \int d^3x \, (\pi\partial_o\phi - \mathcal{H}(\pi,\phi) + J\phi) \, . \tag{65}$$

Here J, the external current, corresponds to ρ and N is some normalization constant. This path integral is evaluated in the same way as we got the right side (63), meaning that (62)-(63) gives a way to calculate a path integral.

The theory defines functional derivative as

$$\frac{\delta}{\delta\rho(t')}\left(\int_{-\infty}^{\infty} dt\rho(t)\phi(t)\right) = \phi(t'). \tag{66}$$

The definition seems to be nothing else than

$$y = \int^t \rho(t)dt \quad \rightarrow \quad dy = \rho(t)dt \tag{67}$$

$$\frac{d}{dt}\int^t \phi(t)\rho(t)dt|_{t=t'} = \frac{d}{dt}\int^t \phi(t(y))dy|_{t=t'} = \phi(t)|_{t=t'} = \phi(t'). \tag{68}$$

Then

$$\frac{d}{dt}|_{t=t'} \exp\left(\alpha\int^t \phi(t)dy\right) = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\alpha\int^t \phi(t)dy\right)^n|_{t=t'} \tag{69}$$

$$= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n-1)!}\phi(t')\left(\alpha\int^{t'} \phi(t)dy\right)^{n-1} \tag{70}$$

and so

$$\frac{d}{dt}|_{t=t'} \exp\left(\alpha\int^t \phi(t)dy\right)|_{\rho(t)=0} \tag{71}$$

$$= \frac{d}{dt}\sum_{n=0}^{\infty}\frac{\alpha^n}{(n-1)!}\phi(t')\left(\alpha\int^{t'} \phi(t)dy\right)|_{\rho(t)=0} = \alpha\phi(t'). \tag{72}$$

Notice that if $t_n > ... > t_1$, then

$$\frac{d}{dt}|_{t=t'_n}\cdots\frac{d}{dt}|_{t=t'_1} \exp\left(\alpha\int^t \phi(t)dy\right)|_{\rho(t)=0} \neq \alpha^n\phi(t'_n)...\phi(t'_1) \tag{73}$$

because after the first differentiation the bound of the integration gets set to t'_1, not to ∞. But if we do the derivations in the inverse order, then

$$\frac{d}{dt}|_{t=t'_1}\cdots\frac{d}{dt}|_{t=t'_n} \exp\left(\alpha\int^t \phi(t)dy\right)|_{\rho(t)=0} = \alpha^n\phi(t'_n)...\phi(t'_1). \tag{74}$$

It seems that the functional derivative indeed is derivation by y:

$$\frac{\delta}{\delta\rho(t_1')}\cdots\frac{\delta}{\delta\rho(t_n')}\exp\left(\alpha\int_{-\infty}^{\infty}\phi(t)dy\right)|_{\rho(t)=0}=\alpha^n\phi(t_n')...\phi(t_1') \tag{75}$$

$$=\frac{d}{dt}|_{t=t_1'}\cdots\frac{d}{dt}|_{t=t_n'}\exp\left(\alpha\int^{t}\phi(t)dy\right)|_{\rho(t)=0}. \tag{76}$$

There is still the path integral in the generating functional $W[J]$. I do not quite see that it gives the expectation value of the operator $\hat{\phi}$, but the following equation can also be considered as definition:

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}W[J]|_{J(x)=0}=<0|T(\hat{\phi}(x_1))...T(\hat{\phi}(x_n))|0> . \tag{77}$$

The method seems to work like this: the generating functional is expanded as a series and the functional derivative is supposed to pick up one of the terms of the series, the one where J is derivated away. $T()$ in (77) is the time ordering function, hat means an operator and $<0|$ and $|0>$ mean the ground state. Thus, we get the ground state to ground state transition propability.

The idea of Feynman's path integral method seems to be the following. We have a generating functional and a way to pick up an expectation value for ground state to ground state transition probability from the generating functional by taking functional derivatives. We also have a way to evaluate the path integral to something that can derivated, the expression in (63). This method is applied by inserting the Hamiltonian or Lagrangean to the exponent in the generating functional. In the scalar field case explained in [3] the first functional derivative gives the expectation value of the classical equation in [3] and the the higher terms give quantum corrections. Something of this type, but we will see in Section 6 that this is not how the method works. The method does not work as at all and the right side (63) is not used for anything.

There are some issues, like:

Issue 5 The method seems to assume that a particle moves for some time as a wave function, we can track the expectation value for it by constricting it with some times $t_1, .., t_n$. Also canonical quantization makes this assumption. I do not think it is justified. Quantum effects happen in the interaction of a field or a particle with matter and there is no reason to assume that either fields or particles travel anywhere as waves. Particle-wave dualism is an incorrect idea, it has led to several paradoxes in quantum mechanics, see [1].

Issue 6 The second issue is that the terms than one picks up from the generating functional by functional derivation are terms in a perturbation series and in Feynman's method they are 4-dimensional Fourier transforms and need to be transformed back from the momentum space to the space-time coordinates. The terms tend to diverge and the method to cope with this divergence by renormalization is what it is often said to be: subtracting infinities from infinities and getting a finite number. Or what can one say of dimensional renormalization (see [3]): as the integral in the Feynman propagator (i.e., the Fourier inverse transform of $1/(p^2-\mu^2)$ diverges at zero, the dimension over which $1/(p^2-\mu^2)$ is integrated is decreased to fractional dimensions, no justification for decreasing the dimension is proposed.

There is still one small issue that can be mentioned. As is clear from the calculation (20)-(23), the solution (26) with the Bessel function J_1 is certainly not derived by inverse Fourier transform. It must be derived from a power series. But treatments of the Green function of the Klein-Gordon operator always present the time integration of the inverse Fourier transform of $1/(p^2-\mu^2)$. It is made by considering ω as a complex variable and changing an integral on the time axis by a contour integral in the complex time space by adding a semicircle to the lie on the time coordinate. This contour goes through two poles and these treatments of how to calculate the Green function explain that it is necessary to go around these poles and then to use the residue theorem. In equation (108) of this article the term avoiding the pole is the $i\epsilon$ term in Feynman's propagator. This is incorrect:

Problem 4 In the calculation of the Feynman propagator, the function $1/(p^2-\mu^2)$ is changed in order for the contour integral to avoid a pole in the time integration, but if the function is modified, then it is no longer the correct function. There is no need for making this error. It is quite easy to integrate the function along the time axis: the function splits to two parts that are symmetric with respect to a vertical line at the origin and they cancel, there only remains the two poles and the 1-dimensional Green function can be calculated in several sound ways, like (34).

5. About the Dirac equation

The Dirac equation is used in quantum field theory as the free spin $\frac{1}{2}$ field. While this equation is not directly connected with quantization of a scalar field, Direc's equation is important for both the canonical and Feynman's path integral quantization methods and relevant in an evaluation of Feynman's path integral quantization method. Let us digress to clarify the mentioned problem in the momentum substitution.

Momentum is classically $p = mv = m\frac{dx}{dt}$ and could have been substituted as $p = m\frac{\partial x}{\partial t}$, but Schrödinger chose to substitute it with a space derivative. Assume a mass is moving to the direction of $\vec{r} = \sum_{i=1}^{3} x_i\vec{e}_i$. As it is moving to the same direction all the time

$$\frac{dx_i}{dr} = \frac{x_i}{r} \tag{78}$$

and

$$dr = \sum_{i=1}^{3} \frac{\partial r}{\partial x_i} r = \sum_{i=1}^{3} \frac{x_i}{r} dx_i = \sum_{i=1}^{3} \frac{dx_i}{dr} dx_i. \tag{79}$$

Thus

$$dr^2 = \sum_{i=1}^{3} dx_i^2 \quad \rightarrow \quad \left(\frac{dr}{dt}\right)^2 = \sum_{i=1}^{3} \left(\frac{dx_i}{dt}\right)^2 \tag{80}$$

and

$$p^2 = \sum_{i=1}^{3} p_i^2. \tag{81}$$

But with space derivatives this is different. When the direction of the movement stays the same

$$\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{82}$$

There is a difference between a field equation and a dynamic equation. Consider a mass moving to the direction of the x_1-axis, thus $x_2 = x_3 = 0$ all the time and $r = x_1$. Assume that some property on the x_1-axis decreases as $1/r$, we calculate its second derivative. The dynamic equation has

$$\frac{\partial^2}{\partial r^2}\frac{1}{r} = \frac{\partial^2}{\partial x_1^2}\frac{1}{x_1} = \frac{2}{x_1^3} \tag{83}$$

but the field equation gives zero

$$\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}\frac{1}{r} = 0. \tag{84}$$

The field equation is not calculating along the x_1-axis and the operator is not

$$\frac{\partial^2}{\partial r^2} \tag{85}$$

it is

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}. \tag{86}$$

Schrödinger wrote his equation as one-dimensional, so it actually is a dynamic equation and as there is no predefined coordinate system, the equation means that the momentum substitution should be (85). The Schrödinger equation has a wave function, but it is a one-dimensional wave. There is no suggestion that it is a field, the interpretation of the wave function as a propability distribution came later from Max Born.

Had Dirac understood the momentum substitution as in (85), he could still have discovered antimateria since

$$-\hbar^2 \frac{\partial^2}{\partial r^2} + \mu^2 = \left(-i\hbar\frac{\partial}{\partial r} + \mu\right)\left(i\hbar\frac{\partial}{\partial r} + \mu\right). \tag{87}$$

But instead he followed Wolfgang Pauli in linearization of the equation. Pauli noticed that

$$M = \begin{pmatrix} p_1 & p_2 + ip_3 \\ -p_2 + ip_3 & -p_1 \end{pmatrix} \tag{88}$$

satisfies

$$M^2 = (p_1^2 + p_2^2 + p_3^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{89}$$

Dirac, wanting to make his equation to comply with Special Relativity, decided to have four components in p and for that he needed a 4×4 matric. In order that the square of the matrix is $h = p_0^2 - p_1^2 - p_2^2 - p_3^2 + \mu^2$ he needed a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{90}$$

with 2×2 matrices A, B, C, D. Then

$$M^2 = \begin{pmatrix} A^2 + BC & AB + BD \\ CA + DC & CB + D^2 \end{pmatrix} = h \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} . \tag{91}$$

This requires only that two equations hold

$$D = -B^{-1}AB \quad , \quad C = hB^{-1} - B^{-1}A^2. \tag{92}$$

Setting $A = p_1 1_2$ gives $D = -A$ and $C = (h - p_1^2)B^{-1}$. If

$$B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \tag{93}$$

then

$$C = \frac{h - p_1^2}{b_1 b_4 - b_2 b_3} \begin{pmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{pmatrix} \tag{94}$$

and all that is needed is to have

$$h - p_1^2 = p_0^2 - p_2^2 - p_3^2 + \mu^2 = b_1 b_4 - b_2 b_3. \tag{95}$$

This is easily done if the number b_i are complex. Thus

$$b_1 b_4 = (p_0 + i\mu)(p_0 - i\mu) = p_0^2 + \mu^2 \tag{96}$$

$$b_2 b_3 = (p_2 + ip_3)(p_2 - ip_3) = p_2^2 + p_3^2. \tag{97}$$

Then

$$M = \begin{pmatrix} p_1 & 0 & p_0 + i\mu & p_2 + ip_3 \\ 0 & p_1 & p_2 - ip_3 & p_0 - i\mu \\ p_0 - i\mu & -p_0 - i\mu & -p_1 & 0 \\ -p_2 + ip_3 & p_0 + i\mu & 0 & -p_1 \end{pmatrix} . \tag{98}$$

One set of Dirac matrices can be selected from M by expressing it as four matrices that are multiplied by p_0, p_1, p_2, p_3 and μ. There is no need to explain this trick with a Grassman algebra.

The trick works as a linearization, but the question is what is the sense? A complex 4-space has eight basis vectors and we have here used five of them, in the form of matrices. Why should four momentums, which are orthogonal in the space-time coordinates, be mapped to an 4-dimensional complex space. Also, the placing of p_i is not at all symmetric in M, though we would expect that p_i are completely symmetric.

There are more questions, like: why are there four components in p when in the relativistic kinetic energy formula and the apparent kinetic energy formula from Compton scattering (see [2]) there are only three components in p? And we can ask why the momentum substitution cannot be to a time derivative as a space derivative causes mainly problems.

6. On quantization of scalar gravity

We finally get to the topic of this article. We follow [3] pages 60 and 61 directly. The book quantizes the scalar theory with the Lagrangean

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{99}$$

where

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\nu \phi)(\partial^\nu \phi) - \frac{1}{2}\mu^2 \phi^2. \tag{100}$$

This Lagrangean gives the Euler-Lagrange equation ([3] sets $c = 1$ and we let it be so here)

$$\left(\frac{\partial^2}{\partial x_0^2} - \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} + \mu^2 \right) \phi = 0. \tag{101}$$

This equation is the Klein-Gordon equation

$$\left(\Box + \mu^2 \right) \phi = 0. \tag{102}$$

In the book

$$\mathcal{L}_1 = -\frac{\lambda}{4!}\phi^4 \tag{103}$$

Then $\mathcal{L}_1 = \mathcal{L}_1(\phi)$ and there is a small variable λ that allows developing the following exponential function into a power series

$$\exp\left(i \int \mathcal{L}_1(y) \right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(i \int \mathcal{L}_1(y) \right)^n. \tag{104}$$

We will only take the first two terms. Equation (6.6) in [3] says (after adding the forgotten integral sign)

$$W[J] = \exp\left(i \int \mathcal{L}_1 \left(-i \frac{\delta}{\delta J(x)} \right) \right) W_0[J]. \tag{105}$$

Inserting the power series for the exponent allows calculating $W[J]$

$$W[J] = W_0[J] + i\frac{\lambda}{4!} \int dx \frac{\delta W_0[J]}{\delta J(x)} + O(\mu^4). \tag{106}$$

Equation (6.3) in [3] tells

$$W_0[J] = N \exp\left(-\frac{1}{2}\int dx\ dy\ J(x)\Delta_F(x-y)J(y)\right) \tag{107}$$

where the Feynman propagator $\Delta(x-y)$ is

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2-\mu^2+i\epsilon}. \tag{108}$$

Thus

$$W[J] = W_0[J] - \frac{\lambda}{4!}\int dy \int dx J(x)\Delta_F(x-y)J(y)W_0[J] + O(\lambda^2). \tag{109}$$

Let us change

$$\mathcal{L}_1 = -\mu^2\phi \tag{110}$$

because with this $\mathcal{L}_1$ the Lagrangean is simply

$$\mathcal{L} = \frac{1}{2}(\partial_\nu\phi)(\partial^\nu\phi). \tag{111}$$

Also with this choice $\mathcal{L}_1 = \mathcal{L}_1(\phi)$ and there is a small variable μ^2 that allows developing the exponential function into a power series. Then

$$W[J] = W_0[J] - \mu^2\int dy \int dx J(x)\Delta_F(x-y)J(y)W_0[J] + O(\lambda^2). \tag{112}$$

If the quantization of the book's example is correct, this should also be correct. The Euler-Lagrange equation for $\mathcal{L}$ is

$$\Box\phi = 0. \tag{113}$$

and it has the time independent solution $\phi = 1/r$, Newtonian gravitation field. As is the case with the book's example. we get a quantum correction term to the mass after making renormalization.

The question is whether this has any sense. I do not think it has any sense at all. We can check how [3] derives the equation (6.6) in [3], given here as (105). It does not come from the right side of (63). It comes in the following way. Firstly, (62) is evaluated as

$$W[J] = N'\int \mathcal{D}\phi(x)\exp\ i\hbar^{-1}\int dt\int d^3x\,(\mathcal{L}+J\phi) \tag{114}$$

in (4.25) in [3]. We will skip the logic of going from (62) to (114). Let us write (114) in a simpler form in order to follow the derivation starting at (5.25) in [3] more easily

$$W[J] = \int \mathcal{D}\phi(x)\exp\ i\int dx\,(\mathcal{L}+J\phi)\,. \tag{115}$$

Inserting

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1(\phi) \tag{116}$$

gives

$$W[J] = \int \mathcal{D}\phi(x) \left(\exp\ i\int dx\mathcal{L}_1(\phi)\right)\left(\exp\ i\int dx\,(\mathcal{L}_0 + J\phi)\right). \tag{117}$$

From the definition of functional derivative the authors of [3] get

$$-i\frac{\delta}{\delta J(x)}\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) = \phi(x)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) \tag{118}$$

and from this they conclude that (at least when $\mathcal{L}_1$ is a power of ϕ)

$$\mathcal{L}_1\left(-i\frac{\delta}{\delta J(x)}\right)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) = \mathcal{L}_1(\phi)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) \tag{119}$$

and more generally, the nth term in an exponential

$$\frac{1}{n!}\mathcal{L}_1\left(\left(-i\frac{\delta}{\delta J(x)}\right)^n\right)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) \tag{120}$$

$$= \frac{1}{n!}\mathcal{L}_1((\phi)^n)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi)\,. \tag{121}$$

Summing the terms into an exponential gives

$$W[J] = \int \mathcal{D}\phi(x)\exp\ i\int dx\mathcal{L}_1\left(-i\frac{\delta}{\delta J(x)}\right)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi)\,. \tag{122}$$

Finally the authors of [3] notice that the operator $\frac{\delta}{\delta J(x)}$ does not depend on ϕ and can be moved outside the path integral yielding (105), equation (6.6) in [3]

$$W[J] = \exp\ i\int dx\mathcal{L}_1\left(-i\frac{\delta}{\delta J(x)}\right)\int \mathcal{D}\phi(x)\exp\ i\int dx\,(\mathcal{L}_0 + J\phi) \tag{123}$$

$$W[J] = \exp\ i\int dx\mathcal{L}_1\left(-i\frac{\delta}{\delta J(x)}\right)W_0[J]. \tag{124}$$

If this calculation is correct, then (113) quantized Newton's and Nordström's gravity, but I do not think this method is correct.

7. Conclusions

The reason for investigating Feynman's path integral quantization method was to see if it could be used in quantization of gravitation because the gravitational field must be scalar or the local speed of light in vacuum is not *c*. My small study showed that both the canonical quantization and Feynman's path integral quantization have several problems which might be called serious and unfixable errors, i.e., if the problems are fixed, there is no quantization method left. It seems that the most reasonable understanding of what is quantization was in the old quantum theory. At that time quantization meant that energy and angular momentum have (in a certain sense) discrete values.

7. References

[1] Jormakka, J., Three fatal errors in quantum physics, ResearchGate, 2025.

[2] Jormakka, J., Apparent momentum in Compton scattering, ResearchGate, 2025.

[3] Bailin, D., Love, A., Introduction fo Gauge Field Theory, Graduate student series in physics, Adam Hilger, Bristol and Boston, Univ. Sussex Press, 1986.