The Electron-Positron Pair Creation in Magnetic Field

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Abstract

The probability of the emission of the electron-positron pairs is calculated from the mass operator formalism introduced by Milton at al. (1981). This formalism is used to estimate the rate of the e^-, e^+ pair production by virtual synchrotron photons. The rate is governed by $eH \ll m^2, E \gg m$ with $\exp\left[-4\sqrt{3}(m^2/eH)(m/E)\right]$, and therefore the e^-e^+ process is beyond the possibility of experimental observation.

1 Introduction

Pair production is the creation of a subatomic particle and its antiparticle from a neutral boson, including creation of the electron and a positron, a muon and an antimuon, or, a proton and an antiproton. Pair production often refers to a photon creating an electron-positron pair near a nucleus. From energy conservation, during the process of pair production follows that incoming energy of the photon must be above a threshold of at least the total rest mass energy of the two particles created. Conservation of energy and momentum are the principal constraints on the process. All other conserved quantum numbers must sum to zero, or, they have opposite values of each other.

For high energy photon (MeV scale and higher), pair production is the dominant mode of photon interaction with matter. These interactions were first observed by Patrick Blackett leading to the 1948 Nobel Prize in Physics. If the photon is near an atomic nucleus N, the energy of a photon can be converted into an electron-positron pair: $\gamma + N(Ze) \to N(Ze) + e^- + e^+$.

The photon energy is converted to particle mass in accordance with the Einstein equation, $E = mc^2$, where E is energy, m is mass and c is the speed of light. The photon must have higher energy than the sum of the rest mass energies of an electron and positron (2 x 511 keV = 1.22 MeV), resulting in a photon-wavelength of 1.2132 picometer) for the production to occur. Thus, pair production does not occur in medical X-ray imaging because these X-rays only contain energy $E \sim 150$ keV. The photon must be near a nucleus in order to satisfy conservation of momentum, as an electron-positron pair produced in free space cannot satisfy conservation of both energy and momentum. Because of this, when pair production occurs, the atomic nucleus receives some recoil. The reverse of this process is electron-positron annihilation.

It is well known the electron-positron pair production by virtual synchrotron radiation is crucial phenomenon. Although the effect certainly occurs in principle, it is suppressed to an exponentially small rate under all conceivable circumstances. The reason is not only the cost of production of the energy of the pair, but more essentially the difficulty of producing the virtual time-like photon.

2 Proper time mass operator method

The starting point of our discussion is the proper-time mass-operator approach of Schwinger (1973; 1951) as applied to synchrotron radiation by particles with spin 1/2 (Schwinger, 1989, chap. 5, sec. 6; Tsai et al., , 1973; Milton 1980).

The relation between $\mathcal{M}$ and the decay rate γ is (Milton et al., 1981)

$$\gamma = -\frac{2m}{E}\mathrm{Im}\mathcal{M}, \tag{1}$$

where the imaginary part reflects the desired final state, including the e^-e^+ pair. Here E is the synchrotron-electron energy.The mass operator

$$\mathrm{Im}\mathcal{M} = ie^2 \int \frac{(dk)}{(2\pi)^4}\gamma^\mu D'(k)\frac{1}{\gamma(\pi-k)+m-i\varepsilon}\gamma_\mu \tag{2}$$

differs from that for ordinary synchrotron radiation in that in place of the free photon propagator we have its contribution from the e^-e^+ state:

$$D'(k) = \int_{(2m)^2}^{\infty} dM^2 \frac{a(M^2)}{k^2+M^2-i\varepsilon}, \tag{3}$$

where

$$a(M^2) = \frac{\alpha}{3\pi}\frac{1}{M^2}\left(1+\frac{2m^2}{M^2}\right)\left(1-\frac{4m^2}{M^2}\right)^{1/2}. \tag{4}$$

Here we are neglecting small effects due to the binding of the e^-e^+ pair to the magnetic field. (This is discussed in the last section). It is evident that the desired mass operator is simply obtained from $\mathcal{M}'$, the mass operator for synchrotron radiation of a photon with mass M:

$$\mathcal{M} = \int dM^2 a(M^2)\mathcal{M}'(M). \tag{5}$$

The modification of a photon mass following from M is (Tsai et al., 1973; Milton 1980):

$$\mathcal{M}'(M) = \frac{\alpha}{2\pi}m\int_0^\infty \frac{dx}{x}\int_0^1 du \quad \times$$

$$\exp\left[-i\frac{m^2}{eH}x\left(u+\frac{1-u}{u}\frac{M^2}{m^2}\right)\right] \quad \times \quad \{A\times(B+C-(1+u))\}, \tag{6}$$

where

$$A = \Delta^{-1/2} \exp\left(-i[\beta - (1-u)x]\frac{E^2 - m^2}{eH}\right) \tag{6a}$$

$$B = \cos(\beta - x) - i\zeta' \frac{E}{m}\sin(\beta - x) + u\cos(\beta + x) - i\zeta'\frac{E}{m}\sin(\beta + x). \tag{6b}$$

$$C = (1-u)\frac{E^2 - m^2}{m^2}\left(\frac{1-u}{\Delta}\cos(\beta - x) + \frac{u}{\Delta}\frac{\sin x}{x}\cos\beta - \cos(\beta + x)\right), \tag{6c}$$

where

$$\Delta = (1-u)^2 + 2u(1-u)\frac{\sin 2x}{2x} + u^2\left(\frac{\sin x}{x}\right)^2 \tag{7}$$

$$e^{-i(\beta - x)} = \left(\frac{D_+}{D_-}\right)^{1/2}, \tag{8a}$$

$$D_\pm = 1 - u + u\frac{\sin x}{x}e^{\pm ix}, \tag{8b}$$

$$\Delta = D_+ D_- \tag{8c}$$

and $\zeta' = \pm 1$ is the eigenvalue of $\gamma^0\sigma_3$. (The 3 axis is the direction of $\mathbf{H}$) .

Equation (6) is exact in H, but formidable. We therefore approximate it in the interesting terrestrial limit of weak fields and high energies (for the electron, $m^2/e = 4.4 \times 10^{13} G$)

$$\frac{eH}{m^2} \ll 1, \frac{e}{m} \gg 1 \tag{9}$$

Then the controlling exponent is ($x \ll 1$)

$$\Theta \approx \frac{m^2}{eH}x\left[u + \frac{1-u}{u}\frac{M^2}{m^2} + \frac{1}{3}u(1-u)^2x^2\frac{E^2}{m^2}\right] = \frac{3}{2}\xi\left(y + \frac{1}{3}y^3\right) \tag{10}$$

with

$$\xi = \frac{u}{1-u}\frac{1}{\Upsilon}\left(1 + \frac{1-u}{u^2}\frac{M^2}{m^2}\right)^{3/2}, \tag{11a}$$

$$y = (1-u)\frac{E}{m}x\left(1 + \frac{1-u}{u^2}\frac{M^2}{m^2}\right)^{-1/2} \tag{11b}$$

The important values of x are set by $x \sim (m/E) \times (1-u)^{-1}$. Appearing here is the parameter which determines the onset of quantum phenomena (Sokolov et al., 1953)

$$\Upsilon = \frac{3}{2}\frac{eH}{m^2}\frac{E}{m} \tag{12}$$

The integrand in (6) is now expanded as in eq. (112) by Schwinger (1989), and we find in eq. (118) by Schwinger (1989)

$$\mathrm{Im}'\mathcal{M} = -\frac{\alpha}{2\pi}\frac{1}{\sqrt{3}}m\int_0^1 du\{(1+u)\int_t^\infty K_{5/3}(\eta)d\eta + E + F\}, \tag{13}$$

where

$$E = \left[\frac{2u(u-2/3)}{1-u} + \frac{2(1-2/3u)}{u^2}\frac{M^2}{m^2}\right] K_{2/3}(\xi) \tag{13a}$$

$$F = \zeta' u \left(1 + \frac{(1-u)}{u^2}\frac{M^2}{m^2}\right)^{1/2} K_{1/3}(\xi) \tag{13b}$$

The situation is becoming much more transparent, since for nearly any conceivable experimental arrangement Υ is at best a few percent. (For $E = 1TeV$ and $H = 10MG, \Upsilon = 0.75$, but this is rather extreme. So to get an indication of the behavior, we use the asymptotic behavior

$$K_\nu(\xi) \sim \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}, \xi \to \infty \tag{14}$$

to approximate eq. (13) by (for an unpolarized electron)

$$\mathrm{Im}\mathcal{M}' \sim -\frac{\alpha}{2\pi}\frac{m}{\sqrt{3}}\int_0^1 du \left\{\frac{1}{3} - u + \frac{2}{3}\frac{1}{1-u} + \frac{2(1-2u/3)}{u^2}\frac{M^2}{m^2}\right\}\left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}. \tag{15}$$

To keep things simple, we make the further reasonable approximation that $M^2 \gg m^2$, in which case the minimum of $\xi(u)$ occurs at

$$u_0 = 1 - \frac{2m^2}{M^2}, \tag{16}$$

where ξ has the value

$$\xi_0 = \xi(u_0) = \frac{3^{3/2}}{2}\frac{1}{\Upsilon}\left(\frac{M^2}{m^2}\right) \gg 1 \tag{17}$$

At this same point ξ has as its second derivative

$$\xi_0'' = \frac{3^{1/2}}{2}\frac{1}{\Upsilon}\left(\frac{M^2}{2m^2}\right)^3. \tag{18}$$

Evaluating eq. (15) by an approximate Gaussian integration, and inserting the result in eq. (5) and eq. (1), we find

$$\gamma_{e^+e^-} \sim -\frac{\alpha}{3\pi}\frac{m^2}{E}\int_{(2m)^2}^\infty \frac{dM^2}{M^2}\frac{4}{3\sqrt{3}}\alpha\frac{m^2}{M^2}\Upsilon e^{-\xi_0} =$$

$$\frac{2}{3}\frac{\alpha^2}{\pi}\frac{m^2}{E}\int_{\zeta_0}^\infty \frac{d\zeta}{\zeta^2}e^{-\zeta}, \quad \zeta_0 = \frac{6\sqrt{3}}{\Upsilon} \tag{19}$$

or,

$$\gamma_{e^+e^-} \sim -\frac{\alpha^2}{162\pi}\frac{m^2}{E}\Upsilon^2 e^{-6\sqrt{3}/\Upsilon}. \tag{20}$$

The conclusion seems inescapable: Even if we could arrange for $\Upsilon \sim 1$, the pair production rate remains exponentially small, completely beyond experimental detection.

3 Vacuum polarization by Tsai

It is quite plausible that the effect of the interaction of the produced e^-e^+ pair with the magnetic field is small. Here we prove this, using the results of Tsai (1974). There, vacuum polarization in the external field is calculated,

$$\mathcal{M}_{\mu\nu} = -ie^2 tr \int \frac{(dp)}{(2\pi)^4} \gamma_\mu \mathcal{G}(p) \gamma_\nu \mathcal{G}(p-k) + c.t. \tag{21}$$

where $\mathcal{G}(p)$ is the exact (in H) electron propagation function, and c.t. stands for a local contact term. The result is

$$\mathcal{M}_{\mu\nu} = \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} \int_{-1}^1 \frac{dv}{2} \left(e^{-is\phi} [PN_0 - QN_1 + RN_2] + c.t. \right), \tag{22}$$

where

$$P = \left(g_{\mu\nu} k^2 - k_\mu k_\nu \right) \tag{22a}$$

$$Q = [g^{||}_{\mu\nu} k^2_{||} - (k_\mu)_{||}(k_\nu)_{||}] \tag{22b}$$

$$R = [g^{\perp}_{\mu\nu} k^2_{\perp} - (k_\mu)_{\perp}(k_\nu)_{\perp}] \tag{22c}$$

and N_1, N_2, N_0 are functions of $zseH$ and v given in published articles (Tsai, 1974a; Tsai, 1974b; Tsai et al, 1974), $||$, $\perp$ refer to the 0-3, and 1-2 subspaces, respectively,

$$c.t. = e^{-ism^2}(1-v^2)(g_{\mu\nu}k^2 - k_\mu k_\nu) \tag{23}$$

and

$$\varphi = m^2 + \frac{1}{4}(1-v^2)k^2_{||} + \frac{\cos zv - \cos sz}{2z \sin z} k^2_\perp. \tag{24}$$

When $H \to 0, z \to 0$,

$$N_0 \to 1 - v^2, \quad N_{1,2} \to 0, \tag{25}$$

and

$$\mathcal{M}_{\mu\nu} \to (g_{\mu\nu}k^2 - k_\mu k_\nu) \frac{\alpha}{2\pi} \int_0^\infty \frac{ds}{s} \int_{-1}^1 \frac{dv}{2} (1-v^2) e^{-ism^2} \left\{ e^{-is\frac{1}{4}(1-v^2)k^2} - 1 \right\}, \tag{26}$$

the familiar $O(\alpha)$ photon polarization tensor, equivalent to eq. (3). The question is: How large are the magnetic field corrections to this? The controlling factor is the exponential; for small fields we can. expand the exponent in powers of $z \sim eH/m^2$.

$$s\phi \sim t(y + \frac{1}{3}y^3), \tag{27}$$

where

$$t = \frac{4[m^2 + \frac{1}{4}(1-v^2)k^2]^{3/2}}{eH\,|\mathbf{k}_\perp|\,(1-v^2)} \tag{28}$$

and

$$y = \frac{z}{4}\,|\mathbf{k}_\perp|\,(1-v^2)[m^2 + \frac{1}{4}(1-v^2)k^2]^{-1/2}. \tag{29}$$

Real photon ($k^2 = 0$) pair creation can be significant because t can be small for high-energy photons. (Tsai et al., 1974). Here, however, $k^2 \sim 4m^2$ and $|\mathbf{k}_\perp| \sim 2m$ (the virtual synchrotron photon cannot possibly be of high energy), so

$$t \sim \frac{m^2}{eH} \gg 1, \tag{30}$$

leading to exponentially small corrections to eq. (26). In other wards, for any possible terrestrial field, $y \ll 1$, so only the first term in eq. (27) is significant, which leads to eq. (26).

4 Discussion

The electron-positron pair production by virtual synchrotron radiation is the crucial phenomenon. Although the effect certainly occurs in principle, it is suppressed to an exponentially small rate under all conceivable circumstances. The reason is not only the cost of production of the energy of the pair, but more essentially the difficulty of producing the virtual time-like photon. The standard explanation of the pair production differs from the new Schwinger approach. It can be found in the famous monograph and textbook on quantum electrodynamics (Akhiezer et al., 1965; Berestetskii et al., 1982). The introduction to the Schwinger source theory is presented in the well-known monographs (Dittrich, 1978; Schwinger, 1969; 1970; 1973; 1989).

The purpose of this paper was to present a complete and explicit result by using a different approach that is an extension of the simple and transparent method proposed by Tsai and Erber (1974) to calculate the photon mass operator in an external homogeneous magnetic field.

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