The Fundamental Structure of Collatz/Syracuse Trajectories

Laurent NEDELEC

E-mail : nedeleclaurent@protonmail.com

Abstract

This new text on the Collatz/Syracuse problem is a continuation of the document published in February 2025 on viXra (Laurent Nedelec : *An algorithmic approach to solving the Collatz/Syracuse problem.* viXra 2502.0056). It explains why the probabilities of divergence for Syracuse trajectories are extremely small. The same conclusion is obtained for non-trivial cycles : their existence is nearly impossible. These two results reinforce the conclusion of the previous text—reached through different methods—namely that the Collatz conjecture is, with very high probability, true.

In this new work, we first analyze the structure of the alternations between even and odd iterations within Syracuse trajectories. We arrive at the conclusion that every Syracuse trajectory is an infinite alternation of even and odd iterations of finite length, and that no trajectory can consist of an infinite succession of solely even or solely odd iterations. We then show that N* has an

equiprobable structure with respect to even and odd iterations in Syracuse trajectories. Next, we examine how this equiprobability within the structure of N* leads trajectories to be globally decreasing. Finally, we address the implications of these results for the questions of divergent trajectories and non-trivial cycles.

Introduction

The Collatz problem (also known as the (3n + 1) problem or the Syracuse problem) was first proposed by Lothar Collatz in the 1930s. Its statement is remarkably simple : the Collatz function associates each positive integer *n* with a unique successor *n′* such that :

- n′ = n / 2 if *n* is even
- n′ = (3n+1) / 2 if *n* is odd

The successive results of iterating the Collatz function from any given integer *n* are referred to as the « Syracuse trajectory of *n* ».

The Collatz problem is to determine whether all Syracuse trajectories, starting from any positive integer, eventually reach the value 1 in a finite number of iterations. The Collatz conjecture is the hypothesis that the answer is positive, meaning that all Syracuse trajectories reach the value 1.

Despite its simple formulation, this problem has remained unsolved for over 80 years. The primary difficulty lies in the fact that the operations of multiplication by (3n+1) / 2 or division by 2, depending on the parity of the numbers obtained during the successive iterations, are difficult to predict based on the initial value. These operations produce unpredictable results, resembling random phenomena.

PART 1

Definitions related to the Collatz problem

1) The Collatz or Syracuse function

Let's define the function C, known as the Collatz or Syracuse function, which associates with any integer n € N* a unique integer C(n) € N* such that :

if *n* is even, C(n) = n / 2

if *n* is odd, C(n) = (3n + 1) / 2

2) The trajectories T_n

For any $n \in N^*$ and $i \in N$, we will define a sequence T_n (Syracuse trajectory with the initial value *n*) composed of integers, which we will represent in the form $S_i(n) \in N^*$ with :

$S_0(n) = n$

$S_{i+1}(n) = C(S_i(n))$ where C is the function defined previously.

Thus, we can express T_n as follows : for any *n* and *i* belonging to N*,

$T_n = \{ n ; S_1(n) ; ... ; S_i(n) ; S_{i+1}(n) ; ...\}$

For example, we have :

$T_1 = \{ 1; 2 ; 1 ; 2 ; 1 ; ...\}$

$T_3 = \{ 3 ; 5 ; 8 ; 4 ; 2 ; 1 ; 2 ; ...\}$

T_1 is called the trivial cycle. It is the infinite alternation of 1 and 2. All trajectories that reach 1 then enter this trivial cycle.

3) The set of verified integers V

We will define *n* as a « verified integer » if there exists an integer *k* such that $S_k(n) = 1$.

We will call V the set of verified integers.

Proving the Collatz conjecture involves proving that V = N*.

4) Verification of a trajectory

The verification of a trajectory T_n consists of determining whether this trajectory converges to 1, meaning if there exists an integer *k* such that $S_k(n) = 1$.

5) Even and odd iterations

In the following text, the general terms « odd iteration » or « even iteration » will refer to an odd or even value, respectively, of any S_k (n) belonging to any T_n (thus leading to an odd iteration of the form $C(S_k (n)) = (3S_k (n) + 1)/2$ or an even iteration of the form $C(S_k (n)) = (S_k (n))/2$, depending on the parity of S_k (n)). For example, saying « there are three consecutive even iterations in this trajectory » means that three even values occured in succession, leading to three even iterations according to the Syracuse function.

PART 2

The characteristics of integers of the form $n_{(1)}(a,i)$ and $n_{(2)}(b,j)$

1) Every even integer can be written uniquely in the form :
$n_{(2)}(b,j) = b \cdot 2^j$, with j € N* , and b = 2p+1 with p € N.

From this, we deduce that every odd integer can be written uniquely in the form :
$n_{(1)}(a,i) = a \cdot 2^i - 1$, with i € N*, and a = 2q+1 with q € N.

$n_{(1)}(a,i)$ and $n_{(2)}(b,j)$ can be consecutive when a = b and i = j. But most of the time, $n_{(1)}(a,i)$ and $n_{(2)}(b,j)$ will simply be considered as arbitrary odd and even integers, unrelated to each other, expressed in the particular forms $n_{(1)}(a,i) = a \cdot 2^i - 1$ or $n_{(2)}(b,j) = b \cdot 2^j$.

We can observe that :

a) with $n_{(1)}(a,i) = a \cdot 2^i - 1$:

$S_1(a\cdot 2^i - 1) = 3\cdot a\cdot 2^{i-1} - 1$
$S_2(a\cdot 2^i - 1) = 9\cdot a\cdot 2^{i-2} - 1$
$S_k(a\cdot 2^i - 1) = 3^k\cdot a\cdot 2^{i-k} - 1$
$S_i(a\cdot 2^i - 1) = a\cdot 3^i - 1$

b) with $n_{(2)}(b,j) = b\cdot 2^j$:
$S_1(b\cdot 2^j) = b\cdot 2^{j-1}$
$S_2(b\cdot 2^j) = b\cdot 2^{j-2}$
$S_k(b\cdot 2^j) = b\cdot 2^{j-k}$
$S_j(b\cdot 2^j) = b$

We can deduce that :
a) $S_i(a\cdot 2^i - 1) = a\cdot 3^i - 1$
b) $S_j(b\cdot 2^j) = b$

for any *i* and *j* € N*. These properties can be easily proven by simple inductive arguments. We shall not present them here so as not to unnecessarily burden the text.

Since *a* and *b* are by definition odd, we can deduce that :
$S_i(a\cdot 2^i - 1)$ is even and $S_j(b\cdot 2^j)$ is odd.

2) Thus, we reach the following result :

RESULT 1

If we perform *i* iterations on an odd integer of the form $n_{(1)}(a,i) = a\cdot 2^i - 1$ and *j* iterations on an even integer of the form $n_{(2)}(b,j) = b\cdot 2^j$ with the Collatz function, then we obtain :

1) $S_i(a\cdot 2^i - 1) = a\cdot 3^i - 1$
2) $S_j(b\cdot 2^j) = b$

Since *a* and *b* are by definition odd, we obtain :
$S_i(a\cdot 2^i - 1)$ is even and $S_j(b\cdot 2^j)$ is odd.

We may therefore conclude that :
$n_{(1)}(a,i) = a\cdot 2^i - 1$, which is odd, is the first term of a continuous sequence of

i odd values, and leads to an even integer *n* of the form $n = a \cdot 3^i - 1$.

$n_{(2)}(b,j) = b \cdot 2^j$, which is even, is the first term of a sequence of *j* even values, and leads to an odd integer *n'* of the form $n' = b$.

3) Implications for Syracuse trajectories

Thus we can deduce from this result that all Syracuse trajectories are an alternation of sequences of even iterations and sequences of odd iterations.

Indeed, we see that starting from any odd initial value $n_{(1)}(a,i) = a \cdot 2^i - 1$, there will follow *i* consecutive odd iterations in the Syracuse trajectory (with *i* possibly equal to 1), until the values reach an even integer that can be written in the form $n_{(2)}(b,j) = b \cdot 2^j$ which will be the starting point of *j* consecutive even iterations (with *j* possibly equal to 1), until reaching an odd integer that can be written in the form $n_{(1)}(c,k) = c \cdot 2^k - 1$ which will be the starting point of *k* consecutive odd iterations (with *k* possibly equal to 1), until reaching an even integer…

The process is the same for an even starting point. Thus all Syracuse trajectories, starting from any initial value, follow this pattern of alternating continuous sequences of even iterations with continuous sequences of odd iterations (the length of each such sequence possibly being equal to 1). When the parity changes, the first even or odd integer of the sequence—expressed in the form $n_{(1)}(a,i) = a \cdot 2^i - 1$ or $n_{(2)}(b,j) = b \cdot 2^j$—determines, through the coefficients *i* and *j*, the number of iterations of the same parity that will follow.

This alternation is, in a sense, fundamental to Syracuse trajectories, even for the trivial cycle T_1 , which alternates one even iteration with one odd iteration indefinitely, or when a particular trajectory reaches the trivial cycle.

An infinite continuous sequence of iterations of a given parity is not possible in any Syracuse trajectory, without exception; there is always necessarily an alternation even/odd.

PART 3

Calculation of integers of the form $p = a \cdot 2^i$ on an interval $[1;2^k[$

We know that any even integer *p* can be expressed in a unique way as $p = a \cdot 2^i$, with i € N* , and *a* an odd integer.

In Part 4, we will try to quantify the probability P(i) that a randomly chosen even integer *p* yields a given index *i* in the expression $p = a \cdot 2^i$, with *a* representing any odd integer whatsoever. Before this, we introduce the following definition :

DEFINITION

We define $p_k(i)$ as the number of even integers of the form $p = a \cdot 2^i$ with *a* representing any odd integer whatsoever, p € $[1;2^k[$, i € [1,k-1] and $k \geq 3$.

According to this definition, we therefore have, to take a few examples :
$p_3(1) = 2$ (since in [1;8[only 2 and 6 have the form p = 2a)
$p_3(2) = 1$ (only 4 have the form p = 4a)
$p_4(1) = 4$ (since in [1;16[only 2,6,10 and 14 have the form p = 2a)
$p_4(2) = 2$ (only 4 and 12 have the form p = 4a)
$p_4(3) = 1$ (only 8 have the form p = 8a)
$p_5(1) = 8$ (since in [1;32[only 2,6,10,14,18,22,26 and 30 have the form p = 2a)
$p_5(2) = 4$ (only 4,12,20 and 28 have the form p=4a)
$p_5(3) = 2$ (only 8 and 24 have the form p=8a)
$p_5(4) = 1$ (only 16 have the form p=16a)

RESULT 2

If we define $p_k(i)$ as the number of even integers of the form $p = a \cdot 2^i$ with *a* representing any odd integer whatsoever and $p € [1;2^k[$, then we have :

$p_k(i) = 2^k / 2^{i+1}$ for all $i € [1,k-1]$ and $k \geq 3$

The proof by induction of this result, which is somewhat lengthy, is given in the **Appendix** at the end of the text so as not to overload the main exposition. It is possible that other, simpler and faster proofs may exist.

PART 4

Transition to the probabilities $P(i) = 1 / 2^i$

Let us consider an interval $[1;2^k[$. There are $2^{k-1} - 1$ even integers in this interval.

We defined $p_k(i)$ as the number of even integers of the form $p = a \cdot 2^i$ with *a* representing any odd integer whatsoever, $p € [1;2^k[$, $i € [1,k-1]$ and $k \geq 3$.

We have shown in the **Appendix** that :
$p_k(i) = 2^k / 2^{i+1}$

The probability P(i) for an arbitrary even integer to be of the form $p = a \cdot 2^i$ on the interval $[1;2^k[$ is equal to the number of cases in which the property holds, divided by the number of possible cases, that is :
$P(i) = (2^k / 2^{i+1}) \cdot (1 / (2^{k-1} - 1)) = (1 / 2^i) \cdot (1 / (1 - (1 / 2^{k-1})))$
with $i € [1,k-1]$ and $k \geq 3$

We therefore see that the larger *k* becomes, the larger the possible value of *i*, the larger the interval $[1;2^k[$ on which the probabilities P(i) are studied, and the closer the factor $(1/(1 - (1/2^{k-1})))$ becomes to 1. Thus, when *k* tends to infinity, we obtain a probability $P(i) = 1 / 2^i$.

We therefore obtain the following result :

RESULT 3

The probability P(i) that *p*, an arbitrary even integer taken at random, is of the form $p = a \cdot 2^i$, with *a* representing any odd integer whatsoever and i € N*, is equal to :

$P(i) = 1 / 2^i$

And we obtain the following additional result :

The probability P'(i) that *m*, an arbitrary odd integer taken at random, is of the form $m = a \cdot 2^i - 1$, with *a* representing any odd integer whatsoever and i € N*, is equal to :

$P'(i) = 1 / 2^i$

Indeed, the odd integer *m* is defined with respect to the next higher even integer. This even integer has a probability P(i) exactly determined to be of the form $a \cdot 2^i$. It is therefore natural that the integer *m* immediately below it has a probability P'(i), equal to P(i), of being of the form $m = a \cdot 2^i - 1$.

PART 5

The equiprobable structure of N* concerning the Syracuse trajectories

1) The equiprobable structure of N*

We will explain what is meant by the equiprobable structure of N* for the even and odd iterations in Syracuse trajectories. First of all, we may note that the total number of even and odd iterations under the Syracuse function, at the global level of N*, is exactly the same : indeed, each sequence beginning with an odd integer $n_{(1)}(a,i) = a\cdot 2^i - 1$ and followed by the even integer $n_{(2)}(a,i) = a\cdot 2^i$ contains exactly the same number of odd and even iterations. If we extend this scheme to all of N*, we see that, at the global level of N*, there are exactly as many possibilities for even iterations as for odd iterations for the Syracuse trajectories.

We have also seen in **Part 4** that the probability P(j) for an arbitrary even integer *n* to be of the form $n_{(2)}(b,j) = b\cdot 2^j$, with *b* representing any odd integer whatsoever and j € N*, is :

$P(j) = 1 / 2^j$

We have deduce that the probability P'(i) for an arbitrary odd integer *n'* to be of the form $n_{(1)}(a,i) = a\cdot 2^i - 1$, with *a* representing any odd integer whatsoever and i € N*, has the same form :

$P'(i) = 1 / 2^i$

Thus, each time the sequence of values in an arbitrary trajectory changes parity—moving from the set of odd numbers to the set of even numbers, or from the set of even numbers to the set of odd numbers—we can estimate the probability that the first even or odd integer at such a parity change is of a particular form $n_{(1)}(a,i)$ or $n_{(2)}(b,j)$. The probabilities of occurrence of the coefficients *i* and *j*, given by P′(i) and P(j), provide a likely estimate of the length of the successive odd or even iterations that will appear in Syracuse trajectories between two parity changes.

Passing to probabilities in Syracuse trajectories is justified by the fact that, starting from an arbitrary even integer, it is difficult to know *a priori* the value of the index *j* in the expression $n_{(2)}(b,j) = b\cdot 2^j$ without actually performing the successive divisions by 2 on the integer in question until reaching an odd value. Since the construction of $n_{(1)}(a,i) = a\cdot 2^i - 1$ is based on the integer $a\cdot 2^i$ the problem of determining the index *i* is exactly the same. The method proposed here, which is simpler and faster, therefore provides a probabilistic estimate for the indices *i* and *j*, rather than an exact determination of these indices.

This probability law first shows us the symmetry, within Syracuse trajectories, in the probabilities of the appearance of a succession of iterations of the same parity, even or odd, of a given length : indeed we have P'(i) = P(i) = 1 / 2^i. We may therefore speak of equiprobability.

Furthermore, successions of even or odd iterations of a given parity are much more likely to be short than long. Indeed, a succession of only one iteration of the same parity has probability 1 / 2, a succession of two iterations of the same parity has probability 1 / 4… Thus, the probabilities of observing long continuous successions of iterations of the same parity become increasingly small as the length of these continuous successions grows.

This probability law can be compared to the probability law governing the heads-or-tails game with a fair coin. The law that governs continuous runs of heads or tails of length *i* follows the same probability : 1 / 2^i. As we have seen in **Parts 3 and 4**, the distribution of the various $n_{(1)}(a,i) = a\cdot 2^i - 1$ and $n_{(2)}(b,j) = b\cdot 2^j$ is highly symmetrical within N*, and this produces the symmetric probabilities P(i) and P′(i) equal to 1 / 2^i. Yet this probability law is identical to probability laws concerning phenomena that are far more random, such as coin tossing.

The difference between a coin toss and the alternation of even/odd iterations in Syracuse trajectories lies in the random or non-random nature of the process. Indeed, the trajectories are precisely determined by the successive iterations of the Syracuse function, and the integer sequences thus formed cannot truly be described as random. Nevertheless, the trajectories resemble pseudo-random behaviour overall. The pseudo-random characteristic comes from the fact that, starting from an arbitrary initial value, the generated Syracuse trajectory will produce successions of even or odd iterations that are difficult to predict, without any apparent pattern, despite the fact that a certain number of initial iterations may be easily determined.

This pseudo-random characteristic appears in the various successive values taken by the trajectories (a large part of the pseudo-random nature of the values taken by the trajectories seems to be due to the addition of one unit after the multiplication by 3 in the construction of the odd iterations of the Syracuse trajectories). But the pseudo-random aspect also lies in the fact that, at each change of parity, the number of iterations of a given parity that will follow is determined by the indices *i* and *j* in the corresponding $n_{(1)}$ and $n_{(2)}$, as we have seen in the previous sections. Yet we have seen that it is easier to estimate the probability of the magnitude of these indices *i* and *j* than to compute them exactly. In a certain sense, we are faced with a double layer of indeterminacy which progressively blurs the predictability of the evolution of the trajectories, at each iteration and at each parity change in the trajectories.

We can therefore almost regard the even and odd iterations of the Syracuse trajectories as independent events of a random type.

We may thus conclude from these observations that the Syracuse trajectories evolve within a very symmetrical equiprobable structure throughout N*, yet yielding results that are close to randomness, with probabilities of appearance of continuous successions of *i* even or odd iterations that are perfectly identical and equal to $1 / 2^i$.

2) We will present two elements that could contradict the equiprobability principle we have just described.

a) A bias that one might find in this equiprobability law would lie in the determination of the starting values. Indeed, these can be chosen arbitrarily with very large values for the coefficients *i* or *j* in the starting terms $n_{(1)}(a,i) = a \cdot 2^i - 1$ and $n_{(2)}(b,j) = b \cdot 2^j$. It is therefore possible to create significant initial imbalances, both in the number of even iterations and in the number of odd iterations. This imbalance will take some time to resolve, but eventually the equilibrium will re-establish itself, because continuous runs of even or odd iterations that follow the equiprobability principle more closely (that is, short continuous sequences, with a relative balance between even and odd iterations) are very likely to appear after the initial imbalance.

This pattern of an initial imbalance may also arise at particular moments during the iterations of various Syracuse trajectories, when one or several terms have a very large coefficient *i* or *j* in the expressions $n_{(1)}(a,i) = a \cdot 2^i - 1$ or $n_{(2)}(b,j) = b \cdot 2^j$. In the same way, such local imbalances will gradually be balanced out between even and odd iterations as the successive iterations proceed, due to the fact—according to the equiprobability law we have defined—that continuous runs of a single parity tend to be short and balanced between even and odd iterations.

We may note that for trajectories whose starting value is, for example $n = 2^j$, the even iterations lead directly to 1 and therefore create an obvious imbalance in favor of the even iterations. But we should also note that once the trajectory reaches 1, it enters an infinite alternation of one even iteration followed by one odd iteration, characteristic of the trivial cycle T_1 . Thus, in a natural way, the even/odd balance is restored over the course of the infinite iterations of the trivial cycle.

b) A second issue that could disturb this equiprobability principle would be the existence of a non-trivial cycle with a numerical imbalance between even and odd iterations. Such a non-trivial cycle, continuing indefinitely with this even/odd imbalance, would contradict the equiprobability principle. The question of non-trivial cycles will be addressed in **Part 7**.

PART 6

The question of divergent trajectories

If we consider the principle of equiprobability between even and odd iterations, the more iterations a Syracuse trajectory undergoes, the closer the even/odd ratio becomes to 1. We can observe that each odd iteration produces a multiplication by a factor of approximately 3 / 2 of the preceding term, while each even iteration produces a division by 2 of the preceding term. Thus, as the number of iterations increases, the numbers of even and odd iterations will theoretically approach numerical equality, leading to an overall multiplication by a factor approaching $(3/4)^{h/2}$ of the starting value as *h*—the total number of iterations—grows. The fact that the ratio $(3/4)^{h/2}$ is far below 1 indicates that trajectories will be globally decreasing, at least when the number of iterations becomes large.

The larger the starting values are, the higher the probability of having a large number of iterations. As we have previously noted, we may artificially choose extremely large starting points with a large succession of odd iterations at the beginning—for example several billions of odd iterations (it is easy to imagine extremely large values of i in the expression $a \cdot 2^i - 1$). This would imply extremely large starting points and extremely large numbers of odd iterations, and such a pattern might continue indefinitely. But the equiprobable structure we have highlighted suggests that a plausible hypothesis is that, after these large initial rises and as soon as an even value is reached, the trajectories return to successions that more faithfully reflect the principle of equiprobability—that is, relatively balanced and relatively short successions of even and odd iterations. This would eventually lead—although the process may take an extremely long time—to an even/odd balance and a global decrease of the trajectories according to the factor $(3/4)^{h/2}$, until reaching 1.

We may even imagine several long continuous successions of odd iterations, interrupted by only a few even iterations, which could create genuine threats of divergence. But it must be noted that the probability that a trajectory encounters very long continuous successions of odd iterations during its evolution is low, and the probability of encountering several such long successions in a row is even lower, although this remains entirely possible. We must also remember that the equiprobable structure implies that long continuous sequences of odd iterations have exactly the same probability of occurring as long continuous sequences of even iterations.

A divergent trajectory would imply very large imbalances between even and odd iterations and would completely contradict the principle of equiprobability. We may therefore deduce that it would require a great deal of “bad luck” to obtain trajectories that constantly chain together long successions of odd values. A hypothetical divergent trajectory would need to possess a great deal of “strength,” and this for an infinite amount of time, in order to overcome this “gravity,” associated with the factor $(3 / 4)^{h/2}$, which tends to pull trajectories toward smaller and smaller values.

Thus, the probabilities of the existence of a divergent trajectory are extremely small.

PART 7

The question of non-trivial cycles

The question of non-trivial cycles was partly addressed in the previous text (see Laurent Nedelec : *An algorithmic approach to solving the Collatz /Syracuse problem*. viXra 2502.0056). We may recall that a result by S. Eliahou concerning the structure of non-trivial cycles in Syracuse trajectories shows that the minimal length of any possible non-trivial cycle is measured in billions of terms. In 2011, he estimated the minimal length of a non-trivial cycle to be 17 billion terms (see S. Eliahou : *Le problème 3n+1, y a-t-il des cycles non-triviaux ? (Third part).* Images des mathématiques, CNRS). In 2025, it seems that this value has risen to 355 billion terms (see David Barina : *Improved verification limit for the convergence of the Collatz conjecture*. The Journal of Supercomputing). Incidentally, an error appeared in the previous text, page 30 : instead of "If the upper bound of A(n) exceeds 2^{1000} then the lower bound for a non-trivial cycle would be approximately $3.45 \cdot 10^{500}$", one should read $3.45 \cdot 10^{150}$ and not $3.45 \cdot 10^{500}$ (see S. Eliahou : *The 3x+1 problem : new lower bounds on non-trivial cycle lengths*).

The increase in the minimal possible size of a non-trivial cycle is due to the fact that the more initial integers are computationally verified to satisfy the conjecture, the larger this minimal bound becomes. Researchers have managed to verify the conjecture for the first roughly 10^{20} integers. For all integers up to 10^{20}, we know that the maximal flight time is 3000 iterations, meaning that at most 3000 iterations are needed to reach 1 for every integer below 10^{20} (see E. Rosendaal : *ericr.nl "3x+1 delay records"*). This is undoubtedly linked to the equiprobable structure of N* and to the factor $(3 / 4)^{h/2}$ mentioned earlier. Syracuse trajectories indeed seem to be pulled downward very quickly, and hence towards 1. This relatively small upper bound of 3000 iterations to reach 1 —even for starting values in the hundreds of billions of billions—suggests that the first trajectory reaching 355 billion iterations without falling below its starting value (since values below the starting point are already verified and therefore will lead to 1; see the previous text, Part 2, section 2, for more details) will, in all likelihood, have an extremely large starting value.

As the verification range expands, the first trajectory containing a possible non-trivial cycle must necessarily begin, at minimum, at the starting point of the trajectory that first reaches 355 billion iterations without descending below its initial value. This also means that every value below that starting point will be verified. Yet we have seen that the starting value of the first trajectory reaching 355 billion iterations without falling below its initial value would, with high probability, be extremely large. This would therefore imply that the set of verified initial integers would also be extremely large. And the larger the set of verified initial integers becomes, the larger the minimal bound for the length of a non-trivial cycle becomes, as explained earlier, reaching colossal values. We would then have to find the first trajectory whose flight time (without falling below its initial value) equals this new minimal cycle length, which would push us even farther into N*, and so on indefinitely.

Thus, as the set of verified integers expands, the minimal bound for a non-trivial cycle becomes increasingly unreachable, largely due to the decreasing factor $(3\ /\ 4)^{h/2}$, which causes Syracuse trajectories to have relatively small flight times. It therefore seems that the larger the initial set of verified integers becomes, the smaller the probability of encountering any first non-trivial cycle. The fact that no non-trivial cycle has been found among the first 10^{20} integers suggests that we are unlikely to find one anywhere else in N*.

CONCLUSION

We may draw a connection, in this conclusion, with the first text published a year ago (Laurent Nedelec : *An algorithmic approach to solving the Collatz/Syracuse problem.* viXra 2502.0056), which introduced the notions of the set of verified integers, the axis of verified integers, and inverse graphs. We observed that the continuous growth of G(1), the inverse graph of 1, and the fact that all bounded trajectories without a non-trivial cycle were automatically verified, made the set of non-verified integers increasingly discontinuous.

We also noted that several results could be attached to the concept of the axis of verified integers. The results of this new text show that the equiprobable structure of N* with respect to the even and odd iterations of Syracuse trajectories leads naturally to the decreasing factor $(3 / 4)^{h/2}$. This factor acts like a weight, a gravity, pulling the various trajectories toward the value 1 relatively quickly. This confirms the results previously obtained, namely that the probabilities of encountering a divergent trajectory or a non-trivial cycle are extremely small.

For a trajectory to be divergent, it would have to evolve within the limits of a maximal bound without ever reaching an already verified integer, then exceed this bound, evolve again within a new maximal bound without being equal to a verified integer, and so on ad infinitum. It would require tremendous "strength" and highly unfavorable probabilities to sustain such growth within the context of an axis of verified integers that is increasingly dense, of the equiprobable structure of N*, and of the decreasing factor $(3 / 4)^{h/2}$.

Regarding non-trivial cycles, we have seen that the initial set of integers verified by computer is currently approximately 10^{20}, and that the maximal number of iterations before reaching 1, for all integers in the interval $[1,10^{20}]$, is 3000. Yet the minimal length of a potential non-trivial cycle above $[1,10^{20}]$ exceeds 355 billion terms. This small maximum value of 3000 iterations required to reach 1 for all integers in $[1,10^{20}]$ suggests that, during successive verifications, trajectories above the upper bound of the initial set of already verified integers have far greater probability of rapidly dropping below their starting value—and thus of reaching 1—than of performing at least 355 billion iterations without being equal to a verified integer. Indeed, the decreasing factor $(3 / 4)^{h/2}$ applies equally to potential non-trivial cycles. The probabilities of the existence of a non-trivial cycle decrease as the initial set of verified integers increases, as discussed in **Part 7** of this new text, and this pattern can be continued indefinitely.

We may therefore conclude that the probabilities of the existence of one or several divergent trajectories, and of one or several non-trivial cycles—which we already considered to be almost zero after the first text—are even smaller after this second text. Everything therefore suggests that the Collatz conjecture holds true.

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APPENDIX

In this appendix, we provide the inductive proof of the result presented in **Part 3**. This proof is somewhat long, and it is possible that other, simpler and faster proofs could be constructed.

We defined $p_k(i)$ as the number of even integers of the form $p = a \cdot 2^i$ where *a* denotes any odd integer whatsoever, p € $[1;2^k[$, i € [1,k-1] and $k \geq 3$.

Let us assume that the proposition stating that the number of integers of the form $p = a \cdot 2^i$, with *a* denoting any odd integer whatsoever, in the interval $[1;2^{k-1}[$ is equal to the number of integers of the form $p = a \cdot 2^i$ in the interval $[2^{k-1};2^k[$ is true at rank *k*, for any i € [1,k-2] and for $k \geq 3$.

Let us also assume that the second proposition :
$p_k(i) = 2^k / 2^{i+1}$ is true at rank *k*, for any i € [1,k-1] and for $k \geq 3$.

Let us prove that these two propositions remain true at rank *k+1*, with i € [1,k-1] for the first proposition and i € [1,k] for the second proposition.

Note : in the first part of the proposition, concerning the equality between the two distinct intervals, we defined i € [1,k-2]. In the case i = k-1, we have p_k (k-1) = 1, which means that we cannot divide the number of case into two equal parts on the intervals $[1;2^{k-1}[$ and $[2^{k-1};2^k[$. This constitutes a special case. This point is discussed in **section 2) e)** of this proof.

1) Initialization of the induction

Let us prove that the propositions hold for k = 3 and k = 4.

a) For k = 3

We study the even integers in the interval [1,8[.
2 and 6 are of the form 2a.
4 is of the form 4a.
Thus we have $p_3\,(1) = 2 = 2^3 / 2^{1+1}$ and $p_3\,(2) = 1 = 2^3 / 2^{2+1}$

Moreover, the number of integers of the form p=2a in the interval [1,4[is equal to the number of integers of the form p=2a in the interval [4,8[.

Therefore, both propositions are verified for k=3.

b) For k = 4

We study the even integers in the interval [1;16[.
2;6;10;14 are of the form 2a.
4 and 12 are of the form 4a.
8 is of the form 8a.
Thus we have :
$p_4\,(1) = 4 = 2^4 / 2^{1+1}$
$p_4\,(2) = 2 = 2^4 / 2^{2+1}$
$p_4\,(3) = 1 = 2^4 / 2^{3+1}$

Moreover, the number of integers of the form p = 2a in the interval [1;8[is equal to the number of integers of the form p = 2a in the interval [8;16[.
And the number of integers of the form p = 4a in the interval [1;8[is equal to

the number of integers of the form p = 4a in the interval [8;16[.
Therefore, both propositions are verified for k=4.

2) Inductive proof at rank $k + 1$

Let us assume that the first proposition is true at rank k, for any i € [1,k-2] and for $k \geq 3$.

Let us also assume that the second proposition is true at rank k, for any i € [1,k-1] and for $k \geq 3$.

Let us prove that these two propositions remain true at rank $k+1$, with i € [1,k-1] for the first proposition and i € [1,k] for the second proposition.

We shall assume as evident that all even integers in $[2^k;2^{k+1}[$ arise from multiplying by 2 all the even and odd integers in $[2^{k-1};2^k[$.

a) Let us examine the case i = 1 at rank $k + 1$.

All odd integers in $[2^{k-1};2^k[$ produce, by multiplication by 2, all the even integers in $[2^k;2^{k+1}[$ of the form p = 2a with a representing any odd integer whatsoever.

There are 2^{k-2} odd integers in $[2^{k-1};2^k[$. Therefore, there are 2^{k-2} even integers of the form $2a$ in the interval $[2^k;2^{k+1}[$.

The number of even integers of the form $2a$ in the interval $[1;2^k[$ is given by the recurrence hypothesis :
$p_k(1) = 2^k / 4$

If we add the value $p_k(1)$, coming from $[1;2^k[$, to the number of integers of the form $2a$ coming from $[2^k;2^{k+1}[$, then for the interval $[1;2^{k+1}[$ we obtain :
$p_{k+1}(1) = (2^k / 4) + 2^{k-2} = 2^{k+1} / 4$

This verifies the formula $p_{k+1}(i) = 2^{k+1} / 2^{i+1}$ for i = 1.

We also observe that the first proposition is verified, since there are as many integers of the form p = 2a coming from $[1;2^k[$ as from $[2^k;2^{k+1}[$.

b) Let us examine the case i = 2 at rank *k + 1*

All even integers of the form p = 2a (with *a* representing any odd integer whatsoever) in $[2^{k-1};2^k[$ produce, by multiplication by 2, all the even integers in $[2^k;2^{k+1}[$ of the form p = 4a.

Since we assume by the recurrence hypothesis that :
1) there are as many integers of the form p = 2a in the interval $[1;2^{k-1}[$ as in the interval $[2^{k-1};2^k[$
2) and that on $[1;2^k[$, $p_k(1) = 2^k / 2^2$

We may conclude that there are $2^k / 2^3$ even integers of the form p = 2a in the interval $[2^{k-1};2^k[$, and therefore just as many integers of the form p = 4a in the interval $[2^k;2^{k+1}[$.

The number of even integers of the form 4a in the interval $[1;2^k[$ is given by the recurrence hypothesis :
$p_k(2) = 2^k / 2^3$

If we add the number of integers of the form n = 4a (with *a* representing any odd integer whatsoever) over the two intervals $[1;2^k[$ and $[2^k;2^{k+1}[$, then we obtain :
$p_{k+1}(2) = 2^k / 2^3 + 2^k / 2^3 = 2^{k+1} / 2^3$

This verifies the formula $p_{k+1}(i) = 2^{k+1} / 2^{i+1}$ for i = 2.

We also observe that the first proposition is verified, since there are as many integers of the form p = 4a coming from $[1;2^k[$ as from $[2^k;2^{k+1}[$.

c) Let us examine the general case for all i € [1,k-1] at rank *k + 1*

Let us prove that both properties are verified for every *i* at rank *k+1*, with i € [1,k-1].

All even integers of the form $p = a \cdot 2^{i-1}$ (with *a* representing any odd integer whatsoever) in the interval $[2^{k-1};2^k[$ produce, by multiplication by 2, all the even integers in $[2^k;2^{k+1}[$ of the form $p = a \cdot 2^i$ with i € [1;k-1].

Since we assume by the recurrence hypothesis that :
1) there are as many integers of the form $p = a \cdot 2^{i-1}$ in the interval $[1;2^{k-1}[$ as in the interval $[2^{k-1};2^k[$ for every i € [1;k-1] (indeed, this property was included in the recurrence hypothesis as holding at rank *k* for $p = a \cdot 2^i$ for any i € [1,k-2]. Since we are now considering integers of the form $p = a \cdot 2^{i-1}$ in the interval $[1;2^k[$, there is no difficulty in extending the range to i € [1;k-1]. See **section 2) d)** of this proof).
2) and that on $[1;2^k[$, $p_k(i-1) = 2^k / 2^i$ for all i € [1;k-1].

We may conclude that there are $2^k / 2^{i+1}$ even integers of the form $p = a \cdot 2^{i-1}$ in the interval $[2^{k-1};2^k[$, and therefore as many integers of the form $p = a \cdot 2^i$ in the interval $[2^k;2^{k+1}[$.

The number of even integers of the form $a \cdot 2^i$ in the interval $[1;2^k[$ is given by the recurrence hypothesis :

$$p_k(i) = 2^k / 2^{i+1}$$

If we add the total number of integers of the form $p = a \cdot 2^i$, coming from the two intervals $[1;2^k[$ and $[2^k;2^{k+1}[$, then we obtain :

$$p_{k+1}(i) = 2^k / 2^{i+1} + 2^k / 2^{i+1} = 2^{k+1} / 2^{i+1}$$

This verifies, at rank *k + 1*, the formula $p_{k+1}(i) = 2^{k+1} / 2^{i+1}$ for all i € [1,k-1].

We also observe that the first proposition is verified at rank *k + 1*, for all i € [1,k-1], since there are as many integers of the form $p = a \cdot 2^i$ coming from $[1;2^k[$ as from $[2^k;2^{k+1}[$.

d) Before addressing the case i = k at rank *k + 1*, let us briefly revisit the case i = k-1, which we have just considered as the last case in the previous section (that is i € [1,k-1]) :

Starting from the two integers of the form $p = a \cdot 2^{k-2}$ in the interval $[1;2^k[$, namely :
1) $p = 2^{k-2}$ in the interval $[1;2^{k-1}[$.
2) $p = 3 \cdot 2^{k-2}$ in the interval $[2^{k-1};2^k[$.

The integer 2) produces, by multiplication by 2, the unique integer of the form $p = a \cdot 2^{k-1}$ in the interval $[2^k;2^{k+1}[$, namely $3 \cdot 2^{k-1}$.

If we add the integer of the form $p = a \cdot 2^{k-1}$ coming from the interval $[1;2^k[$ (namely $p = 2^{k-1}$), we obtain 2 integers of the form $p = a \cdot 2^{k-1}$ in the interval $[1; 2^{k+1}[$.

This verifies, at rank $k + 1$, the formula $p_{k+1}(k-1) = 2^{k+1} / 2^k = 2$ for i = k-1.

We also observe that the first proposition is verified at rank $k + 1$, for i = k-1, since there are as many integers of the form $p = a \cdot 2^{k-1}$ coming from $[1;2^k[$ as from $[2^k;2^{k+1}[$.

e) Let us finally examine the last particular case i = k at rank $k + 1$

For i = k-1 in the interval $[1;2^k[$, we have only one case :
Indeed, $p_k(k-1) = 2^k / 2^k = 1$.
There is only one integer of the form $p = a \cdot 2^{k-1}$ in the interval $[1;2^k[$, which is $p = 2^{k-1}$ (since $p = 3 \cdot 2^{k-1}$ does not belong to $[1;2^k[$).
Thus, there is only one integer of the form $p = a \cdot 2^k$ in the interval $[1;2^{k+1}[$, namely $p = 2^k$.

This verify, at rank $k + 1$ and for i = k, the formula $p_{k+1}(k) = 2^{k+1} / 2^{k+1} = 1$.

This completes the proof that, assuming that the two propositions are true at rank *k,* for all k ≥ 3, with i € [1,k-2] for the first proposition and i € [1,k-1] for the second proposition, then these two propositions are true at rank *k+1*, with i € [1,k-1] for the first proposition and i € [1,k] for the second proposition.

We therefore see that we have covered all even integers in the interval $[2^k ; 2^{k+1}[$ by analysing the multiplication by 2 of all even and odd integers in the interval $[2^{k-1};2^k[$. We began by multiplying by 2 all the odds in $[2^{k-1};2^k[$, and then progressively all the even integers of the form $a \cdot 2^i$ up to i = k – 1.

We finally may note that the set of all $p_k(i)$ for i € [1,k-1] indeed covers the total number of even integers in the interval $[1;2^k[$, equal to $2^{k-1} - 1$, since :

$$S = 1 + 2 + 4 + 8 + 16 + \ldots + 2^{k-2} = 2^{k-1} - 1$$

Indeed, $2S = 2 + 4 + 8 + 16 + \ldots + 2^{k-2} + 2^{k-1}$

And $2S - S = S = 2^{k-1} - 1$

We therefore obtain the following result :

If we define $p_k(i)$ as the number of even integers of the form $p = a \cdot 2^i$ where *a* denotes any odd integer whatsoever, $p \in [1;2^k[$, $i \in [1,k-1]$ and $k \geq 3$.

Assuming that the proposition stating that the number of integers of the form $p = a \cdot 2^i$, with *a* denoting any odd integer whatsoever, in the interval $[1;2^{k-1}[$ is equal to the number of integers of the form $p = a \cdot 2^i$ in the interval $[2^{k-1};2^k[$ is true at rank *k*, for any $i \in [1,k-2]$ and for $k \geq 3$.

Assuming also that the second proposition :
$p_k(i) = 2^k / 2^{i+1}$ is true at rank *k*, for any $i \in [1,k-1]$ and for $k \geq 3$.

We have shown, after initialization, that these two propositions hold at rank *k+1*, for every $i \in [1,k-1]$ for the first proposition and every $i \in [1,k]$ for the second proposition.

The two propositions are therefore generalizable to all N*.