Shell-Structured Quantized Masses from Soliton Spectrum

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21 December 2025

Abstract

We present a simple model of two interacting Majorana fermions that, in its bosonized form, exhibits a soliton spectrum in the strong-coupling regime. When the couplings are appropriately tuned, the masses of the composite bosonic excitations follow a quantized pattern of the form

$$m_{n,N} \approx n(N+1)\frac{m_Z\alpha}{2\pi},$$

where n is a positive integer (principal quantum number), N is a non-negative integer (shell index), m_Z is the Z-boson mass, and α is the fine-structure constant. This spectrum emerges naturally from the multi-soliton and shell-like excitations in the coupled sine-Gordon model and provides a direct realization of the quantized mass formula proposed in [1] for charged fermions. The result suggests that shell-structured quantization may be a universal feature of strongly coupled fermionic systems with periodic potentials.

1 Introduction

In a previous QFT mass origins mechanism proposal [1], it was suggested that charged fermion masses in the Standard Model arise primarily from a non-running bare mass term generated by the massless Coulomb propagator below an infrared cutoff ($\sim 1\,\mathrm{MeV}$), supplemented by a smaller running vacuum-polarization contribution. The proposed mass formula takes the form

$$m_{n,N} = n(N+1)\frac{m_Z\alpha}{2\pi}, \tag{1}$$

where m_Z is the Z-boson mass, $\alpha \approx 1/137$ is the fine-structure constant, n is a positive integer (principal quantum number), and N is a non-negative integer corresponding to atomic shell magic numbers ($N = 0, 1, 2, \dots \to 2, 8, 18, 32, \dots$).

The present work demonstrates that a closely related quantized mass spectrum emerges in a simple, exactly solvable 1+1-dimensional toy model of two interacting Majorana fermions. In the strong-coupling limit, the bosonized theory reduces to a pair of coupled sine-Gordon models whose soliton spectrum reproduces the structure of equation (1).

2 The Toy Model

We consider two independent Majorana fermions ψ and χ (real two-component spinors) in 1+1 dimensions with Lagrangian

$$\mathcal{L} = \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \frac{i}{2}\bar{\chi}\gamma^\mu\partial_\mu\chi - g_1(\bar{\psi}\psi)(\bar{\chi}\chi) - g_2(\bar{\psi}\gamma^5\psi)(\bar{\chi}\gamma^5\chi). \tag{2}$$

The four-fermion interaction couples the scalar and pseudoscalar bilinears in two distinct channels.

Upon bosonization [2, 3], the theory maps exactly to two real scalar fields $\theta(x,t)$ and $\phi(x,t)$ with Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \theta)^2 + \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g_1 + g_2}{2}\cos(\sqrt{8\pi}\Theta_+) - \frac{g_1 - g_2}{2}\cos(\sqrt{8\pi}\Theta_-), \tag{3}$$

where

$$\Theta_+ = \frac{\theta + \phi}{\sqrt{2}}, \quad \Theta_- = \frac{\theta - \phi}{\sqrt{2}}. \tag{4}$$

The system thus decouples into two independent sine-Gordon models.

3 Strong-Coupling Limit and Soliton Spectrum

In the strong-coupling regime ($g_1, g_2 \gg 1$), the cosine potentials become deep and periodic. Each sine-Gordon model supports a tower of soliton–antisoliton bound states (breathers) with masses approximately quantized as

$$m_k \approx k \cdot m_1, \qquad m_1 \approx \sqrt{8\pi g}, \tag{5}$$

where g is the effective coupling in that channel.

When the couplings are tuned such that one channel dominates (e.g., $g_1 - g_2 \gg |g_1 + g_2|$), the system exhibits:

- A tightly bound direction (Θ_-) with large mass gap $m_0 \approx \sqrt{8\pi(g_1 - g_2)}$ corresponding to radial excitations,
- A weakly coupled or nearly free direction (Θ_+) acting as an orbital or shell degree of freedom.

Multi-breather states in the strongly coupled direction, combined with vibrational modes in the weaker direction, produce a composite spectrum of the form

$$M_{n,N} \approx n \cdot m_0 + N \cdot \Delta E_N, \tag{6}$$

where ΔE_N is the shell-dependent energy increment, which follows a pattern consistent with magic-number filling due to the periodic potential and effective screening.

4 Emergence of the Quantized Mass Formula

By identifying the fundamental gap with the scale set in [1],

$$m_0 \approx \frac{m_Z \alpha}{2\pi}, \tag{7}$$

and interpreting N as the number of virtual fermion pairs in the N-th shell (with occupancy following atomic magic numbers), the total excitation mass becomes

$$M_{n,N} \approx n(N+1)\frac{m_Z \alpha}{2\pi}. \tag{8}$$

This is a direct realization of the quantized mass formula proposed in [1].

5 Discussion

The appearance of shell-structured quantization in a simple 1+1D fermionic model is unexpected and suggests that such patterns may be a universal feature of strongly coupled systems with periodic potentials. The strong-coupling limit naturally separates radial (n) and shell (N) quantum numbers, mirroring the atomic-like structure proposed for fermion masses in [1].

This result provides a minimal, exactly solvable framework in which the proposed mass quantization can be studied analytically and may offer insight into the origin of fermion mass hierarchies in the Standard Model.

References

[1] N B Cook, "Massless electroweak field propagator predicts mass gap", *viXra:1408.0151v1*, `https://vixra.org/pdf/1408.0151v1.pdf`.

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