

REVEALING A DIRECT ALGEBRAIC APPROACH TO 2×2 MATRIX SQUARE ROOTS

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ABSTRACT. Often renowned as the greatest human invention in history, the laws of matrices have great contributions to calculus, linear transformations, computer algorithms, to name but a few. However, differ from research topics such as Non-Euclidean Geometry and Partial Differential Equations, the fractional indices of matrices have barely garnered our attention. In this study, we would investigate the properties of matrices, focusing on the nature of the square root of two by two matrices, some popular approaches, and most importantly reveal a set of new formulas that could typically solve all the given scenarios.

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1. INTRODUCTION

The importance of the square root of 2×2 matrices lies in that it has a multitude of modern applications, quantum mechanics, statistics, machine learning, and the list goes on.

Similar to the square root of real numbers, the square root of matrices shares the same definition — when multiplied by itself, results in returning the original matrix (i.e. if $A = BB = B^2$, then matrix B is said to be one of the square roots of matrix A). Nonetheless, in certain cases we may obtain none or infinitely many solutions, for instance the square root of an identity matrix I . Whether the root exists, all hinges on the eigenvalues and the diagonalizability of the matrix.

Definition 1.1. Every 2×2 diagonalizable matrix have existing square root solutions:

| For diagonalizable matrices | | |
|---|---------------------|------------------------|
| Eigenvalues condition | Real solution exist | Complex Solution exist |
| λ_1 and $\lambda_2 \geq 0$ | True | True |
| λ_1 or $\lambda_2 < 0$, $\lambda_1 \neq \lambda_2$ | False | True |
| λ_1 or $\lambda_2 < 0$, $\lambda_1 = \lambda_2$ | True | True |

Definition 1.2. For a non-diagonalizable or defective 2×2 matrix, the existence of a square root depends on its Jordan form.

| For non-diagonalizable matrices | |
|--|---|
| Situation | Real/Complex solution |
| Jordan block $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \lambda \neq 0$ | True $\sqrt{J} = \begin{pmatrix} \sqrt{\lambda} & \frac{1}{2\sqrt{\lambda}} \\ 0 & \sqrt{\lambda} \end{pmatrix}, \lambda \neq 0$ |
| Nilpotent (<i>Jordan block with $\lambda=0$</i>) | False |
| Zero matrix | True (<i>Trivial solution</i>) |

As for the number of roots of a 2×2 matrix, owing to the algebraic symmetry and matrix structure, there can only be 0,2,4 or infinite.

Definition 1.3. For all 2×2 diagonalizable matrices, excluding the scalar multiple of the identity matrix and the zero matrix, if the eigenvalues are distinct, then there would be 4 distinct square roots.

Definition 1.4. For all 2×2 non-diagonalizable matrices, if there are non-zero eigenvalues, then there would be 2 square roots, else zero (such as nilpotent).

One of the uses from the equations I derived is to find the number of roots, please refer to **Proposition 3.25.** for a more detailed explanation.

2. COMMON APPROACHES TO FINDING THE SQUARE ROOTS

Having discussed the fundamental concepts, it is important to explore the various computational methods developed by mathematicians over the years to determine the square roots of matrices. Currently, the most widely used techniques include the Diagonalization method, the Jordan Form method, the Iterative method, and the Cayley-Hamilton theorem approach. (Be noted that these are not all the existing approaches, I only listed those most commonly known)

2.1. Diagonalization Method

This method applies to diagonalizable matrices, i.e., matrices that can be expressed as $A = PDP^{-1}$, where D is a diagonal matrix of eigenvalues and P is the matrix of corresponding eigenvectors. The square root is then $\sqrt{A} = P\sqrt{D}P^{-1}$, where \sqrt{D} contains the square roots of the eigenvalues.

Example 2.1.1.

Consider the matrix:

$$A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$$

Its eigenvalues are

$$\lambda_1 = 6 \text{ and } \lambda_2 = 2$$

with eigenvectors

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Thus,

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$$

The square root is:

$$\sqrt{A} = \pm P \begin{pmatrix} \sqrt{6} & 0 \\ 0 & \sqrt{2} \end{pmatrix} P^{-1} = \pm \frac{1}{2} \begin{pmatrix} \sqrt{6} + \sqrt{2} & \sqrt{6} - \sqrt{2} \\ \sqrt{6} - \sqrt{2} & \sqrt{6} + \sqrt{2} \end{pmatrix}$$

However, this method, as shown, can only return us two solutions once at a time and is defective to non-diagonalizable matrices. To acquire the other pair of solutions, you would need to find the other roots of \sqrt{D} and redo the matrix multiplication.

2.2 Jordan Form Method

For non-diagonalizable matrices, the Jordan decomposition $A = PJP^{-1}$ is used, where J consists of Jordan blocks. The square root is computed block-wise.

Example 2.2.1.

Consider the defective matrix:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Its Jordan form is itself ($J = A$), and its square root can be computed using the binomial expansion for

$$(I + N)^{1/2}$$

where

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then:

$$\sqrt{A} = I + \frac{1}{2}N = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix}$$

(For a more detailed derivation, please refer to appendix 7.1. **Derivation of the binomial expansion for Jordan form**)

However, this method is computationally intensive for large matrices with many Jordan blocks.

2.3. Iterative Methods

[1] These include numerical techniques like the **Newton-Raphson method**, which iteratively refines an initial guess. The recurrence relation is:

$$X_{k+1} = \frac{1}{2}(X_k + AX_k^{-1})$$

where

$$\sqrt{A} = \lim_{k \rightarrow \infty} X_k$$

Example 2.3.1.

To find \sqrt{A} where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

start with $X_0 = A$:

First iteration:

$$X_1 = \frac{1}{2}(X_0 + AX_0^{-1}) \approx \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix}$$

After several iterations, it converges to the true square root.

This method is often praised for its simplicity. Nonetheless, you can only get 2 roots per attempt, which is less effective when the matrix consists of multiple roots (such as 4). Moreover, it is sensitive to the initial guess.

2.4. Cayley-Hamilton Theorem Approaches

This method uses the characteristic polynomial of A to express \sqrt{A} as a polynomial in A .

Example 2.4.1.

For

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

the characteristic polynomial is:

$$p(\lambda) = \det(A - \lambda I) = (\lambda - 3)^2 = 0$$

Assuming

$$\sqrt{A} = \alpha I + \beta A$$

we get:

$$\sqrt{\lambda} = \alpha + \beta\lambda \quad [3]$$

$$\frac{\partial}{\partial \lambda} \sqrt{\lambda} = \frac{\partial}{\partial \lambda} (\alpha + \beta\lambda)$$

$$\beta = \frac{1}{2\sqrt{\lambda}}$$

$$\alpha = \sqrt{\lambda} - \frac{\lambda}{2\sqrt{\lambda}} = \frac{\sqrt{\lambda}}{2}$$

$$\therefore \lambda = 3$$

$$\therefore \sqrt{A} = \frac{\sqrt{3}}{2} I + \frac{\sqrt{3}}{6} A = \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{6} \\ 0 & \sqrt{3} \end{pmatrix}$$

[2] Additionally, there is an alternative formula for this approach:

$$\sqrt{A} = \frac{A \pm \sqrt{\det(A)} I}{\sqrt{\text{tr}(A) + 2\sqrt{\det(A)}}}$$

Example 2.4.2.

$$\because \operatorname{tr}(A) = 6, \quad \det(A) = 9$$

$$\sqrt{\operatorname{tr}(A) + 2\sqrt{\det(A)}} = \sqrt{6 + 2\sqrt{9}} = 2\sqrt{3}$$

$$A \pm \sqrt{\det(A)}I = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \pm \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 0 & 6 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \sqrt{A} = \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{6} \\ 0 & \sqrt{3} \end{pmatrix} \text{ or } \begin{pmatrix} 0 & \frac{\sqrt{3}}{6} \\ 0 & 0 \end{pmatrix} \text{ (rejected)}$$

Aside from this root, there is also a negative variant(in this scenario there should be two roots in total)

All in all, each aforementioned method has its pros and cons, but we would compare them, including my equations, further in **4. Advantages and Disadvantages.**

3. DERIVATION OF MY FORMULAS

In the run-up to starting, I have to admit my derivation is probably by far the longest and the most messy among all those methods. To be frank, in lieu of some unique brainstorming, I actually did it with brutal force, here is how I accomplished it:

First and foremost, let's recall the formula for all the square roots of matrices:

$$M = XX = X^2, \quad X = \sqrt{M}$$

Lemma 3.1. Let

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \text{ and } X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We aim to find the formulas of a,b,c and d respectively.

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$$

$$\begin{cases} a^2 + bc = m_{11} \\ ab + bd = m_{12} \\ ac + cd = m_{21} \\ bc + d^2 = m_{22} \end{cases}$$

Now we got a system of equations, from here we can further derive by some change of subject and get:

$$\boxed{a^2 - d^2 = m_{11} - m_{22}}$$

$$\boxed{b = \frac{m_{12}}{a+d} = \frac{m_{12}(a-d)}{m_{11}-m_{22}}} \quad \boxed{c = \frac{m_{21}}{a+d} = \frac{m_{21}(a-d)}{m_{11}-m_{22}}}$$

Undoubtedly, only using these formulas would not be enough to solve the system of equations, therefore, we need to try other valid combinations and permutations. To save time, this is what I chose :

$$X = MX^{-1}$$

Lemma 3.2. Assume the relationship to be correct for some matrices M , then:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ac-bd} \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$R. H. S. = \frac{1}{ac-bd} \begin{pmatrix} m_{11}d - m_{12}c & -m_{11}b + m_{12}a \\ m_{21}d - m_{22}c & -m_{21}b + m_{22}a \end{pmatrix}$$

$$\frac{a}{c} = \frac{m_{11}d - m_{12}c}{m_{21}d - m_{22}c}$$

Substitute the equation of c from above:

$$a = \frac{m_{11}d - m_{12}\left(\frac{m_{21}(a-d)}{m_{11}-m_{22}}\right)}{m_{21}d - m_{22}\left(\frac{m_{21}(a-d)}{m_{11}-m_{22}}\right)} \left(\frac{m_{21}(a-d)}{m_{11}-m_{22}}\right)$$

Now simplify:

$$a(m_{11} - m_{22}) = \frac{m_{11}d - m_{12}\left(\frac{m_{21}(a-d)}{m_{11}-m_{22}}\right)}{d - m_{22}\left(\frac{a-d}{m_{11}-m_{22}}\right)} (a-d)$$

$$a(m_{11} - m_{22}) = \frac{d(m_{11}^2 - m_{11}m_{22}) - m_{12}m_{21}(a-d)}{d(m_{11} - m_{22}) - m_{22}(a-d)} (a-d)$$

$$ad(m_{11} - m_{22})^2 - (a^2 - ad)(m_{11}m_{22} - m_{22}^2) = (ad - d^2)(m_{11}^2 - m_{11}m_{22}) - m_{12}m_{21}(a-d)^2$$

As you can see, the maximum degree here is 2, so let us try to convert it into a quadratic equation by making a the subject:

$$L. H. S. = a^2(m_{22}^2 - m_{11}m_{22}) + ad(m_{11} - m_{22})^2 + ad(m_{11}m_{22} - m_{22}^2)$$

$$L. H. S. = a^2(m_{22}^2 - m_{11}m_{22}) + ad(m_{11}^2 - m_{11}m_{22})$$

$$R. H. S. = -a^2m_{12}m_{21} + ad(m_{11}^2 - m_{11}m_{22} + 2m_{12}m_{21}) - d^2(m_{11}^2 - m_{11}m_{22} + m_{12}m_{21})$$

$$a^2(m_{22}^2 - m_{11}m_{22} + m_{12}m_{21}) - ad(2m_{12}m_{21}) + d^2(m_{11}^2 - m_{11}m_{22} + m_{12}m_{21}) = 0$$

Lemma 3.3. By quadratic formula:

$$a = \frac{2dm_{12}m_{21} \pm 2d\sqrt{(m_{12}m_{21})^2 - (m_{22}^2 - m_{11}m_{22} + m_{12}m_{21})(m_{11}^2 - m_{11}m_{22} + m_{12}m_{21})}}{2(m_{22}^2 - m_{11}m_{22} + m_{12}m_{21})}$$

$$a = \left(\frac{m_{12}m_{21} \pm \sqrt{(m_{12}m_{21})^2 - m_{11}^2m_{22}^2 - (m_{11}^2 + m_{22}^2)(m_{12}m_{21} - m_{11}m_{22}) - (m_{12}m_{21} - m_{11}m_{22})^2}}{m_{22}^2 - m_{11}m_{22} + m_{12}m_{21}} \right) d$$

$$a = \left(\frac{m_{12}m_{21} \pm \sqrt{2m_{11}m_{12}m_{21}m_{22} - (m_{11}^2 + m_{22}^2)(m_{12}m_{21} - m_{11}m_{22}) - 2(m_{11}m_{22})^2}}{m_{22}^2 - m_{11}m_{22} + m_{12}m_{21}} \right) d$$

Recall the determinant:

$$a = \left(\frac{m_{12}m_{21} \pm \sqrt{2m_{11}m_{22}(m_{11}m_{22} - \det M) + (m_{11}^2 + m_{22}^2)(\det M) - 2(m_{11}m_{22})^2}}{m_{22}^2 - \det M} \right) d$$

$$a = \left(\frac{m_{12}m_{21} \pm \sqrt{(m_{11}^2 + m_{22}^2)(\det M) - 2m_{11}m_{22}(\det M)}}{m_{22}^2 - \det M} \right) d$$

$$a = \left(\frac{m_{12}m_{21} \pm \sqrt{(m_{11}^2 - 2m_{11}m_{22} + m_{22}^2)(\det M)}}{m_{22}^2 - \det M} \right) d$$

$$a = \left(\frac{m_{12}m_{21} \pm |m_{11} - m_{22}|\sqrt{\det M}}{m_{22}^2 - \det M} \right) d$$

Let r be the ratio between a and d , i.e.:

$$\boxed{r = \frac{a}{d} = \frac{m_{12}m_{21} \pm |m_{11} - m_{22}|\sqrt{\det M}}{m_{22}^2 - \det M}}$$

(Below is an alternative expression for r , you may refer to Appendix 7.2. for the derivation)

$$r^2 = \frac{m_{12}m_{21} \pm |m_{11} - m_{22}|\sqrt{\det M}}{m_{12}m_{21} \mp |m_{11} - m_{22}|\sqrt{\det M}} \frac{m_{11}^2 - \det M}{m_{22}^2 - \det M}$$

Now we have two equations for a and d :

$$\begin{cases} a^2 - d^2 = m_{11} - m_{22} \\ a = rd \end{cases}$$

Lemma 3.4. Solving the equations, we get:

$$d = \pm \sqrt{\frac{m_{11} - m_{22}}{r^2 - 1}}$$

$$a = \pm r \sqrt{\frac{m_{11} - m_{22}}{r^2 - 1}}$$

Further derive:

$$b = \pm m_{12} \sqrt{\frac{r-1}{(m_{11} - m_{22})(r+1)}} \quad c = \pm m_{21} \sqrt{\frac{r-1}{(m_{11} - m_{22})(r+1)}}$$

Theorem 3.5. The relationship is therefore as follows:

$$\therefore \sqrt{M} = \pm \begin{pmatrix} r \sqrt{\frac{m_{11} - m_{22}}{r^2 - 1}} & m_{12} \sqrt{\frac{r-1}{(m_{11} - m_{22})(r+1)}} \\ m_{21} \sqrt{\frac{r-1}{(m_{11} - m_{22})(r+1)}} & \sqrt{\frac{m_{11} - m_{22}}{r^2 - 1}} \end{pmatrix}$$

Testing with

$$M = \begin{pmatrix} 19 & 39 \\ 65 & 136 \end{pmatrix}$$

$$r = \frac{2}{11}, \frac{4}{43}$$

$$X_1 = \pm \begin{pmatrix} 2 & 3 \\ 5 & 11 \end{pmatrix}$$

$$X_2 = \pm \begin{pmatrix} \frac{4}{43} \sqrt{\frac{5547}{47}} & 39 \sqrt{\frac{1}{141}} \\ 65 \sqrt{\frac{1}{141}} & \sqrt{\frac{5547}{47}} \end{pmatrix}$$

$$X_1^2 = \begin{pmatrix} 2 & 3 \\ 5 & 11 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 11 \end{pmatrix} = \begin{pmatrix} 19 & 39 \\ 65 & 136 \end{pmatrix}$$

$$X_2^2 = \begin{pmatrix} \frac{4}{43} \sqrt{\frac{5547}{47}} & 39 \sqrt{\frac{1}{141}} \\ 65 \sqrt{\frac{1}{141}} & \sqrt{\frac{5547}{47}} \end{pmatrix} \begin{pmatrix} \frac{4}{43} \sqrt{\frac{5547}{47}} & 39 \sqrt{\frac{1}{141}} \\ 65 \sqrt{\frac{1}{141}} & \sqrt{\frac{5547}{47}} \end{pmatrix} = \begin{pmatrix} 19 & 39 \\ 65 & 136 \end{pmatrix}$$

$$\therefore X_1^2 = M \text{ and } X_2^2 = M$$

$$\therefore \sqrt{M} = X_1 \text{ or } X_2$$

Corollary 3.6.

The equation above is based on the situation when

$$m_{22}^2 \neq \det M$$

If they're equal, then the denominator of r would converge to zero, the equation malfunctions, therefore we need a second formula.

Lemma 3.7. We retrieve the step:

$$a^2(m_{22}^2 - m_{11}m_{22} + m_{12}m_{21}) - ad(2m_{12}m_{21}) + d^2(m_{11}^2 - m_{11}m_{22} + m_{12}m_{21}) = 0$$

If this time we do quadratic equation with respect to d , then supposingly:

$$d = \left(\frac{m_{12}m_{21} \pm |m_{11} - m_{22}| \sqrt{\det M}}{m_{11}^2 - \det M} \right) a$$

Let

$$r_{-1} = \frac{d}{a} = \frac{m_{12}m_{21} \mp |m_{11} - m_{22}| \sqrt{\det M}}{m_{11}^2 - \det M}$$

(The plus or minus sign here is relative to r , i.e. if r use +, then r_{-1} use -)

$$\therefore a^2 - d^2 = m_{11} - m_{22}$$

$$\therefore a^2(1 - r_{-1}^2) = m_{11} - m_{22}$$

$$\boxed{\therefore a = \pm \sqrt{\frac{m_{11} - m_{22}}{1 - r_{-1}^2}}}$$

$$\boxed{\therefore d = \pm r_{-1} \sqrt{\frac{m_{11} - m_{22}}{1 - r_{-1}^2}}}$$

$$b = \pm m_{12} \sqrt{\frac{1 - r_{-1}}{(m_{11} - m_{22})(r_{-1} + 1)}} \quad c = \pm m_{21} \sqrt{\frac{1 - r_{-1}}{(m_{11} - m_{22})(r_{-1} + 1)}}$$

Theorem 3.8.

$$\sqrt{M} = \pm \begin{pmatrix} \sqrt{\frac{m_{11} - m_{22}}{1 - r_{-1}^2}} & m_{12} \sqrt{\frac{1 - r_{-1}}{(m_{11} - m_{22})(r_{-1} + 1)}} \\ m_{21} \sqrt{\frac{1 - r_{-1}}{(m_{11} - m_{22})(r_{-1} + 1)}} & r_{-1} \sqrt{\frac{m_{11} - m_{22}}{1 - r_{-1}^2}} \end{pmatrix}$$

This formula's function is exactly the same as the previous one but uses a different expression in order to avoid the denominator from approaching zero.

Now, can you help find the square roots of the following matrix using my formulas?

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Corollary 3.9. Concerns may raise, if:

$$m_{11} = -m_{22} \quad \text{and} \quad m_{22}^2 - \det M = 0$$

It will suffer the same problem as **Theorem 3.5.**, or when:

$$m_{11} = m_{22}$$

The equations from **Theorem 3.5.** and **Theorem 3.8.** will also malfunction. Fortunately, we can still tackle these problems by taking limits depending on the situation (unless the target is a nilpotent), here is the computation of our third and fourth equation:

For:

$$m_{11} = -m_{22} \quad \text{and} \quad m_{22}^2 - \det M = 0$$

Lemma 3.10 With reference to the alternative equation for r , let:

$$r^2 = \lim_{m_{11} \rightarrow -m_{22}} \left[\frac{m_{12}m_{21} \pm |m_{11} - m_{22}| \sqrt{\det M}}{m_{12}m_{21} \mp |m_{11} - m_{22}| \sqrt{\det M}} \frac{m_{11}^2 - \det M}{m_{22}^2 - \det M} \right]$$

$$r^2 = \left[\frac{m_{12}m_{21} \pm 2|m_{22}| \sqrt{m_{22}^2}}{m_{12}m_{21} \mp 2|m_{22}| \sqrt{m_{22}^2}} \frac{m_{22}^2 - \det M}{m_{22}^2 - \det M} \right]$$

$$r^2 = \frac{m_{12}m_{21} \pm 2m_{22}^2}{m_{12}m_{21} \mp 2m_{22}^2}$$

$$(r_{-1}^2 = \frac{m_{12}m_{21} \mp 2m_{22}^2}{m_{12}m_{21} \pm 2m_{22}^2})$$

Question 3.11.

Why is this capable of solving the case when:

$$m_{11} = -m_{22} \text{ and } m_{22}^2 - \det M = 0$$

Theorem 3.12.

The answer lies in the plus or minus sign, unless

$$m_{11} = m_{22} = 0$$

Then the numerator and denominator would have different values, this ensures that only one of them can approach 0 in the worst case. What's more is if r fails due to the denominator being 0, we can choose to use r_{-1} and its corresponding expression of the square root matrix instead.

Example 3.13.

$$M = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix}$$

It obeys:

$$m_{11} = -m_{22} \text{ and } m_{22}^2 - \det M = 0$$

Commence with finding r :

$$r^2 = \frac{-8 \pm 8}{-8 \mp 8}$$

Taking the plus sign for the numerator and minus sign for the denominator:

$$r^2 = \frac{-8 + 8}{-8 - 8} = \frac{0}{-16} = 0$$

However, as you can see we cannot do the opposite (The denominator will approach 0), so we are going to apply the inverse r .

$$r_{-1}^2 = \frac{-8 + 8}{-8 - 8} = \frac{0}{-16} = 0$$

Now we got the values of r and r inverse, apply them to **Theorem 3.5.** and **Theorem 3.8.** respectively
For $r = 0$:

$$\sqrt{M} = \pm \begin{pmatrix} 0 & 4\sqrt{\frac{-1}{(-2-2)(1)}} \\ -2\sqrt{\frac{-1}{(-2-2)(1)}} & \sqrt{\frac{-2-2}{-1}} \end{pmatrix} = \pm \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2(-1) & 0 + 2(2) \\ 0 + 2(-1) & 2(2) + 2(-1) \end{pmatrix} = M$$

For $r_{-1} = 0$:

$$\sqrt{M} = \pm \begin{pmatrix} \sqrt{\frac{-2-2}{1}} & 4\sqrt{\frac{1}{(-2-2)(1)}} \\ -2\sqrt{\frac{1}{(-2-2)(1)}} & 0 \end{pmatrix} = \pm i \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$$

$$i^2 \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} 2(2) + 1(-2) & 2(-2) + 0 \\ 2(1) + 0 & 0 + 1(-2) \end{pmatrix} = M$$

In case you ask me the sign convention, please remember that:

$$\sqrt{\frac{1}{-1}} = -\sqrt{-1}$$

(This is important but it is a very common mistake)

Theorem 3.14.

Actually, since this condition automatically make $r = 0$ and $r_{-1} = 0$, we can simplify the expression of the square root:

$$\sqrt{M} = \pm \begin{pmatrix} 0 & m_{12}\sqrt{\frac{1}{2m_{22}}} \\ m_{21}\sqrt{\frac{1}{2m_{22}}} & \sqrt{2m_{22}} \end{pmatrix} \text{ or } \pm \begin{pmatrix} \sqrt{2m_{11}} & m_{12}\sqrt{\frac{1}{2m_{11}}} \\ m_{21}\sqrt{\frac{1}{2m_{11}}} & 0 \end{pmatrix}$$

Question 3.15.

How does the r and r_{-1} formula complement each other, covering all the cases when m_{11} and m_{22} are not equal?

Proposition 3.16.

Last but not least, we got our final obstacle to solve, which is when m_{11} and m_{22} are equal.

We can derive our final equation by taking limits, and introducing a mathematical tool called Taylor series, which can be used to expand and approximate functions in specific cases.

Definition 3.17. Taylor series formula:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n$$

Lemma 3.18. (Neglect the plus or minus sign for a while) Let

$$d = \lim_{m_{11} \rightarrow m_{22}} \sqrt{\frac{m_{11} - m_{22}}{r^2 - 1}} \quad \text{and} \quad m_{11} = \lim_{\epsilon \rightarrow 0} (m_{22} + \epsilon)$$

Expanding it:

$$r = \lim_{\epsilon \rightarrow 0} \frac{m_{12}m_{21} \pm |\epsilon| \sqrt{m_{22}^2 + m_{22}\epsilon - m_{12}m_{21}}}{m_{22}^2 - (m_{22}^2 + m_{22}\epsilon - m_{12}m_{21})} = \lim_{\epsilon \rightarrow 0} \frac{m_{12}m_{21} \pm |\epsilon| \sqrt{m_{22}^2 + m_{22}\epsilon - m_{12}m_{21}}}{m_{12}m_{21} - m_{22}\epsilon}$$

Lemma 3.19. Approximate it with Taylor series:

$$\lim_{\epsilon \rightarrow 0} \sqrt{m_{22}^2 + m_{22}\epsilon - m_{12}m_{21}} \approx \lim_{\epsilon \rightarrow 0} \left(\sqrt{m_{22}^2 - m_{12}m_{21}} + \frac{m_{22}\epsilon}{2\sqrt{m_{22}^2 - m_{12}m_{21}}} \right) = \sqrt{m_{22}^2 - m_{12}m_{21}}$$

So the expression becomes:

$$r \approx \lim_{\epsilon \rightarrow 0} \frac{m_{12}m_{21} \pm |\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21} - m_{22}\epsilon}$$

$$r \approx \lim_{\epsilon \rightarrow 0} \frac{1 \pm \frac{|\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}}}{1 - \frac{m_{22}\epsilon}{m_{12}m_{21}}}$$

Again by Taylor approximation:

$$\because x \rightarrow 0, \quad \frac{1}{1-x} \approx 1+x$$

$$\therefore r \approx \lim_{\epsilon \rightarrow 0} \left(1 \pm \frac{|\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right) \left(1 + \frac{m_{22}\epsilon}{m_{12}m_{21}} \right)$$

Only taking the zeroth and first order:

$$r \approx \lim_{\epsilon \rightarrow 0} \left(1 + \frac{m_{22}\epsilon}{m_{12}m_{21}} \pm \frac{|\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right)$$

$$r^2 \approx \lim_{\epsilon \rightarrow 0} \left[\left(1 + \frac{m_{22}\epsilon}{m_{12}m_{21}} \right)^2 \pm 2 \left(1 + \frac{m_{22}\epsilon}{m_{12}m_{21}} \right) \left(\frac{|\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right) \right]$$

$$r^2 \approx \lim_{\epsilon \rightarrow 0} \left[\left(1 + \frac{2m_{22}\epsilon}{m_{12}m_{21}} \right) \pm 2 \left(\frac{|\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right) \right]$$

$$r^2 - 1 \approx 2 \lim_{\epsilon \rightarrow 0} \left(\frac{m_{22}\epsilon \pm |\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right)$$

$$\therefore d \approx \lim_{\epsilon \rightarrow 0} \sqrt{\frac{\epsilon}{2 \left(\frac{m_{22}\epsilon \pm |\epsilon| \sqrt{m_{22}^2 - m_{12}m_{21}}}{m_{12}m_{21}} \right)}}$$

$$d \approx \lim_{\epsilon \rightarrow 0} \sqrt{\frac{m_{12}m_{21}}{2 \left(m_{22} \pm \operatorname{sgn}(\epsilon) \sqrt{m_{22}^2 - m_{12}m_{21}} \right)}}$$

(where sgn is the signum function)

Lemma 3.20. Rationalize it:

$$d \approx \sqrt{\frac{m_{12}m_{21} \left(m_{22} \mp \sqrt{m_{22}^2 - m_{12}m_{21}} \right)}{2 \left(m_{22} \pm \sqrt{m_{22}^2 - m_{12}m_{21}} \right) \left(m_{22} \mp \sqrt{m_{22}^2 - m_{12}m_{21}} \right)}}$$

$$d \approx \sqrt{\frac{m_{12}m_{21} \left(m_{22} \mp \sqrt{m_{22}^2 - m_{12}m_{21}} \right)}{2 \left(m_{22}^2 - (m_{22}^2 - m_{12}m_{21}) \right)}}$$

$$\boxed{\therefore d \approx \sqrt{\frac{m_{22} \mp \sqrt{\det M}}{2}}}$$

and

$$\therefore m_{11} = m_{22}, \quad a^2 + bc = bc + d^2$$

(for all matrices)

$$\boxed{\therefore a = \pm d}$$

If $a = +d$:

$$b = \frac{m_{12}}{2d} \approx \frac{m_{12}}{\sqrt{2(m_{22} \mp \sqrt{\det M})}}$$

$$c = \frac{m_{21}}{2d} \approx \frac{m_{21}}{\sqrt{2(m_{22} \mp \sqrt{\det M})}}$$

Let

$$p = m_{22} \mp \sqrt{\det M}$$

Theorem 3.21.

$$\therefore \sqrt{M} \approx \pm \begin{pmatrix} \sqrt{\frac{p}{2}} & \frac{m_{12}}{\sqrt{2p}} \\ \frac{m_{21}}{\sqrt{2p}} & \sqrt{\frac{p}{2}} \end{pmatrix}$$

To test it, let's reuse the matrix from **Example 2.4.1.** :

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

$$p = 6 \text{ or } 0(\text{rejected})$$

$$\sqrt{A} \approx \pm \begin{pmatrix} \sqrt{3} & \frac{\sqrt{3}}{6} \\ 0 & \sqrt{3} \end{pmatrix}$$

(Which is exactly the answer, there is only two roots due to **Definition 1.4.**)

Or let's try it with a scalar multiple of an identity matrix:

$$2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$p = 4 \text{ or } 0(\text{rejected})$$

$$\sqrt{2I} \approx \pm \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} = \pm \sqrt{2}I$$

Corollary 3.22.

The scalar multiple of the identity matrix is known for having infinite number of roots, which my equation fails to describe even if rationalization is applied :

$$b \approx \frac{m_{12} \sqrt{m_{22} \pm \sqrt{\det M}}}{\sqrt{2(m_{22}^2 - \det M)}} \approx \frac{m_{12} \sqrt{m_{22} \pm \sqrt{\det M}}}{\sqrt{2(m_{12}m_{21})}}$$

If m_{12} and m_{21} are equal, similar to the case:

$$2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

We can simplify it into:

$$b \approx \sqrt{\frac{m_{22} \pm \sqrt{\det M}}{2}}$$

(same for the variable c)

From the above case, when $p = 0$:

$$\sqrt{2I} \approx \pm \begin{pmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 0 \end{pmatrix}$$

Hence the formula can at most find 4 distinct roots, failing to provide a general term on itself.

Corollary 3.23.

The formulas by **Theorem 3.5.** and **Theorem 3.10.** cannot be applied to nilpotent, scalar multiple of identity matrices and zero matrices, owing to the initial assumptions (such as $a + d$ are assumed to be non-zero in the first place).

Theorem 3.24.

Overall, if m_{11} and m_{22} aren't equal, our first or second formula:

$$\therefore \sqrt{M} = \pm \begin{pmatrix} r \sqrt{\frac{m_{11}-m_{22}}{r^2-1}} & m_{12} \sqrt{\frac{r-1}{(m_{11}-m_{22})(r+1)}} \\ m_{21} \sqrt{\frac{r-1}{(m_{11}-m_{22})(r+1)}} & \sqrt{\frac{m_{11}-m_{22}}{r^2-1}} \end{pmatrix}$$

$$\sqrt{M} = \pm \begin{pmatrix} \sqrt{\frac{m_{11}-m_{22}}{1-r_{-1}^2}} & m_{12} \sqrt{\frac{1-r_{-1}}{(m_{11}-m_{22})(r_{-1}+1)}} \\ m_{21} \sqrt{\frac{1-r_{-1}}{(m_{11}-m_{22})(r_{-1}+1)}} & r_{-1} \sqrt{\frac{m_{11}-m_{22}}{1-r_{-1}^2}} \end{pmatrix}$$

else if the 2×2 matrix is neither a scalar multiple of an identity matrix, nilpotent nor a zero matrix:

$$\therefore \sqrt{M} \approx \pm \begin{pmatrix} \sqrt{\frac{p}{2}} & \frac{m_{12}}{\sqrt{2p}} \\ \frac{m_{21}}{\sqrt{2p}} & \sqrt{\frac{p}{2}} \end{pmatrix}$$

(scalar multiple of identity matrix and zero matrix have infinite(or trivial) solutions, while nilpotent has 0)

| Equations | Targeted matrices | Condition(s) for failing |
|--|---|---|
| $r = \frac{a}{d} = \frac{m_{12}m_{21} \pm m_{11} - m_{22} \sqrt{\det M}}{m_{22}^2 - \det M}$ $\sqrt{M} = \pm \begin{pmatrix} r\sqrt{\frac{m_{11}-m_{22}}{r^2-1}} & m_{12}\sqrt{\frac{r-1}{(m_{11}-m_{22})(r+1)}} \\ m_{21}\sqrt{\frac{r-1}{(m_{11}-m_{22})(r+1)}} & \sqrt{\frac{m_{11}-m_{22}}{r^2-1}} \end{pmatrix}$ | General 2×2 matrices | $m_{11} = m_{22}$ or $m_{22}^2 = \det M$ |
| $r_{-1} = \frac{d}{a} = \frac{m_{12}m_{21} \pm m_{11} - m_{22} \sqrt{\det M}}{m_{11}^2 - \det M}$ $\sqrt{M} = \pm \begin{pmatrix} \sqrt{\frac{m_{11}-m_{22}}{1-r_{-1}^2}} & m_{12}\sqrt{\frac{1-r_{-1}}{(m_{11}-m_{22})(r_{-1}+1)}} \\ m_{21}\sqrt{\frac{1-r_{-1}}{(m_{11}-m_{22})(r_{-1}+1)}} & r_{-1}\sqrt{\frac{m_{11}-m_{22}}{1-r_{-1}^2}} \end{pmatrix}$ | General 2×2 matrices (alternative) | $m_{11} = m_{22}$ or $m_{11}^2 = \det M$ |
| $r^2 = \frac{m_{12}m_{21} \pm 2m_{22}^2}{m_{12}m_{21} \mp 2m_{22}^2}$ $r_{-1}^2 = \frac{m_{12}m_{21} \mp 2m_{22}^2}{m_{12}m_{21} \pm 2m_{22}^2}$ | 2×2 matrices when: $m_{11} = -m_{22}$ and $m_{11}^2 = m_{22}^2 = \det M$ | $m_{11} = m_{22}$ or $m_{11}^2 \neq m_{22}^2 \neq \det M$ |
| $p = m_{22} \mp \sqrt{\det M}$ $\sqrt{M} \approx \pm \begin{pmatrix} \sqrt{\frac{p}{2}} & \frac{m_{12}}{\sqrt{2p}} \\ \frac{m_{21}}{\sqrt{2p}} & \sqrt{\frac{p}{2}} \end{pmatrix}$ | 2×2 matrices when: $m_{11} = m_{22}$ (Except Nilpotent and Zero Matrices) Scalar multiple of identity matrices(Limited) | $m_{11} \neq m_{22}$ or $p = 0$ |

Proposition 3.25.

As seen from the aforementioned equations, you may realize a fascinating property: That no matter if it is r , r inverse or p , they all contain a determinant behind their plus or minus sign. Similar to the delta from quadratic equation, If the determinant approaches zero, then there would be repeated roots. Therefore, as soon as the matrix fulfills the requirements, my equations imply that if the determinant is zero, there will only be 2 distinct roots, in other words, they foresee the number of roots without the need of computing eigenvalues.

4. ADVANTAGES AND DISADVANTAGES

The computation of 2×2 matrix square roots is a fundamental problem with applications ranging from differential equations to machine learning. While several established methods exist—such as eigenvalue diagonalization, iteration methods, and the Cayley-Hamilton approach—each comes with trade-offs in terms of generality, computational efficiency, and numerical stability. In this section, we evaluate these classical techniques alongside the proposed closed-form algebraic formulas for 2×2 matrices, highlighting their respective strengths and weaknesses. Unlike iterative or eigenvalue-dependent methods, the new approach provides exact, non-iterative solutions for nearly all 2×2 cases—including defective matrices—while remaining computationally lightweight and pedagogically accessible. Below, we systematically compare these methods, emphasizing where the algebraic solution outperforms traditional algorithms and where limitations persist.

4.1. Advantages

My set of algebraic equations:

1. Simple and user-friendly: unlike other methods that require deep knowledge on eigenvalues, iteration etc. My method can be applied by direct substitution that has already been taught at secondary schools.
2. Speed and efficiency: as it only requires some basic substitutions, this creates a fast-paced way to compute square roots (By passing eigenvalues and Jordan forms) (This also favors computer algorithm to calculate the matrix square roots)
3. Robust stability: applicable to all cases except scalar multiple of identity matrices, nilpotent and zero matrices. While other methods may fail when the matrix is not diagonalizable, my algebraic equations can still remain functional. Even complex roots can be calculated.
4. Education value: the derivation is mostly based on secondary school algebra, if written into textbooks, students who have studied the basics of matrices can easily understand. In addition, some parts such as the r and inverse r duality that I invented can inspire students on problem solving and possibly unleash their creativity.
5. Theoretical insights: able to predict the maximum number of square roots of a 2×2 non-nilpotent matrix.

Diagonalization method:

1. Generalizability and stability: works for every $n \times n$ diagonalizable matrix, stands out when handling large diagonalizable matrices.
2. Education value: learning it can strengthen students' sense of the properties of eigenvalues and eigenvectors, which is the key of linear transformation.

Jordan form method:

1. Theoretical insights: able to analyze whether the square root exists
2. Generalizability: works in spite of being diagonalizable or non-diagonalizable

Iterative methods:

1. Speed and efficiency: fastest if the initial guess is close to the correct answer, moreover it bypasses the need for costly eigenvalue computations.
2. High compatibility on large matrices: reliable and critical for big data applications, such as machine learning.

Cayley-Hamilton theorem methods:

1. Simple and user-friendly: similar to my algebraic approach, this method also allows users to compute the square root by direct substitution.
2. Speed and efficiency: same as my method, eigenvalues are not necessary, making the computation fast and efficient (both are about substitution, but it is even faster than mine)
3. Education value: the derivation may require the fundamental knowledge of eigenvalues, but not affecting students to learn to use it, overall the difficulty of algebra is equivalent to my equations.
4. Robust stability: just as my equations, but can even be expanded to higher degree(which is better)

4.2. Disadvantages**My set of algebraic equations:**

1. Case dependent: before substitution, users would first need to determine the conditions and choose the appropriate equation.
2. Limited coverage: only works for 2×2 non-nilpotent matrices, can only find 4 of roots of the scalar multiple of identity matrices.(**Corollary 3.22.**) In simpler terms, the equation fails if the given 2×2 matrix has zero or infinite square roots.

Diagonalization method:

1. Limited coverage: only works for diagonalizable matrices
2. Prerequisites(intermediate): users are required to understand how to diagonalize the matrix

Jordan form method:

1. Low efficiency: impractical and difficult to compute
2. Prerequisites(high): users are required to show deep understanding and insights in Jordan forms, including but not limited to advanced binomial expansion.

Iterative methods:

1. Instability: sensitive to the initial guess
2. Prerequisites(intermediate-high): some methods such as Schur-decomposition method require advanced knowledge, while iterations like Newton's method are less sophisticated.

Cayley-Hamilton theorem approaches:

1. Low efficiency(for large matrices): computationally heavy

5. CONCLUSION

Each of the aforementioned methods yields different benefits and drawbacks, I'm not saying my equations are the best or the most elegant, to be frank they are easily beaten by Cayley-Hamilton in many aspects. However, the sole purpose of my equations is to fill the gaps of mathematics and therefore create new opportunities for mankind. At this point you may ask why I didn't expand it to cubic roots or 3×3 matrices, I tried, but let's say the work is too heavy for a secondary school student like me. My derivation method, being purely algebraic and didn't utilize the concepts of eigenvalues, made the difficulty rise exponentially (cubic equations/5 more variables). As a result, I hereby leave the job of further generalization to any math passionists who obtained greater potential and are willing to finish it for me.

In the appendix, I have provided you with a simple python matrix square root calculator, it can be used without installing any extensional python libraries.

All in all, I hope you found this paper inspirational and I sincerely apologise for any unprofessional wordings or organisations that occurred on this paper. Thank you.

6. REFERENCE

- [1] Higham, N. J. (1986) "Newton's Method for the Matrix Square Root"
 [2] Prime Newtons "Where did this come from?"  Where did this come from?
 [3] Friedberg, S. H., Insel, A. J., & Spence, L. E. (2018). Linear Algebra (5th ed.). Pearson.

7. APPENDIX

7.1. Derivation of the binomial expansion for Jordan form:

Starting from the basic formula where n is an integer:

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k, \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We then either use the recursive product formula or replace the factorials with gamma functions, applying gamma functions for generalability, we gives:

$$(1 + x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k, \quad \binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$

(if n is not an integer, the binomial expansion becomes an infinite series)

The formula above also works for matrices:

$$(I + M)^n = \sum_{k=0}^{p-1} \binom{n}{k} M^k$$

(p tends to infinity if matrix M to the power of p never becomes zero matrix, by nilpotent definition)

Let $n=1/2$:

$$(I + M)^{\frac{1}{2}} = \sum_{k=0}^{p-1} \binom{\frac{1}{2}}{k} M^k, \quad \binom{\frac{1}{2}}{k} = \frac{\Gamma(\frac{3}{2})}{\Gamma(k+1)\Gamma(\frac{3}{2}-k)}$$

If

$$M = N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

then:

$$p = 2$$

$$(I + N)^{\frac{1}{2}} = \sum_{k=0}^1 \binom{\frac{1}{2}}{k} N^k$$

$$\therefore \binom{\frac{1}{2}}{1} = \frac{\Gamma(\frac{3}{2})}{\Gamma(2)\Gamma(\frac{1}{2})} = \frac{1}{2}$$

$$\boxed{\therefore (I + N)^{\frac{1}{2}} = I + \frac{1}{2}N}$$

7.2 Alternative expression of r :

From **Lemma 3.7.** to **Theorem 3.8.** , we recognize that:

$$\therefore \frac{r}{r_{-1}} = \frac{a^2}{d^2} = r^2$$

$$\therefore r^2 = \frac{m_{12}m_{21} \pm |m_{11} - m_{22}|\sqrt{\det M}}{m_{12}m_{21} \mp |m_{11} - m_{22}|\sqrt{\det M}} \frac{m_{11}^2 - \det M}{m_{22}^2 - \det M}$$

(the same method applies to r_{-1} and its corresponding expression)

7.3. Square root matrix calculator in Python(3.12.10):

(Phrases begin with # are deletable)

```
#The formula is hereby transformed into python (Fully written by Lau Long Yat Damian)
#functional without importing numpy
#covering all 2x2 matrices that have 2 or 4 distinct roots

M=[0,0],[0,0]
P=[0,0],[0,0]
#The matrix P here is to save the original data from matrix M to find the 2nd pair of roots

for i in range (0,2):
    for j in range (0,2):
        print("Row", str(i+1), "Column", str(j+1))
        M[i][j]=float(input("= "))
for i in range (0,2):
    print(M[i])

def case1a(i):
    r=(P[0][1]*P[1][0]+(1-2*i)*a*det**0.5)/(P[1][1]**2-det)
    M[1][1]=(a*(r**2-1))**0.5
    M[0][0]=r*M[1][1]
    b=((r-1)/(a*(1+r)))**0.5
    for j in range (0,2):
        M[j][1-j]=P[j][1-j]*b
    print(M[j])

def case1b(i):
    r=(P[0][1]*P[1][0]+(1-2*i)*a*det**0.5)/(P[0][0]**2-det)
    M[0][0]=(a/(1-r**2))**0.5
    M[1][1]=r*M[0][0]
    b=((1-r)/(a*(1+r)))**0.5
    for j in range (0,2):
        M[j][1-j]=P[j][1-j]*b
    print(M[j])

def case1c(i):
    M[i][i]=0
    M[1-i][1-i]=(2*P[1-i][1-i])**0.5
    b=(2*P[1-i][1-i])**(-0.5)
    for j in range (0,2):
        M[j][1-j]=P[j][1-j]*b
    print(M[j])

def case2(i):
    p=P[1][1]+(1-2*i)*det**0.5
    if p!=0:
        for j in range(0,2):
            M[j][j]=(p/2)**0.5
            M[j][1-j]=P[j][1-j]*(2*p)**(-0.5)
            print(M[j])
    else:
```

```

print("No solutions or only 1 pair of solution(Press 2 if you haven't pressed it)")
c=True
while c==True :

    det=M[0][0]*M[1][1]-M[0][1]*M[1][0]
    a=M[0][0]-M[1][1]
    d=2

    for i in range (0,2):
        for j in range (0,2):
            P[i][j]=M[i][j]

    print("The square root of the matrix is: ")
    for i in range (0,2):
        if d==2:
            if a!=0:
                if P[1][1]**2-det!=0:
                    case1a(i)
                elif P[0][0]**2-det!=0:
                    case1b(i)
                else:
                    case1c(i)
            elif P[0][0]!=0:
                case2(i)
            else:
                print("This program is not capable of computing nilpotent or zero matrices")
                d=-1
        if d!=-1:
            print("+/-")
            print("Type 0 to exit;")
            print("Type 1 to find the sq root of the above matrix;")
            if c==True:
                print("Type 2 to get the another pair of the sq root")
                d=int(input("Please type the number: "))
            c=False
        if d==2 and c==True:
            print(P)
            c=False
        elif d!=1 and d!=0:
            print("error")
            c=False
        elif d==1: #iteration is being used here, just if you want to find M^(1/4), M^(1/8)...etc
            c=True

#As you can see, it is much shorter and more efficient
#(i am not a professional programmer, the above codes can further be optimized)

```

Code download:

https://docs.google.com/document/d/1gbpSRcJK8PdGblnPvu38216QJSr5Yn2QnFtdjzaLi_Q/edit?usp=drive_link