# Some triconnected graphs and their families without Hamiltonian cycles

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1978-1979

Further, cycle is always elementary cycle, without repetitions of vertices. Hamiltonian cycle is called H-cycle, Graph with desired connectivity and without H-cycle is called non-H-graph. Number of vertices of graph $G$ is $|G|$ or $n$.

1°

If we are content with cyclic triconnectivity, from each non-H-graph there that has vertex $x$ of third order, and its incident edges are $r, s, t$, we get non-H graph $\tilde{G}$ with greater $n$ with transformation of fig. 1,

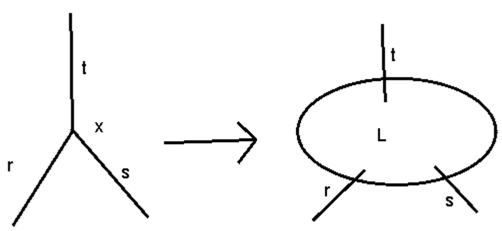



Figure 1:

where $L$ - arbitrary triconnected graph, where edges $r, s, t$ are incident with 3 different vertices.
Really, if $\tilde{G}$ would have H-cycle $\tilde{C}$, it contained just 2 edges from $r, s, t$.
Contracting $L$ back to one vertex, from $\tilde{C}$ we would get in $G$ not existing H-cycle.
Simplest case: $L$ is triangle, then

$$|\tilde{G}| = |G| + 2 \ .$$

Using, e.g. Petersen graph with such $L$, we can get non-H-graphs with $n = 12, 14, 16, ...$

[*]The translation is made from the manuscript (Daži trīssakarīgi grafi un to saimes bez Hamiltona cikliem) http://dspace.lu.lv/dspace/handle/7/2190

[†]Translated from Latvian by Dainis Zeps

2°

Cyclic 4-connected cubic non-plane non-H-graphs are different Adelson-Velsky and Titov constructed cubic graphs [1], which are 4-chromatic against edges, or otherwise - with chromatic class 4.
Really, according theorem of Vizing, cubic graphs have chromatic class 3 or 4. If H-cycle $C$ is there, because of $n$ being always even, edges of $C$ may be colored properly in 2 colors, and with the third – full matching of the left part. Because of this, cubic graphs with chromatic class 4 are non-H-graphs, if only desired relationship is provided, that in [1] is done.
Unplanarity is the consequence of recently proven 4-color hypothesis, if we can trust the literature.

3°

The generalization of Zeps construction allows to get non-H-graph with arbitrary large connectivity and cyclical connectivity. In special cases (see following points) it is possible to obtain plane graphs; general necessary and sufficient conditions of planity seem to be complicate.
The construction itself: in order to get $\nu$-connectivity we take graph $G_0$ with $\beta \geq \nu$ specially marked vertices $b_j, j = 1, 2, ...\beta$. Other vertices and edges $\in G$ may be or not; $G_0$ may, e.g., consist only from isolated vertices $b_j$.
We take $\alpha > \beta$ graphs $G_1, G_2, ..., G_\alpha$. Vertices of these graphs we connect via edges with different $b_j$ to get desired relationship; such edges we will call connecting edges of the obtained graph $G$.
Even if all possible connecting edges are drawn, the obtained graph is non-H-graph. Really, if there were H-cycle $C$, it contained at least 2 connecting edges that would be incident to each vertex of $G_i$, all together $2\alpha > 2\beta$ such edges, each of which would be incident to some $b_j$ - but with this last property may be just $2\beta$ edges of cycle $C$ - contradiction.
In the same way we conclude that in case $\alpha > \beta + 1$ graph $G$ does not have Hamiltonian path. With sufficiently large $\alpha, \beta$ and $\alpha - \beta$ it is possible to obtain graphs with arbitrary large connectivity, for which longest elementary path's relation of $l$ against $n$ is arbitrary small.
Notable special case: $b_j$ are isolated vertices, similarly each $G_i$ is isolated vertex $a_i$. Then $G$ is bigraph, $K_{\alpha,\beta}$ partial graph in sense of Berge (i.e., $G$ has all $K_{\alpha,\beta}$ vertices, not necessarily all edges). For triconnectivity we may start with $K_{4,3}$, and obtain greater bigraphs, non-H-graphs, with growing $\alpha$ and $\beta < \alpha$ by adding new $a_i$ or $b_j$, drawing new appropriate connecting edges (i.e., they are the only edges of $G$), and (or) resetting already existing edges, in order to keep triconnectivity.
In general case obtained graphs $G$, similarly as $K_{4,3}$, will be non-plane graphs.
Let us notify one more special case that we are going to use immediately: connected bigraph with odd number of vertices is always non-H-graph. Such graph has only one proper coloring of vertices in two colors. By taking largest set of one color as $\{a_i\}$, second one as $\{b_j\}$, we will have $\alpha \geq \beta + 1$.

4°

Triconnected plane bichromatic non-H-graphs we get taking dual graphs $G$ of triconnected, plane, biquadratic unigraph $G'$ with odd number of vertices $n'$. As it was told recently in seminar, such $G'$ divide plane into $a$ regions, $n = n' + 2$. This number $a$ is just $n$ of the dual graph, that followingly is odd along with $n'$.
As we have seen, simplest suitable $G'$ is with 9 vertices, thus we will get $G$ with $n = 11$, fig.2.

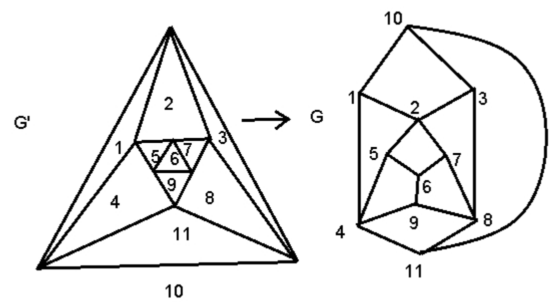
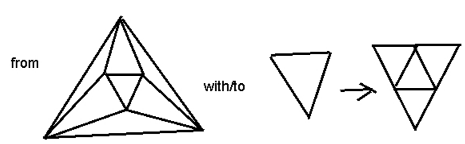



Figure 2:

In the graph $G$, vertices with odd numbers constitute $\{a_i\}$ with $\alpha = 6$, other - $\{b_j\}$ with $\beta = 5$.
Further $G'$ with augmented odd $n'$ we will get by adding:
1) in the graph with odd $n'$ - triangle in some of regions, in this way we get in fig.2 $G'$




$G'$ with $n' = 8$ will give non-isomorphic $G'$ with $n' = 11$, fig.3. and so on.
2) In the graph $G'$ with odd $n'$ $-$ square area in one area (with $\geq 4$ vertices), two triangles (that always possible) and so on.
Surely,it is more convenient to translate these operations into terms of graph $G$ and to work only with them.
In this point considered non-H-graphs has square areas, thus, their cyclic connectivity 4.

5°

If we want to obtain plane non-H-graphs with maximal possible cyclical connectivity 5, then, possibly, smallest $n$ will give non-existence principle of isobar partition of areas[2], especially for cubic graphs. To this is

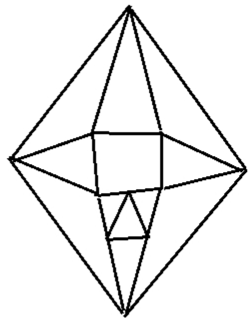


Figure 3:

pointing, e.g., [3]. where, after catalogue being built, it was found that graph [2, fig.3] is plane cubic cyclic 5-connected non-H-graph with minimal possible number of areas. The cubic non-H-graphs, for which this principle don't give ...and which are mentioned in [4] are cyclic 3 and 4 connected and with good $n$: 92, 200 and so on.
One of the consequences of the principle: plane graph is non-H-graph if number of vertices of one is $n_0 \neq 2(mod\, 3)$. Going out from such graph, we get other one with greater $n$, if we replace some elementary cycle $C$ (border of area or not) with ring of pentagons, that gives new $2\nu$ areas of pentagons, keeping all old ones. Exemli gratia, if $C$ is hexagon, replacement is shown on fig.4.

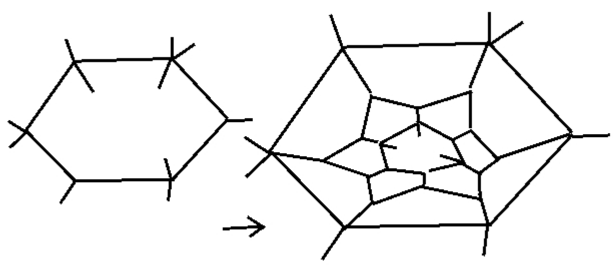


Figure 4:

As you can see, by replacing cycle of length $\nu$ (and always $\nu \geq 5$ for cyclically 5-connected graphs), $3\nu$ new vertices appear - the growth of $n$ is larger than in other considered points.
As source graphs may be taken graphs from [2] (only hexagons in graph from fig.3 should be left together, they can't be replaced); it is not hard to construct other source graphs with $n = 30 - 40$, e.g.
(Subscription: E. Grinbergs, 23.5.78)

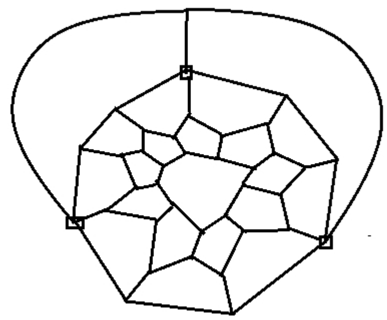


Figure 5: $n = 4 * 9 + 1 = 37$

# References


[1] G.M.Adelson-Velski, V.K.Titov. About edge 4-chromatic cubic graphs, Voprosi kibernetiki. Trudi seminara po kombinatornoi matematike, M. izd. AN SSSR, 1973, 5.-14.

[2] E.Grinbergs, About plane homogeneous graphs degree three without Hamiltonian Cycles, Latv. Jezhegodnik, 4, Riga, 1968, 51.-58.

[3] J. Zaks, Non-Hamitonian non-Grinbergian graphs, Discrete Math. 17, 1977, 317-321.

[4] G.B. Faulkeri, H.H. Younger, Non-hamitonian cubic planar maps, Discrete Math. 7, 1974,67-74, it would be good to order copy, that I don't have - article, as seems, mentioned in RZh Mat 1974 8 V 331)


Supplement to ④

In the biquadratic plane graph it is possible to increase directly number of vertices by 2, transforming triangle (that always is present) to 3 triangles:

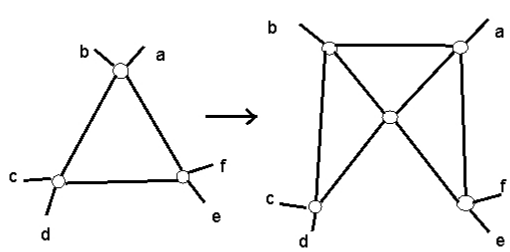



Figure 6: ⓓ

Correspondingly, in the dual graph we transform vertex of degree 3 (beneath - 2) into 3 vertices,

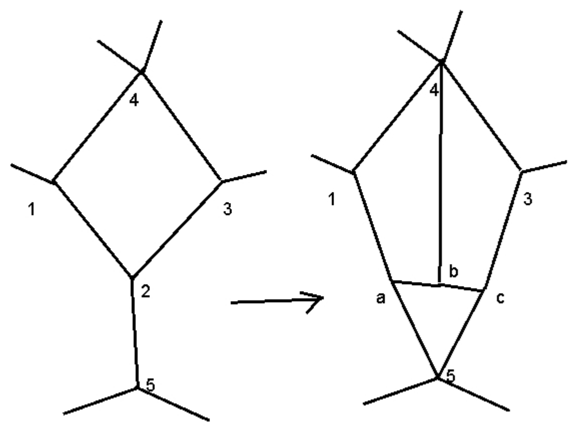



Figure 7: (β)

otherwise, one square into 3 squares; number of vertices of all touching areas doesn't change, similarly degrees of vertices 1 or 3, $a, b, c$ are of degree 3, 4 and 5 increase by 1. If 1, 2, 3, 4 had in correspondence proper coloring with last odd vertex number $n$, then we may put $a = 2, b = n + 2, c = n + 1$, new proper numbering with little changes in the matrix of graph relations, or list.
Analogously, if vertex with odd number is changed. Graph
with $n = 11$ has vertices of degree 3 of two types: 2 and 10, that are not neighbors with vertex of degree 4; vertex of odd number, that is neighbor with vertex of degree 4.
Transforming vertices of degree 3 either of one type or other, we get both non-isomorphic graphs with $n = 13$, that were possible to obtain as dual ones to both biquadratic ones with $n = 11$; both last are possible to receive with transformation $\alpha$ from
that has triangle of two types.
(Subscription: Emanuels Grinbergs, 31.5.78.)

**More about Hamiltonian cycle,**
maybe useful such facts either

(1°) Hipohamiltonian graphs

These are graphs that has no H cycles, but by removing any vertex we get graph with H cycle there, in the book:
R. Busaker, T Saati, Endliche Graphen und Netzen, 63.-67. pp. in a long way show that such graph with minimal $n = 10$ is Petersen graph (not naming it by name and drawing in an awkward way, fig.3.14)
Hipohamiltonian are graphs of such family with $4\nu$ vertices, $\nu$ odd $\geq 5$: we take Möbius ladder with
$\nu$ steps ( in fig. $\nu = 5$, without vertex ($a$) but an edge crossing), the middle points of steps are united in cyclic order with vertices of $\nu$-gon.

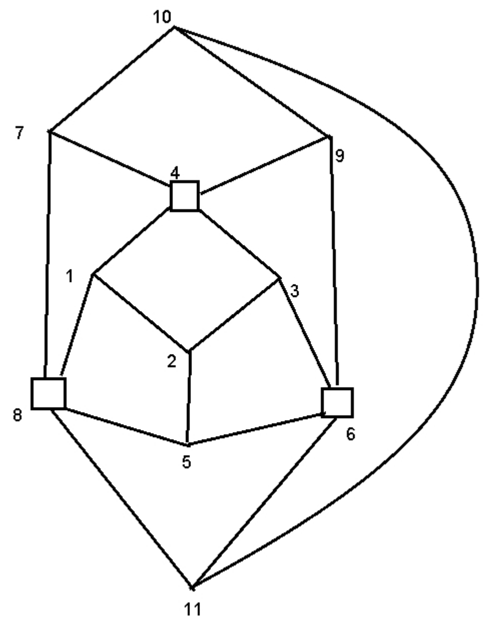
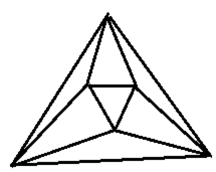



Figure 8:



② 3-H graphs

Since in cubic graph via each edge pass even number of H cycles (Berge, 210 pp, with not too good proof), then minimal possible number $> 0$ of these cycles is 3. Cubic graphs with just 3 cycle we call 3-H-graphs. The following construction of graphs is given, as it seems, by König. The source graph, evidently, is 3-H-graph. The edges of it we will call tracks. New graphs we get by taking in arbitrary number pairs of vertices on both tracks, uniting both with edges, e.g.
Starting to go along both tracks simultaneously from the right side, we may go through vertices and to come until left side, similarly as in picture. Having 3 choices to start, we get just 3 H cycles. Both other passings:
via each edge passing just 2 H cycles.
(Subscr.: E. Gr.)

Some plane graphs
without Hamiltonian cycle

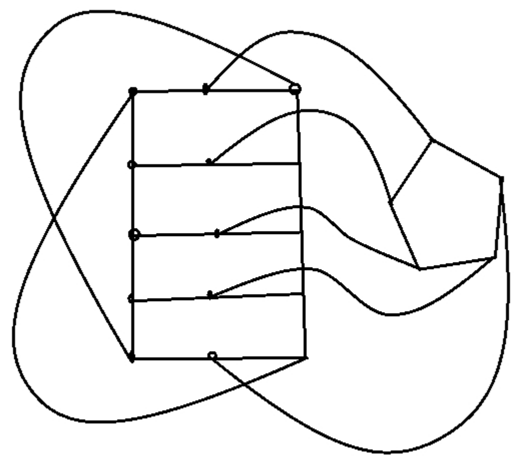


Figure 9:

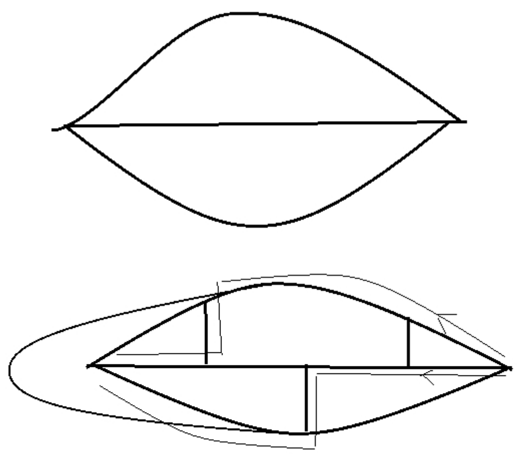


Figure 10:

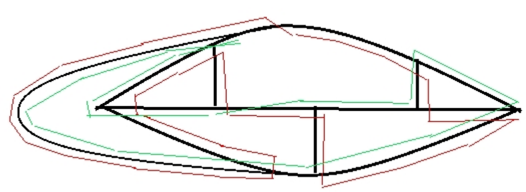


Figure 11:

$1^o$

Seems, smallest cubical graph (Lederberg 1966, Busak 1967) - cyclically triconnected, $n = 38$

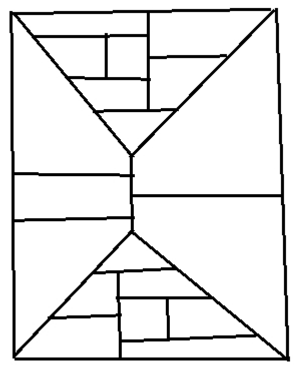


Figure 12:

$2^o$

Up to now, smallest graphs with non suitable isobar partition or without it. Excluding notified areas, all other are pentagons. With square are designated vertices of degree 4, all other of degree 3. All graphs cyclically 5-connected.

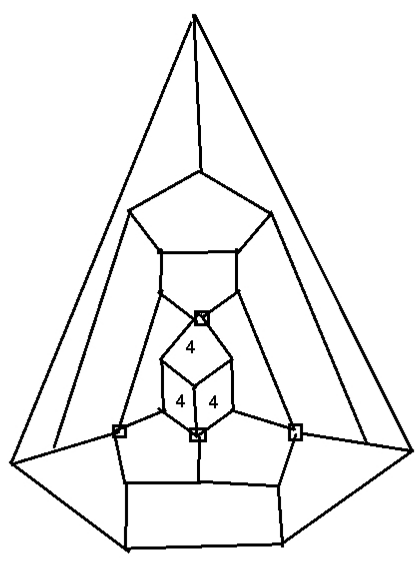



Figure 13: $n = 26$

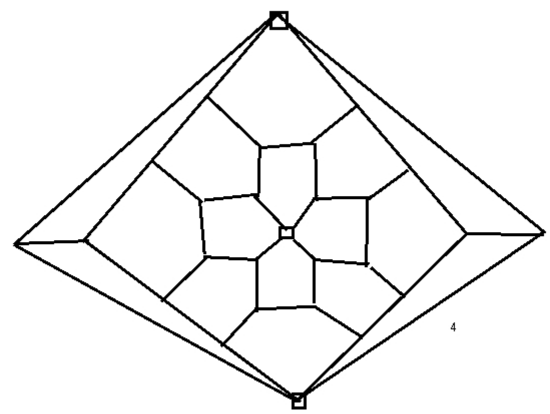



Figure 14: $n = 24$

Outer area is square.
For all these graphs,

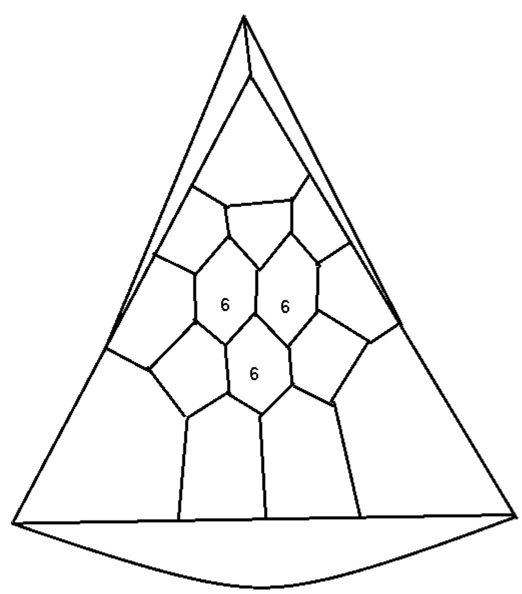



Figure 15: $n = 32$

for any two neighbor vertices $x, y$ hold $h(x, y) = 0$. For some examined pairs of neighbors, on border of area $h(x, y)$ was much greater than 1, even after putting degree two vertex on outgoing edge.
(Subscript. Emanuels Grinbergs 4.5.79.)

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