N-ary anticommutators and Generalized Clifford Algebras in Finsler and Spectral Geometry

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December 2025

Abstract

It is shown that a careful study of the simplest family of generalized Clifford algebras (GCAs) associated with the N-th root of unity in d-dimensions leads to the following generalized anti-commutator with N entries $\{e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_N}\} = e_{i_1} e_{i_2} e_{i_3} \ldots e_{i_N} + permutations = N! \eta_{i_1 i_2 \ldots i_N} e$, where e is the unit element and all the $N!$ terms of the permutations appear with the same positive sign. The components of the rank-N metric are $\eta_{i_1 i_2 \ldots i_N} = 1$, iff $i_1 = i_2 = i_3 \ldots = i_N$, and 0 otherwise. The range of the indices $i_1, i_2, \ldots i_N$ is $1, 2, \ldots, d$. We proceed to explore the N-th norm extensions of the quadratic norm and write down a generalized Finsler-like arc-length based on the rank-N metric $g_{\mu_1 \mu_2 \ldots \mu_N}$. We continue by constructing the different expressions of the Dirac operators associated with the (Generalized) Clifford Spaces corresponding to these GCA's. Dirac operators are essential in the study of Spectral Geometry in Noncommutative Geometry after imposing the correspondence between the geodesic distance and the inverse of the Dirac operator (a fermion propagator). These generalized anti-commutators above are special types of an N-ary algebraic structure. We finalize by displaying the relation among GCAs and the algebras underlying the noncommutative fuzzy torus and discuss applications in condensed matter and quantum groups. We conclude with some remarks on N-ary algebras and their applications in Mathematics and Physics.

Keywords : Clifford Algebras; Generalized Clifford Algebras; Dirac operator; Finsler Geometry; Noncommutative Geometry; Spectral Geometry; N-ary algebras; Fuzzy torus.

1 N-ary anticommutators, Generalized Clifford Algebras and Finsler Geometry

It has been recently shown in [9] how generalized Clifford algebras allows to construct the N-th root of N-order linear differential equations involving massless and massive particles. The N-th higher-order linear differential equation is equivalent, after a factorization and cyclic permutation of the factors, to N first-order differential equations. Explicit solutions were found. Extensive studies on Clifford algebras, their generalizations, and their physical applications were made for about a decade starting 1967, under the name of L-Matrix Theory, by Ramakrishnan and his collaborators [1], [2].

In [9] we focused on a very *special* case of these generalized Clifford algebras (GCA) with ordered ω -commutation relations [1], [2]. In particular, on the complex ternary Clifford algebra in two dimensions and denoted by $Cl_2^{\frac{1}{3}}$ [3], [4], [5], [6] with two generators e_1, e_2, obeying the relations

$$e_1^3 \ = \ e_2^3 \ = \ e; \ \ e_1 \ e_2 \ = \ \omega \ e_2 \ e_1; \ \ \omega \equiv e^{\frac{2\pi i}{3}} \tag{1}$$

where e is the identity element. ω is the primitive complex cubic root of unity satisfying

$$\omega^3 = 1, \ \ 1 + \omega + \omega^2 = 0; \ \ \bar{\omega} = \omega^2; \ \ \omega - \omega^2 = i\sqrt{3} \tag{2}$$

An arbitrary element of the complex ternary Clifford algebra in $d = 2$ is described by the expansion [6]

$$U \ = \ \sum_{j,k=0}^{j,k=2} u_{jk} \ e_1^j \ e_2^k \ = \ u_{00} \ e + u_{10} \ e_1 + u_{01} \ e_2 \ + \ u_{20} \ e_1^2 + u_{02} \ e_2^2 +$$

$$u_{11} \ e_1 \ e_2 + u_{21} \ e_1^2 \ e_2 + u_{12} \ e_1 \ e_2^2 + u_{22} \ e_1^2 \ e_2^2 \tag{3}$$

where the coefficients of (3) are complex-valued. The complex ternary Clifford algebra $Cl_2^{\frac{1}{3}}$ was shown to be isomorphic to the unitary algebra $u(3)$. Basis-free definitions of the determinant, trace, and characteristic polynomial in this special class of generalized Clifford algebras were constructed by [6]. Also, a similar operational procedure to what occurs in ordinary complex Clifford algebras, was introduced in the definition of Hermitian conjugation (or Hermitian transpose) without using the corresponding matrix representations.

In general, for $d =$ even, the generalized Clifford algebra (GCA) associated with the N-th root of unity is isomorphic to the unitary algebra $u(N^{\frac{d}{2}})$ of dimension N^d. [6]. The $d =$ odd case is more complicated because the unitary algebra associated with the generalized Clifford algebra (GCA) is now given by the direct sum of N copies of $u(N^{\frac{d-1}{2}})$. The matrix realization of each one of the e_i generators $(i = 1, 2, 3, \ldots, d)$ are given by $N^{\frac{d-1}{2}} \times N^{\frac{d-1}{2}}$ matrices, hence N copies span the $N \times N^{d-1} = N^d$-dimensional space of the GCA.

From some of the entries of the multiplication table of the ternary Clifford algebra in $d = 2$ denoted by $Cl_2^{\frac{1}{3}}$, like

$$e_2\ e_1 = \omega^2\ e_1\ e_2;\ \ e_2\ e_1^2\ =\ \omega\ e_1^2\ e_2;\ \ e_1\ e_1^2\ =\ e;\ \ e_2\ e_2^2\ =\ e$$
$$e_2\ e_1\ e_2\ =\ \omega^2\ e_1\ e_2^2;\ \ e_2\ e_1^2\ e_2\ =\ \omega\ e_1^2\ e_2^2;\ \ldots \tag{4}$$

one can show after some straightforward algebra that the cube of the generalized Clifford algebra (GCA) valued differential $e_1 dx_1 + e_2 dx_2$ is given by

$$(e_1 dx_1\ +\ e_2 dx_2)^3\ =\ [(dx_1)^3\ +\ (dx_2)^3]\ e\ +\ (1+\omega+\omega^2)\ e_1^2\ e_2\ (dx_1)^2\ dx_2\ +$$
$$(1+\omega+\omega^2)\ e_2^2\ e_1\ (dx_2)^2\ dx_1\ =\ [(dx_1)^3\ +\ (dx_2)^3]\ e \tag{5}$$

and which results from the null sum of the three cubic roots of unity $1 + \omega + \omega^2 = 0$. The coordinate variables x_1, x_2, and those below, are chosen to be real-valued.

In the case of the ternary Clifford algebra in $d = 3$ denoted by $Cl_3^{\frac{1}{3}}$, after some lengthy but straightforward algebra, one ends up with

$$(e_1 dx_1\ +\ e_2 dx_2\ +\ e_3 dx_3)^3\ =\ e\ [(dx_1)^3\ +\ (dx_2)^3\ +\ (dx_3)^2] \tag{6}$$

as a result of the following relations obtained after the rearrangement of factors

$$(1+\omega+\omega^2)\ e_1^2\ e_2\ (dx_1)^2\ dx_2\ =\ 0,\ \ (1+\omega+\omega^2)\ e_2^2\ e_1\ (dx_2)^2\ dx_1\ =\ 0 \tag{7a}$$

$$(1+\omega+\omega^2)\ e_1^2\ e_3\ (dx_1)^2\ dx_3\ =\ 0,\ \ (1+\omega+\omega^2)\ e_3^2\ e_1\ (dx_3)^2\ dx_1\ =\ 0 \tag{7b}$$

$$(1+\omega+\omega^2)\ e_2^2\ e_3\ (dx_2)^2\ dx_3\ =\ 0,\ \ (1+\omega+\omega^2)\ e_3^2\ e_2\ (dx_3)^2\ dx_2\ =\ 0 \tag{7c}$$

$$2\ (1+\omega+\omega^2)\ e_1\ e_2\ e_3\ dx_1\ dx_2\ dx_3\ =\ 0 \tag{7d}$$

An example of the rearrangement of factors is

$$e_2\, e_3\, e_1\ =\ \omega^{-2}\, e_1\, e_2\, e_3\ =\ \omega\, e_1\, e_2\, e_3;\ \ e_2\, e_1\, e_3\ =\ \omega^{-1}\, e_1\, e_2\, e_3\ =\ \omega^2\, e_1\, e_2\, e_3,\ \ldots \tag{8}$$

such that the 6 terms $e_1 e_2 e_3 +$ permutations lead to $2(1 + \omega + \omega^2) e_1 e_2 e_3 = 0$ as shown in eq-(7d) . In a similar fashion one obtains the terms in eqs-(7) whose contribution is zero. Hence out of the 27 terms which appear in the cube $(e_1 dx_1 + e_2 dx_2 + e_3 dx_3)^3$, due to the relations in eqs-(7), only 3 terms remain as displayed by eq-(6).

Therefore, from eq-(6) one learns that the cube of the GCA-valued differential $(e_i dx_i)$ is given by

$$(e_i dx_i)^3 \ = \ (e_1 dx_1 \ + \ e_2 dx_2 \ + \ e_3 dx_3)^3 \ = \ = \ e \ \eta_{i_1 i_2 i_3} \ dx^{i_1} \ dx^{i_2} \ dx^{i_3} \ =$$
$$e \ [(dx_1)^3 + (dx_2)^3 + (dx_3)^3] \tag{9}$$

with $\eta_{i_1 i_2 i_3} = 1$ iff $i_1 = i_2 = i_3$, and 0 otherwise. As a result of eq-(9) one has the inequality

$$(ds)^3 \ = \ \Big(\ (dx_1)^2 \ + \ (dx_2)^2 \ + \ \ldots \ + \ (dx_d)^2 \ \Big)^{\frac{3}{2}} \ \neq \ (dx_1)^3 + (dx_2)^3 + \ldots + (dx_d)^3 \ \Rightarrow$$
$$ds_{(3)} \ \equiv \ [\ (dx_1)^3 \ + \ (dx_2)^3 \ + \ \ldots \ + \ (dx_d)^3 \ ]^{\frac{1}{3}} \ \neq$$
$$ds_{(2)} \ \equiv \ [\ (dx_1)^2 \ + \ (dx_2)^2 \ + \ \ldots \ + \ (dx_d)^2 \ ]^{\frac{1}{2}} \tag{10}$$

between the quadratic $ds_{(2)}$ and cubic norm $ds_{(3)}$. Absolute values $|\ldots|$ in eq-(10) are not required because the variables x_i are chosen to be real-valued.

One can extend the results in eqs-(5,9) to the most general case of the complex generalized Clifford algebra associated to the N-th root of unity $\omega \equiv e^{2\pi i/N}$ in d-dimensions, with d generators $e_1, e_2, e_3, \ldots, e_d$. One then arrives at the most general relation for all d and N

$$(e_1 dx_1 \ + \ e_2 dx_2 \ + \ \ldots \ + \ e_d dx_d)^N \ = \ e \ [(dx_1)^N \ + \ (dx_2)^N \ + \ \ldots \ + \ (dx_d)^N] \ =$$
$$e \ [\ \eta_{11\ldots1} \ (dx_1)^N \ + \ \eta_{22\ldots2} \ (dx_2)^N \ + \ \ldots \ + \ \eta_{dd\ldots d} \ (dx_d)^N \ ] \tag{11}$$

due to the algebraic constraint $1 + \omega + \omega^2 + \omega^3 + \ldots + \omega^{N-1} = 0$; the commutation relations $e_i e_j = \omega e_j e_i$ with $i < j$; and $e_i^N = e$ where $i, j = 1, 2, 3, \ldots, d$. We refer the reader to section 7 of [2] for the full details of how to obtain the rigorous derivation of eq-(11).

To sum up, given a generalized Clifford algebra (GCA) associated with the N-th root of unity, involving d generators $e_1, e_2, \ldots, e_d$, and the unit element e, we can infer from the eqs-(5,9,11) derived above that the following generalized anti-commutator with N entries gives

$$\{e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_N}\} \ = \ e_{i_1} e_{i_2} e_{i_3} \ldots e_{i_N} \ + \ permutations \ = \ N! \ \eta_{i_1 i_2 \ldots i_N} \ e \tag{12a}$$

where all the $N!$ terms of the permutations appear with the same positive sign. The components of the rank-N metric are $\eta_{i_1 i_2 \ldots i_N} = 1$, iff $i_1 = i_2 = i_3 \ldots = i_N$, and 0 otherwise. The range of the indices $i_1, i_2, \ldots i_N$ is $1, 2, \ldots, d$. Therefore, we may *encode* the GCA algebra $Cl_d^{1/N}$ in terms of the N-ary algebra depicted by the anticommutators in eq-(12a).

Before proceeding, we should mention that the ternary Clifford algebra depicted above in eq-(12a), with 6 terms appearing in the generalized anti-commutator when $N = 3$, is *not* the same as the one formulated by Kerner [7] involving the *cyclic* anticommutator requiring 3 terms only $Q_a Q_b Q_c + Q_b Q_c Q_a +$

$Q_c Q_a Q_b = 3\rho_{abc}\mathbf{1}$, which is one of the possible ternary extensions of $Q_a Q_b + Q_b Q_c = 2\eta_{ab}\mathbf{1}$. Another ternary extension involves the inclusion of ω, ω^2 factors $Q_a Q_b Q_c + \omega Q_b Q_c Q_a + \omega^2 Q_c Q_a Q_b$. By inspection, in the case of the GCA algebra $Cl_3^{1/3}$ one finds that $e_1 e_2 e_3$ + cyclic permutation $= (1+2\omega) e_1 e_2 e_3$ which is neither zero nor is proportional to the unit element e. Whereas, by including all the 6 terms of the permutation $e_1 e_2 e_3 + \ldots = 0$, one obtains a vanishing result as displayed in eq-(7d).

The existence and particular properties of the cubic Grassmann and Clifford algebras studied by Kerner [7] were used to define cubic roots of linear differential operators which clearly $differ$ from the operators found in our previous work [9]. For more on cubic forms and algebras with cubic constitutive relations see [8].

The curved space version of eq-(12a) requires the introduction of a vierbein (frame fields) e^i_μ ($\mu = 1, 2, \ldots d; i = 1, 2, \ldots, d$) where $e_\mu = e^i_\mu e_i$ are the d-dim curved space GCA generators such that the generalized anti-commutator with N entries in eq-(12a) can be rewritten in terms of the curved space GCA generators as

$$\{e_{\mu_1}, e_{\mu_2}, e_{\mu_3}, \ldots, e_{\mu_N}\} \ = \ e_{\mu_1} e_{\mu_2} e_{\mu_3} \ldots e_{\mu_N} + permutations \ = \ N! \, g_{\mu_1 \mu_2 \ldots \mu_N} \, e \tag{12b}$$

The curved space rank-N metric $g_{\mu_1 \mu_2 \ldots \mu_N}$ is defined in terms of the flat space rank-N metric $\eta_{i_1 i_2 \ldots i_N}$ as

$$g_{\mu_1 \mu_2 \ldots \mu_N} \ = \ e^{i_1}_{\mu_1} \ e^{i_2}_{\mu_2} \ e^{i_3}_{\mu_3} \ldots e^{i_N}_{\mu_N} \ \eta_{i_1 i_2 i_3 \ldots i_N}. \tag{12c}$$

and is just a generalization of the relation $g_{\mu\nu} = e^i_\mu e^j_\nu \eta_{ij}$ between the tangent space η_{ij} and curved space $g_{\mu\nu}$ metric via the vierbein (frame fields) e^i_μ.

Despite that the N-th norm $ds_{(N)}$ is not equal to the quadratic norm $ds_{(2)}$

$$ds_{(N)} \ \equiv \ [\ (dx_1)^N \ + \ (dx_2)^N \ + \ \ldots \ + \ (dx_d)^N \ ]^{\frac{1}{N}} \ \neq$$

$$ds_{(2)} \ \equiv \ [\ (dx_1)^2 \ + \ (dx_2)^2 \ + \ \ldots \ + \ (dx_d)^2 \ ]^{\frac{1}{2}} \tag{13}$$

one can still define the following integral associated with the real-valued trajectories $x_i = x_i(\tau), i = 1, 2, 3, \ldots, d$ of a particle moving in d-dim real Euclidean space as

$$\int \ ds_{(N)} \ = \ \int \ d\tau \ [\ (\frac{dx_1}{d\tau})^N \ + \ (\frac{dx_2}{d\tau})^N \ + \ \ldots \ + \ (\frac{dx_d}{d\tau})^N \ ]^{\frac{1}{N}} \tag{14}$$

Like in Finsler geometry, one may define a generalized arc length as

$$\int \ ds_{(N)} \ = \ \int \ d\tau \ \mathcal{L} \ = \ \int \ d\tau \ [\ g_{i_1 i_2 i_3 \ldots i_N}(x^i, \frac{dx^i}{d\tau}) \ \frac{dx^{i_1}}{d\tau} \ \frac{dx^{i_2}}{d\tau} \ \ldots \ \frac{dx^{i_N}}{d\tau} \ ]^{\frac{1}{N}} \tag{15a}$$

Under scalings of the velocities $\frac{dx^i}{d\tau} \rightarrow \lambda \frac{dx^i}{d\tau}$ the integrand scales as $\mathcal{L} \rightarrow \lambda \mathcal{L}$; i.e. the integrand $\mathcal{L}$ is a homogeneous function of unit weight if the components of the rank N metric tensor $g_{i_1 i_2 i_3 \ldots i_N}(x^i, \frac{dx^i}{d\tau})$ are homogeneous functions of zero weight such that $\dot{x}^i \frac{\partial}{\partial \dot{x}^i}(g_{i_1 i_2 i_3 \ldots i_N}) = 0$, with $\dot{x}^i \equiv \frac{dx^i}{d\tau}$, and leading to

$$g_{i_1 i_2 i_3 \ldots i_N}(x^i, \dot{x}^i) \ = \ \frac{1}{N!} \ \frac{\partial^N \mathcal{L}^N}{\partial \dot{x}^{i_1} \partial \dot{x}^{i_2} \ldots \partial \dot{x}^{i_N}} \tag{15b}$$

The analog of the analytical continuation (Wick rotation) from a Euclidean to Lorentzian signature is attained now by performing the transformation $e_1 \to (-1)^{1/N} e_1 = \tilde{e}_1 \Rightarrow \tilde{e}_1^N = -e$, such that

$$(\tilde{e}_1 dx_1 \ + \ e_2 dx_2 \ + \ \ldots \ + \ e_d dx_d)^N \ = \ e \ [- \ (dx_1)^N \ + \ (dx_2)^N \ + \ \ldots \ + \ (dx_d)^N] \tag{16}$$

with $\eta_{11\ldots1} = -1$ and $\eta_{22\ldots2} = \eta_{33\ldots3} = \ldots = \eta_{dd\ldots d} = 1$.

2 Dirac operators, Generalized Clifford Spaces and Spectral Geometry

A Clifford Space (C-space) associated with a real quadratic Clifford algebra in d-dimensions is characterized by the Clifford-algebra-valued coordinate $\mathbf{X}$, a polyvector, which admits the following decomposition

$$\mathbf{X} \ = \ x \ \mathbf{1} \ + \ x_\mu \ \gamma^\mu \ + \ x_{\mu_1 \mu_2} \ \gamma^{\mu_1 \mu_2} \ + \ \ldots \ x_{\mu_1 \mu_2 \ldots \mu_d} \ \gamma^{\mu_1 \mu_2 \ldots \mu_d} \tag{17}$$

where x is a scalar, x_μ is a vector, $x_{\mu_1 \mu_2} = -x_{\mu_2 \mu_1}$ is a bivector, $x_{\mu_1 \mu_2 \mu_3}$ is a trivector (antisymmetric in all of its indices), and so forth. In order to avoid introducing combinatorial numerical factors one may impose the ordering prescription $\mu_1 < \mu_2 < \mu_3 \ldots$. To match physical units in the terms of eq-(17) one is required to introduce suitable powers of a fiducial length scale which can be chosen to be the Planck scale L_P and which can be set to unity after adopting a geometric system of units $\hbar = c = G = 1$.

The reversal operation $\tilde{\mathbf{X}}$ on $\mathbf{X}$ is defined by reversing the order of the indices of the bivector $\gamma^{\mu_1 \mu_2}$ generator, trivector $\gamma^{\mu_1 \mu_2 \mu_3}$ generator, $\ldots$ giving $\gamma^{\mu_2 \mu_1}$, $\gamma^{\mu_3 \mu_2 \mu_1}, \ldots$, respectively. Given the reversal operation, the quadratic norm-squared of the Clifford-valued differential $\mathbf{dX}$ is given by the scalar part of the Clifford geometric product of $\mathbf{d}\tilde{\mathbf{X}}$ with $\mathbf{dX}$

$$||\mathbf{dX}||^2 \ = \ \langle \mathbf{d}\tilde{\mathbf{X}} \ \mathbf{dX} \rangle \ = \ dx^2 \ + \ dx_\mu \ dx^\mu \ + \ dx_{\mu\nu} \ dx^{\mu\nu} \ + \ \ldots$$
$$dx_{\mu_1 \mu_2 \ldots \mu_d} \ dx^{\mu_1 \mu_2 \ldots \mu_d}; \quad L_P = 1 \tag{18}$$

The brackets $\langle \ldots \rangle$ in eq-(18) denote extracting the scalar part of the geometric product.

In the case of complex Clifford algebras the coordinates are complex-valued and the norm squared is now given by

$$||\mathbf{dY}||^2 \ = \ \langle \mathbf{dY}^\dagger \ \mathbf{dY} \rangle \ = \ dy^* dy \ + \ (dy_\mu)^* \ (dy^\mu) \ + \ (dy_{\mu\nu})^* \ (dy^{\mu\nu}) \ + \ \ldots$$

$$(dy_{\mu_1\mu_2\ldots\mu_d})^* \ (dy^{\mu_1\mu_2\ldots\mu_d}) \quad (19)$$

where $\mathbf{dY}^\dagger$ is the Hermitian conjugate of $\mathbf{dY}$ obtained by a reversal operation followed by a complex conjugation of the coordinates.

In the most general scenario, the norm squared $||\mathbf{Y}||^2$ is preserved under the transformations

$$\mathbf{Y} \to \mathbf{Y}' \ = \ \exp(\mathbf{R}) \ \mathbf{Y} \ \exp(-\mathbf{R}) \ \Rightarrow \langle \mathbf{Y'}^\dagger \ \mathbf{Y}' \rangle \ = \ ||\mathbf{Y}'||^2 \ =$$
$$\langle \exp(\mathbf{R}) \ \mathbf{Y}^\dagger \ \exp(-\mathbf{R}) \ \exp(\mathbf{R}) \ \mathbf{Y} \ \exp(-\mathbf{R}) \ \rangle \ = \ \langle \exp(\mathbf{R}) \ \mathbf{Y}^\dagger \ \mathbf{Y} \ \exp(-\mathbf{R}) \ \rangle \ =$$
$$\langle \mathbf{Y}^\dagger \ \mathbf{Y} \rangle \ = \ ||\mathbf{Y}||^2; \ \ \mathbf{R}^\dagger = -\mathbf{R} \quad (20)$$

resulting from the cyclicity property of the bracket operation $\langle ABC \rangle = \langle BCA \rangle = \langle CAB \rangle$ and involving the unitary transformations via the Clifford-valued operator $\mathbf{U}^\dagger = \mathbf{U}^{-1}$ defined in terms of the rotor operator as $\mathbf{U} = \exp(\mathbf{R})$, such that the rotor $\mathbf{R}$ is anti-Hermitian $\mathbf{R}^\dagger = -\mathbf{R}$. Given a Clifford-valued rotor $\mathbf{R}$ polyvector

$$\mathbf{R} \ = \ \xi \ \mathbf{1} \ + \ \xi_\mu \ \gamma^\mu \ + \ \xi_{\mu_1\mu_2} \ \gamma^{\mu_1\mu_2} \ + \ \ldots \ \xi_{\mu_1\mu_2\ldots\mu_d} \ \gamma^{\mu_1\mu_2\ldots\mu_d} \quad (21)$$

the condition $\mathbf{R}^\dagger = -\mathbf{R}$ will impose certain constraints on the complex-valued parameters $\xi, \xi_\mu, \xi_{\mu_1\mu_2}, \ldots$, which are the C-space extension of the rotation and boost rapidity parameters of the Lorentz transformations associated with the group $SO(d-1,1)$ in a d-dim Minkowski spacetime. For example, given $\xi_{\mu\nu}\gamma^{\mu\nu}$, under complex conjugation and reversal it gives $-(\xi_{\mu\nu})^*\gamma^{\mu\nu}$, and after equating it with $-\xi_{\mu\nu}\gamma^{\mu\nu}$ one learns that $(\xi_{\mu\nu})^* = \xi_{\mu\nu} =$ real. Hence, in this way one finds that the ξ's parameters are either real or purely imaginary. For instance, ξ_μ is purely imaginary. In the case of GCA's the constraints among the parameters are more complicated as we shall see below.

The study of Clifford spaces (C-spaces) associated with ordinary quadratic Clifford algebras, and the construction of an Extended Relativity theory in such spaces, based on the generalizations of the translations, rotations and boosts transformations in Minkowski spacetime, can be found in [14]. The C-space transformations of the polyvector-valued coordinates were interpreted by [14] as generalized transformations that map line, area, volume, hyper-volume intervals among each other and which correspond collectively to the evolution of points (worldlines), strings (worldsheets), membranes (world volumes), $p-$branes (world hyper-volumes), respectively. One of the interesting features of the curved C-space geometry is that the scalar curvature in C-space could be decomposed into the sum of the ordinary Riemannian scalar curvature plus higher powers of the curvature tensor (higher curvature gravity) [14]. For a recent and mathematically rigorous treatment of the formal geometry on differentiable graded manifolds involving polyvectors and polydifferential forms see [15].

Turning attention to Generalized Clifford Algebras (GCA), a generalized Clifford Space associated with a *complex* ternary Clifford algebra in $d = 2$ dimensions is characterized by the complex ternary Clifford-algebra-valued coordinate

$$\mathbf{Y} \ = \ e \ y_{00} \ + \ e_1 \ y_{10} \ + \ e_2 \ y_{01} \ + \ e_1 \ e_2 \ y_{11} \ +$$

$$e_1^2 \ y_{20} \ + \ e_2^2 \ y_{02} \ + \ e_1^2 \ e_2 \ y_{21} \ + \ e_1 \ e_2^2 \ y_{12} \ + \ e_1^2 \ e_2^2 \ y_{22} \qquad (22)$$

comprised of the following $3^2 = 9$ *complex*-valued coordinate components

$$Y_M \ = \ \{ \ y_{00}, y_{10}, y_{01}, y_{11}, y_{20}, y_{02}, y_{21}, y_{12}, y_{22} \ \}, \ \ Y_M \in \mathbf{C} \qquad (23)$$

Once again, concerning the physical units of the components of eq-(22), one would then require again to introduce suitable judicious powers of L_P in front of the components of $\mathbf{Y}$ in eq-(22) in order to match physical units. For example, y_{00} is dimensionless; y_{10}, y_{01} have units of length $[L]$. y_{11}, y_{20}, y_{02} have units of $[L^2]$. y_{21}, y_{12} have units of $[L^3]$. And y_{22} has units of $[L^4]$. Setting $L_P = 1$ simplifies matters so that we don't have to write explicitly the powers of L_P.

There is a one-to-one correspondence among the Y_M components in eq-(23) with the tensorial coordinates of different ranks. y_{00} is a scalar. y_{10}, y_{01} correspond to the vector components z_1, z_2. y_{11}, y_{20}, y_{02} correspond to the second rank tensor components z_{12}, z_{11}, z_{22}. y_{21}, y_{12} correspond to the rank-3 tensor components z_{112}, z_{122}. And y_{22} to the rank-4 tensor components z_{1122}. Clearly, there is a difference between the nature of the components in eqs-(22,23) with those belonging to a quadratic Clifford algebra-valued polyvector comprised of a scalar, pseudo-scalar, vector, and antisymmmetric tensors of different ranks.

The Hermitian conjugate $\mathbf{Y}^\dagger$ of $\mathbf{Y}$ is defined in [6] by replacing the complex coordinates in eqs-(22,23) by their complex conjugates $Y_M \to (Y_M)^*$, and by replacing the generators, and their products, by their inverses as follows

$$e_1 \to e_1^{-1}, \ \ e_2 \to e_2^{-1}, \ \ e_1 \ e_2 \to (e_1 \ e_2)^{-1}, \ \ e_1^2 \to (e_1^2)^{-1}, \ \ldots, \ e_1^2 \ e_2^2 \to (e_1^2 \ e_2^2)^{-1} \qquad (24)$$

In the very special case of quadratic Clifford algebras $e_i^2 = e$ for all $i = 1, 2, \ldots, d$; $e_i e_j = -e_j e_i, i < j$, replacing the products of the generators by their inverses is tantamount of performing the reversal operation. For instance,

$$(e_1 \ e_2)^{-1} \ = \ e_2 \ e_1 \Rightarrow (e_1 \ e_2)^{-1} \ e_1 \ e_2 \ = \ e_2 \ e_1 \ e_1 \ e_2 \ = \ e_2 \ e \ e_2 \ = \ e, \ \ldots \qquad (25)$$

as a result of

$$(e_1)^{-1} = e_1, \ \ (e_2)^{-1} = e_2, \ \ (e_1 \ e_2)^{-1} \ = \ (e_2)^{-1} \ (e_1)^{-1} \ = \ e_2 \ e_1 \qquad (26)$$

since $e_1^2 = e_2^2 = e$.

The Hermitian conjugation operation allows to define the norm-squared of $\mathbf{Y}$ in terms of the following inner product given by [6]

$$||\mathbf{Y}||^2 \ = \ \mathbf{Y} \cdot \mathbf{Y} \ \equiv \ \langle \mathbf{Y}^\dagger \ \mathbf{Y} \rangle_0 \ = \ |y_{00}|^2 \ + \ |y_{10}|^2 \ + \ |y_{01}|^2 \ + \ \ldots \ |y_{22}|^2 \qquad (27)$$

Let us study the conditions on the rotor parameters in the case of GCA's such that $\mathbf{R}^\dagger = -\mathbf{R}$ under the Hermitian operation involving the complex conjugation of the coefficients and inversion of the generators and their products. The rotor operator associated with a *complex* ternary Clifford algebra in $d = 2$ dimensions is characterized by the complex ternary Clifford-algebra-valued quantity

$$\mathbf{R} \; = \; e\,\xi_{00} \; + \; e_1\,\xi_{10} \; + \; e_2\,\xi_{01} \; + \; e_1\,e_2\,\xi_{11} \; + \\ e_1^2\,\xi_{20} \; + \; e_2^2\,\xi_{02} \; + \; e_1^2\,e_2\,\xi_{21} \; + \; e_1\,e_2^2\,\xi_{12} \; + \; e_1^2\,e_2^2\,\xi_{22} \tag{28}$$

where all the parameters in (28) are dimensionless. The Hermitian conjugate is

$$\mathbf{R}^\dagger \; = \; e\,(\xi_{00})^* \; + \; (e_1)^{-1}\,(\xi_{10})^* \; + \; (e_2)^{-1}\,(\xi_{01})^* \; + \; (e_1\,e_2)^{-1}\,(\xi_{11})^* \; + \\ (e_1^2)^{-1}\,(\xi_{20})^* + (e_2^2)^{-1}\,(\xi_{02})^* + (e_1^2\,e_2)^{-1}\,(\xi_{21})^* + (e_1\,e_2^2)^{-1}\,(\xi_{12})^* + (e_1^2\,e_2^2)^{-1}\,(\xi_{22})^* \tag{29}$$

Let us provide some examples of the constraints among the parameters in order to obey the condition $\mathbf{R}^\dagger = -\mathbf{R}$. One finds, for example, that given

$$(e_1\,e_2)^{-1} \; = \; \omega\,e_1^2\,e_2^2 \;\Rightarrow\; (\xi_{11})^*\,\omega\,e_1^2\,e_2^2 \; = \; -\,\xi_{22}\,e_1^2\,e_2^2 \Rightarrow\; (\xi_{11})^* \; = \; -\,\omega^2\,\xi_{22} \tag{30}$$

it leads to a constraint between the complex valued coefficients ξ_{11} and ξ_{22}. From

$$(e_1)^{-1} \; = \; e_1^2 \;\Rightarrow\; (\xi_1)^*\,e_1^2 \; = \; -\,\xi_{20}\,e_1^2 \;\Rightarrow\; (\xi_1)^* \; = \; -\,\xi_{20} \tag{31}$$

it leads to a constraint between the complex valued coefficients ξ_1 and ξ_{20}; and so forth. In this way one finds the constraints among all the rotor parameters in order for the rotor to satisfy the anti-Hermitian $\mathbf{R}^\dagger = -\mathbf{R}$ condition. The scalar parameter ξ_{00} obeys $(\xi_{00})^* = -\xi_{00}$; i.e it is pure imaginary.

Having discussed the basic geometrical properties of Clifford spaces and their generalizations, we finalize with some remarks about Dirac operators in spectral geometry [10]. The Dirac operator associated with the quadratic Clifford algebras has been essential in Connes' work in Noncommutative Geometry over the past decades [10], [11]. More recent references can be found in [12], [13]. The basic data of spectral geometry is encoded in a spectral triple $\mathcal{T} = (\mathcal{A}, \mathcal{H}, D)$, consisting of an algebra $\mathcal{A}$ that captures the topological data, a Hilbert space $\mathcal{H}$, and a generalized Dirac operator $D : \mathcal{H} \to \mathcal{H}$ which encodes the metric data.

Starting from a Riemannian manifold (M, g) with a spin structure, we can represent the algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}$ of square-integrable sections of the complex spin bundle $S \to M$; i.e. spinors. The curved space Dirac operator $D = i\gamma^\mu \nabla_\mu$ acts on these spinors. We may then recover the Riemannian line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ from the Dirac propagator via the correspondence $ds \leftrightarrow D^{-1}$, and the geodesic distance formula can now be rewritten in a purely algebraic fashion as follows [10]

$$d(x,y) \;=\; sup_{f\in\mathcal{A}} \;\{|f(x)-f(y)| \mid ||D,f|| \leq 1\} \tag{32}$$

The intuition behind this formula (32) was recently explained by [12] and is that one can always find a function $f : M \rightarrow \mathbf{C}$ with a finite difference of its values on two points $x, y \in M$ such that $|f(x)-f(y)| = d(x,y)$. This is achieved by looking at all functions whose gradients on M are bounded from above by 1, and then selecting the one which maximizes the distance between the two points. The Dirac operator D provides a suitable gradient. Self-adjointness of D guarantees that $ds^2 \geq 0$ [12]. The Dirac operator was essential in constructing the spectral action whose asymptotic expansion generated the curvature invariants of the gravitational and Yang-Mills theories and leading to a unification prospect of gravity with the Standard Model. See [10], [11], and the recent references [12], [13] for all the technical details.

A Dirac-like operator $\mathcal{D}$ based on the GCA in d-dimensions and corresponding to the N-th root of unity, $Cl_d^{1/N}$, was defined in [2], [9] as

$$\mathcal{L} \;\equiv\; e_1\partial_1 \;+\; e_2\partial_2 \;+\; \ldots \;+\; e_d\partial_d \tag{33a}$$

with $\partial_1 = \frac{\partial}{\partial x^1}, \partial_2 = \frac{\partial}{\partial x^2}, \ldots, \partial_d = \frac{\partial}{\partial x^d}$, and where we have omitted the explicit i factors, and it obeys the relation [2], [9]

$$\mathcal{L}^N \;=\; (e_1\partial_1 + e_2\partial_2 + \ldots + e_d\partial_d)^N \;=\; e\,[(\partial_1)^N + (\partial_2)^N + \ldots + (\partial_d)^N] \tag{33b}$$

Next we shall define more general Dirac-like operators than $\mathcal{L}$.

The analog of the Dirac operator in C-space comprised of real-valued coordinates is given by

$$\mathbf{D} \;=\; i\frac{\partial}{\partial x} \;+\; i\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} \;+\; i\gamma^{\mu_1\mu_2}\frac{\partial}{\partial x^{\mu_1\mu_2}} \;+\; \ldots \;+\; i\gamma^{\mu_1\mu_2\ldots\mu_d}\frac{\partial}{\partial x^{\mu_1\mu_2\ldots\mu_d}} \tag{34}$$

and it is understood that the first term $i(\partial/\partial x)$ contains an implicit factor of the unit element of the Clifford algebra. Given the above $\mathbf{D}$ operator one can explore the correspondence $||\mathbf{dX}|| \leftrightarrow \mathbf{D}^{-1}$ between the displacement $\mathbf{dX}$ and the inverse $\mathbf{D}^{-1}$ of the Dirac operator in the C-space associated with the quadratic Clifford Algebra. Because the C-space associated with a quadratic real Clifford algebra in d-dimensions is 2^d-dimensional, instead of having the algebra $\mathcal{A}$ of functions $f : M \rightarrow \mathbf{C}$, one has now an algebra of functions from a 2^d-dim space $\mathcal{M}$ to the complex numbers $\mathcal{M} \rightarrow \mathbf{C}$. And one can proceed in a similar fashion to write the geodesic distance formula in a purely algebraic fashion as in eq-(32).

Let us proceed now with the study of the Dirac operators in the Generalized Clifford spaces associated with GCA's. The analog of the Dirac operator involving real coordinates and corresponding to a ternary Clifford algebra in $d = 2$ dimensions is given by

$$\mathbf{D} \;=\; i\;e\frac{\partial}{\partial x_{00}} \;+ i\;e_1\frac{\partial}{\partial x_{10}} \;+ i\;e_2\frac{\partial}{\partial x_{01}} \;+ i\;e_1e_2\frac{\partial}{\partial x_{11}} \;+$$

$$i\ e_1^2 \frac{\partial}{\partial x_{20}} \ + i\ e_2^2 \frac{\partial}{\partial x_{02}} \ + i\ e_1^2 e_2 \frac{\partial}{\partial x_{21}} \ + i\ e_1 e_2^2 \frac{\partial}{\partial x_{12}} \ + i\ e_1^2 e_2^2 \frac{\partial}{\partial x_{22}} \tag{35}$$

The GCA-valued differential is

$$\mathbf{DX} \ = \ e\ dx_{00} \ + \ e_1\ dx_{10} \ + \ e_2\ dx_{01} \ + \ e_1\ e_2\ dx_{11} \ + \\ e_1^2\ dx_{20} \ + \ e_2^2\ dx_{02} \ + \ e_1^2\ e_2\ dx_{21} \ + \ e_1\ e_2^2\ dx_{12} \ + \ e_1^2\ e_2^2\ dx_{22} \tag{36}$$

and from eq-(36) one can infer that the quadratic norm is

$$(d\Upsilon)^2 \ = \ ||\mathbf{DX}||^2 \ = \ (dx_{00})^2 \ + \ (dx_{10})^2 \ + \ (dx_{01})^2 \ + \ (dx_{11})^2 \ + \\ (dx_{20})^2 \ + \ (dx_{02})^2 \ + \ (dx_{21})^2 \ + \ (dx_{12})^2 \ + \ (dx_{22})^2 \tag{37}$$

such that one can explore the correspondence $d\Upsilon \leftrightarrow \mathbf{D}^{-1}$ between the displacement $d\Upsilon$ and the inverse $\mathbf{D}^{-1}$ of the generalized Dirac operator in eq-(35) associated with the Generalized Clifford Algebra $Cl_2^{1/3}$. The generalized Clifford-space associated with a GCA $Cl_d^{1/N}$ is N^d-dimensional, so instead of having the algebra $\mathcal{A}$ of functions $f : M \to \mathbf{C}$, one has now an algebra of functions from a N^d-dim space $\mathcal{M}$ to the complex numbers $\mathcal{M} \to \mathbf{C}$. And, once again, one can proceed in a similar fashion to write the geodesic distance formula in a purely algebraic fashion as in eq-(33).

3 Fuzzy Torus, Noncommutative Spaces and GCAs

An element $\mathbf{A}$ of the GCA $Cl_2^{1/N}$ in $d = 2$ associated with the N-th root of unity can be expanded in powers of the two generators e_1, e_2 as follows

$$\mathbf{A} \ = \ \sum_{j,k=0}^{j,k=N-1} \ u_{jk}\ e_1^j\ e_2^k \tag{38}$$

The GCA $Cl_2^{1/N}$ is N^2-dimensional and is isomorphic to the $u(N)$ algebra of unitary $N \times N$ matrices. A $N \times N$ matrix representation of the e_1, e_2 generators is given by the shift V and clock U matrices, respectively. For instance, when $N = 3$, $\omega = e^{2\pi i/3}$, the V and U matrices are given by

$$V \ = \ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{39a}$$

$$U \ = \ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \tag{39b}$$

and obey the relations

$$VU \ = \ \omega \ UV, \ \ U^3 = V^3 = \mathbf{1}_{3\times 3}, \ \ UU^\dagger = U^\dagger U = \mathbf{1}_{3\times 3}, \ \ VV^\dagger = V^\dagger V = \mathbf{1}_{3\times 3} \tag{40}$$

The Weyl braiding matrix W is given by

$$W \ = \ \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \tag{41}$$

and it obeys the relationship $W^{-1}VW = U \Rightarrow VW = WU$.

For other values of N corresponding to the GCA $Cl_2^{1/N}$ one has

$$VU \ = \ \omega \ UV \ = \ e^{2\pi i/N} \ VU, \ \ U^N = V^N = \mathbf{1}_{N\times N},$$
$$UU^\dagger = U^\dagger U = \mathbf{1}_{N\times N}, \ \ VV^\dagger = V^\dagger V = \mathbf{1}_{N\times N} \tag{42}$$

The $N \times N$ clock matrix is $U = \text{diag}\ (1, \omega, \omega^2, \omega^3, \ldots, \omega^{N-1})$. And the $N \times N$ shift matrix V has the same form as in eq-(39a) with 1's off-the-main diagonal and 1 in the very last row and first column.

The relations (42) are the same as the algebraic relations used to define the noncommutative fuzzy torus $\mathbf{T}^2_\theta$ in terms of the noncommutative (dimensionless) coordinates (Hermitian operators) $\hat{u}, \hat{v}$ obeying [16], [12]

$$[\hat{u}, \hat{v}] \ = \ \frac{2\pi i}{N} \ \mathbf{1} \ = \ i \ \theta \ \mathbf{1}, \ \ \theta \equiv \frac{2\pi}{N} \tag{43}$$

where $\mathbf{1}$ is the Hermitian unit operator commuting with $\hat{u}, \hat{v}$ and that *must* be introduced in the right hand side of (43). Because the commutator of two Hermitian operators is anti-Hermitian, the right hand side (43) must contain an i factor times the unit Hermitian operator $\mathbf{1}$. The noncommutativity parameter θ is identified with the angle $\theta = \frac{2\pi}{N}$. After setting the correspondence

$$U \leftrightarrow e^{i\hat{u}}, \ \ V \leftrightarrow e^{i\hat{v}}, \tag{44}$$

and recurring to the Baker-Campbell-Hausdorff formula in the special case that the commutator of two operators $[X, Y]$ is central; i.e $[X, [X, Y]] = [Y, [X, Y]] = 0$, one finds the following correspondence between the algebra involving the operators $\hat{u}, \hat{v}, \mathbf{1}$ and the algebra involving the $N \times N$ matrices U, V [1]

$$e^{i\hat{v}} \ e^{i\hat{u}} \ = \ e^{[\hat{u}, \hat{v}]} \ e^{i\hat{u}} \ e^{i\hat{v}} \ = \ e^{\frac{2\pi i}{N}\mathbf{1}} \ e^{i\hat{u}} \ e^{i\hat{v}} \ = \ e^{i\theta\mathbf{1}} \ e^{i\hat{u}} \ e^{i\hat{v}} \Leftrightarrow$$
$$VU \ = \ \omega \ UV \ = \ e^{2\pi i/N} \ VU \ = \ e^{i\theta} \ VU \tag{45}$$

after establishing the correspondence between the exponential $\exp(i\theta\mathbf{1})$ involving the unit operator $\mathbf{1}$ with the phase factor $\exp(2\pi i/N) = \exp(i\theta)$. Therefore,

[1]This results from $\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y])$, and $\exp(Y)\exp(X) = \exp(Y + X - \frac{1}{2}[X, Y])$ after multiplying the latter terms by $\exp([X, Y])$, when $[X, Y]$ is central, (it commutes with X and Y)

the GCA $Cl_2^{1/N}$ captures the noncommutative algebra associated with the fuzzy torus for all values of N.

A more general family of GCAs is obtained from the relations

$$e_i\ e_j\ =\ \omega_{ij}\ e_j\ e_i,\quad i<j;\quad e_1^N=e_2^N=e_3^N=\ldots=e_d^N=e,\quad i,j=1,2,\ldots,d \tag{46}$$

with

$$\omega_{ij}\ =\ \exp(\frac{2\pi i}{N}T_{ij}),\quad \omega_{ij}^N=1,\quad T_{ji}=-T_{ij}\ \Rightarrow\ \omega_{ji}\ =\ \omega_{ij}^{-1} \tag{47}$$

and where the entries T_{ij} of the antisymmetric $d\times d$ matrix $\mathbf{T}$ are comprised of integers[2]. One finds that in this more general case the numerical relations obtained in section **1** would have to be *modified* because one does not have any longer at our disposal a unique value for ω given by the N-th root of unity but a set of many different values of ω_{ij}.

The GCAs defined in eqs-(46,47) capture the algebra of the noncommutative coordinates $\hat{x}_1,\hat{x}_2,\hat{x}_3,\ldots,\hat{x}_d$ associated with the fuzzy d-dim torus $\mathbf{T}^d_\Theta$ of topology $S^1\times S^1\times\ldots\times S^1$, and given by

$$[\ \frac{\hat{x}_k}{R_k},\ \frac{\hat{x}_l}{R_l}\ ]\ =\ -i\ \frac{\Theta_{kl}\ L_P^2}{R_kR_l}\ \mathbf{1}\ =\ -i\ \frac{2\pi}{N}\ T_{kl}\ \mathbf{1};\quad k,l=1,2,3,\ldots,d \tag{48}$$

where Θ_{kl} is an antisymmetric $d\times d$ matrix whose entries are all *constant*. The noncommutative coordinates $\hat{x}_1,\hat{x}_2,\hat{x}_3,\ldots,\hat{x}_d$ are associated with the d covering spaces of the d circles given by d real lines, respectively. After imposing the correspondence $e_k\to e^{i\hat{x}_k/R_k}, e_l\to e^{i\hat{x}_l/R_l}$, where R_k,R_l are the radii of the respective circles, and using the Baker-Campbell-Hausdorff formula one ends up with the following correspondence (dictionary) between the algebra involving the $\hat{x}_k,\hat{x}_l,\mathbf{1}$ operators and the algebra involving the e_k,e_l generators of the GCA

$$\exp(i\hat{x}_k/R_k)\ \exp(i\hat{x}_l/R_l)\ =\ \exp(i\frac{\Theta_{kl}L_P^2}{R_jR_k}\mathbf{1})\ \exp(i\hat{x}_l/R_k)\ \exp(i\hat{x}_k/R_l)\ \Leftrightarrow$$

$$e_k\ e_l\ =\ \exp(\frac{2\pi i}{N}T_{kl})\ e_l\ e_k\ \Rightarrow\ \frac{2\pi}{N}T_{kl}\ \leftrightarrow\ \frac{\Theta_{kl}L_P^2}{R_jR_k}\mathbf{1} \tag{49}$$

The correspondence $e_j^N\ =\ e\ \to\ e^{iN\hat{x}_j/R_j}\ =\ \mathbf{1}\ \to\ e^{iNx_j/R_j}\ =\ 1$, implies $N\theta_j=2N_j\pi\Rightarrow\theta_j\equiv\frac{x_j}{R_j}=\frac{2N_j\pi}{N}\Rightarrow x_j=\frac{N_j}{N}(2\pi R_j)$, with $j=1,2,\ldots,d$, and $N_1,N_2,\ldots,N_d$ are integers with $N_j\le N$. A given array of chosen integers $\{N_1,N_2,\ldots,N_d\}$ corresponds to a given array of points $x_1,x_2,\ldots,x_d$ on d real lines covering the d circles $S^1\times S^1\ldots\times S^1$ of the fuzzy torus $\mathbf{T}^d_\Theta$.

[2]The integer entries of the matrix $\mathbf{T}$ must not divide N if one is to assume that the generators e_i do not commute

The **c**-number $\theta_j = \frac{N_j}{N}(2\pi) \leq 2\pi$ is the angle defined by $\frac{x_j}{R_j}$ where x_j is the *classical* coordinate variable associated with the noncommutative coordinate operator $\hat{x}_j$. For example, setting $N_j = 1$ for all values of $j = 1, 2, \ldots, d$, one has that the classical coordinates of a point P inside a d-dim cell are $P = \frac{2\pi}{N}(R_1, R_2, \ldots, R_d)$. For other values of N_j one has that the coordinates of the array of points inside the d-dim cell are $\frac{2\pi}{N}(N_1 R_1, N_2 R_2, \ldots, N_d R_d)$.

It is well known that to define an angle operator $\hat{\theta}$ in Quantum Mechanics is problematic due that angles wrap around circles but a standard operator requires a non-periodic, continuous spectrum, leading to contradictions. Secondly, if one tries to precisely define an angular operator $\hat{\theta}$ it would require an infinite angular momentum uncertainty due to the uncertainty principle. Namely, a particle on a circle or sphere cannot have a perfectly defined angle and angular momentum simultaneously. For this reason one must have the correspondence described in the manner as shown above where the angles are classical commuting quantities (**c**-numbers).

There are three interesting cases to explore from the last terms in eq-(49) :

Case (i) : If one sets the radii of the circles $S^1 \times S^1 \times \ldots \times S^1$ of the torus $\mathbf{T}^d_\Theta$ to be all equal to the Planck length L_P, then the noncommutativity antisymmetric matrix is given by $\Theta_{kl} = \frac{2\pi}{N} T_{kj}$. In the $N \to \infty$ limit, for finite matrix entries in T_{jk}, then $\Theta_{kl} \to 0$ and one recovers the classical limit leading to commutative coordinates. Thus $1/N$ plays in this case the role of the discretized version of Planck's constant $\hbar$.

Case (ii) : When the areas are *quantized* in Planck length units as $R_k R_l = N L_P^2 \Rightarrow 2\pi T_{kl} = \Theta_{kl}$, and in this case the matrix entries of Θ_{jk} no longer bear a relation with N such that Θ_{jk} is no longer zero in the $N \to \infty$ limit.

Case (iii) : The $L_P \to 0$ limit also leads to $N \to \infty$ when $R_1 R_2 \neq 0$. In units of $\hbar = c = 1$, the Planck length-squared in $D = 4$ spacetime dimensions is $L_P^2 = G_N$. Hence, $L_P \to 0$ implies $G_N \to 0$. The author [27] has recently emphasized that the quantum gravity regime is characterized by a finite Newton constant G_N, with the semiclassical limit corresponding to $G_N \to 0$. Geometric concepts fundamental to classical gravity and quantum field theory in curved spacetime—such as spacetime geometry, causal structure, notions of time, and local regions—are sharply defined only in the $G_N \to 0$ limit. The challenge in quantum gravity is to understand how geometry becomes "fuzzy"as one moves away from the semiclassical regime [27] .

Furthermore, Liu argued [27] that this latter perspective is the natural one in the context of Maldacena's AdS/CFT duality, where quantum gravity in an asymptotically anti-de Sitter (AdS) spacetime is described by a conformal field theory (CFT) living on its boundary. The quantum gravity regime with a finite G_N corresponds to a CFT with a finite value of the N parameter characterizing the number of degrees of freedom, and the semi-classical $G_N \to 0$ limit corresponds to taking $N \to \infty$. Despite that the meaning of the parameter N is $different$ from the one studied in this work, in both cases the $G_N \to 0$ limit corresponds to a $N \to \infty$ limit.

Concluding, when the entries of the antisymmetric matrix Θ_{kl} are comprised of *constants*, one has found a direct relation between the GCAs described by eqs-(46,47) and the noncommutative algebra of the fuzzy d-dim torus $\mathbf{T}^d_\Theta$ described by the first line in eq-(49).

GCA's also appear in condensed matter physics. Given an electron of charge e and mass m moving in a crystal lattice under the influence of an external constant homogeneous magnetic field $\vec{B}$, the Hamiltonian is not invariant under the lattice translation group, but now the invariance group is the so-called magnetic translation group with its generators τ_1, τ_2, τ_3 obeying a GCA [2] (section 11)

$$\tau_j \; \tau_k \; = \; \exp(i\Theta_{jk}) \; \tau_k \; \tau_j \tag{50}$$

where Θ_{jk} is proportional to the magnetic flux $\vec{B}\cdot(\vec{a}_j \times \vec{a}_k)$ through the plane $\vec{a}_j \times \vec{a}_k$ spanned by the primitive lattice vectors $\vec{a}_j, \vec{a}_k = \vec{a}_1, \vec{a}_2, \vec{a}_3$. The generators are plane-wave like $\tau_j = \exp(i\vec{a}_j \cdot (\vec{p} - e\vec{A}))$, where $\vec{A}$ is the magnetic vector potential and $\vec{p}$ is the electron momentum.

GCAs have also found applications in the study of quantum groups when q is a primitive root of unity [17]. An $N \times N$ linear transformation matrix M acting on the noncommutative N-dimensional Manin vector space and its dual is a member of the quantum group $GL_q(N)$ if its noncommuting elements M_{jk} satisfy certain commutation relations that are already GCA-like, or Heisenberg-Weyl-ike [2], [17]. This formalism was extended to the two-parameter quantum group $GL_{p,q}(2)$ and the two-parameter quantum supergroup $GL_{p,q}(1|1)$ [18].

4 Conclusions and Outlook

To summarize the results of this work :

In section **1** a careful study of the simplest family of GCAs associated with the N-th root of unity in d-dimensions allowed us to show that the following generalized anti-commutator with N entries leads to

$$\{e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_N}\} \; = \; e_{i_1} e_{i_2} e_{i_3} \ldots e_{i_N} \; + \; permutations \; = \; N! \; \eta_{i_1 i_2 \ldots i_N} \; e$$

where all the $N!$ terms of the permutations appear with the same positive sign. The components of the rank-N metric are $\eta_{i_1 i_2 \ldots i_N} = 1$, iff $i_1 = i_2 = i_3 \ldots = i_N$, and 0 otherwise. The range of the indices $i_1, i_2, \ldots i_N$ is $1, 2, \ldots, d$. The curved space version of the generalized anti-commutator relation was provided by eq-(12b). We proceeded to eq-(11) which provided the justification to explore the N-th norm extensions of the quadratic norm and to write down a generalized Finsler-like arc-length based on the rank-N metric and displayed in eqs-(15a,15b).

In section **2** we constructed the different expressions of the Dirac operators associated with the (Generalized) Clifford Spaces in connection with the study of Spectral Geometry in Noncommutative Geometry, and which is based on the correspondence between the geodesic distance and the inverse of the Dirac operator (a fermion propagator). It remains to study further the versions of the spectral actions in the (Generalized) Clifford Spaces built from these Dirac operators and to explore their physical significance.

The generalized anti-commutators (12a,12b) are special types of an N-ary algebraic structure. N-ary algebras have all sorts of applications in Mathematics and Physics [19]. Nambu mechanics is based on a Jacobian which is an extension of the Poisson bracket [20]. The N-ary bracket was essential in the construction of closed String Field Theory [21] and whose mathematical framework relies on the notion of *operads* developed by [22], and on strong homotopy algebras Lie algebras L_∞ [23]. Tensorial coordinates are natural in (Generalized) Clifford Spaces as shown in eqs-(17,22,23). They also appear in the work of [24] based on the infinite-dimensional E_{11} algebra.

In section **3** we displayed the relations (45,49) among the GCAs and the algebras underlying the noncommutative fuzzy torus in $d \geq 2$ and discussed applications in condensed matter and quantum groups [2]. The authors [25] have shown that metastring theory gives rise to a quantum structure coined "modular space-time". These modular spaces are compact cells in phase space and carry units of symplectic flux, exactly in the same way as the relations displayed in eqs-(45,50), which are special cases of the more general equations depicted in eq-(49). The findings by [25] were based on the work on modular variables in Quantum Theory by [26]. For all these reasons described in this work, we hope that GCA's will play an important role in future developments in Mathematics and Physics.

Acknowledgements

We thank M. Bowers for very kind assistance.

Data Availability Statement : The manuscript contains no data.

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