Universal Divisibility Framework: A Unified Theory of Divisibility Across Integer, Rational, Real, and Complex Domains

Ansh Mathur*
S.R.M Institute of Science and Technology, Kattankulathur
Chennai, Tamil Nadu, India

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Abstract

This paper introduces the Universal Divisibility Framework (UDF), a comprehensive mathematical theory that extends the classical notion of divisibility from integers to rationals, reals, and complex numbers. The framework is built upon the Universal Divisibility Function $d(a, b, c) = \lfloor a/c \rfloor (b \bmod c) - \lfloor b/c \rfloor (a \bmod c)$, which provides a unified criterion for divisibility across multiple number systems. We establish the Universal Divisibility Theorem, proving that for $a, b \in \mathbb{R}$ with $b \neq 0$, and an integer c satisfying $\lfloor b/c \rfloor = \pm 1$, we have $b \mid a$ if and only if $d(a, b, c) \equiv 0 \pmod{b}$. This framework not only recovers all classical integer divisibility rules as special cases but also eliminates false positives that arise when traditional rules are naively extended to non-integer domains. We provide explicit divisibility formulas for numbers 1–1000, demonstrate applications to Diophantine equations and matrix algebras, and discuss implications for computational number theory and cryptography.

1 Introduction

1.1 Historical Context and Limitations of Classical Divisibility

Divisibility has been a central concept in number theory since antiquity, with documented rules appearing in ancient Greek, Indian, and Arabic mathematics. The classical definition—for integers a, b with $b \neq 0$, we say b divides a (written $b \mid a$) if there exists an integer k such that $a = kb$—has spawned numerous heuristic rules for specific divisors (e.g., divisibility by 2, 3, 5, 11). However, this framework suffers from several fundamental limitations:

1. **Domain restriction**: Classical divisibility is confined to integers, leaving non-integer divisibility questions (e.g., "does 2.5 divide 7.5?") formally undefined.
2. **False positives**: When traditional rules are extended beyond integers, they often yield incorrect results (e.g., the rule for divisibility by 13 incorrectly suggests 3.25 is divisible by 13).
3. **Lack of unification**: Existing rules appear ad hoc—digit sums for 3, alternating sums for 11, chunking methods for 7—with no underlying unifying principle.
4. **Difficulty with composites**: Constructing rules for composite numbers is nontrivial and often avoided in elementary treatments.

*Corresponding author: am3274@srmist.edu.in

1.2 Research Objectives and Contributions

This work addresses these limitations by developing a universal framework for divisibility. Our primary objectives are:

1. To define a single function that characterizes divisibility across multiple number systems.
2. To derive all classical divisibility rules as special cases of this unified framework.
3. To eliminate false positives through precise parameter constraints.
4. To extend divisibility testing to composite numbers systematically.
5. To provide explicit formulas for divisibility by numbers 1–1000.

The principal contributions of this paper are:

- The Universal Divisibility Function $d(a, b, c)$ and accompanying theorem.
- Proof of equivalence between classical divisibility and the UDF condition under appropriate parameter choices.
- Extension of divisibility to $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and matrix algebras.
- A complete parameterization for divisibility by numbers 1–1000.
- Applications to Diophantine equations, primality testing, and cryptography.

2 The Universal Divisibility Framework

2.1 Definition and Basic Properties

Definition 1 (Universal Divisibility Function)**.** *For numbers a, b ($b \neq 0$) and an auxiliary integer c, define:*

$$d(a, b, c) = \left\lfloor \frac{a}{c} \right\rfloor (b \bmod c) - \left\lfloor \frac{b}{c} \right\rfloor (a \bmod c).$$

Equivalently, using $x \bmod y = x - y\lfloor x/y \rfloor$, we obtain the simplified form:

$$d(a, b, c) = b \left\lfloor \frac{a}{c} \right\rfloor - a \left\lfloor \frac{b}{c} \right\rfloor.$$

The parameter c acts as a "modular base" that can be optimized for computational efficiency. When $c = 10^k$, the computation involves only digit manipulation, making it suitable for mental arithmetic.

2.2 The Universal Divisibility Theorem

Theorem 2 (Universal Divisibility Theorem)**.** *Let $a, b \in \mathbb{R}$ with $b \neq 0$, and let c be an integer such that $\lfloor b/c \rfloor = \pm 1$. Then:*

$$b \mid a \iff d(a, b, c) \equiv 0 \pmod{b}.$$

Moreover,

$$d(a, b, c) \equiv - \left\lfloor \frac{b}{c} \right\rfloor a \pmod{b}.$$

Proof. Starting from the simplified form:

$$d(a,b,c) = b\left\lfloor \frac{a}{c} \right\rfloor - a\left\lfloor \frac{b}{c} \right\rfloor.$$

Reducing modulo b:

$$d(a,b,c) \equiv -a\left\lfloor \frac{b}{c} \right\rfloor \pmod{b}.$$

If $\lfloor b/c \rfloor = \pm 1$, then multiplication by this factor is invertible modulo b. Thus:

$$d(a,b,c) \equiv 0 \pmod{b} \iff a \equiv 0 \pmod{b},$$

which is exactly $b \mid a$. ☐

Remark 3. *The condition $\lfloor b/c \rfloor = \pm 1$ ensures the coefficient of a in the congruence is ± 1, making the equivalence exact. This condition can often be satisfied by choosing c close to b in magnitude.*

2.3 Examples and Illustration

Example 4 (Divisibility by 7)**.** *Classical rule: double the last digit and subtract from the rest. In UDF, take $b = 7$, $c = 10$. Then $\lfloor 7/10 \rfloor = 0$, violating our condition. Instead, use negative representation: $b = -7$, $c = 10$. Then $\lfloor -7/10 \rfloor = -1$, satisfying the condition. Compute:*

$$d(a,-7,10) = -7\lfloor a/10 \rfloor - a(-1) = a - 7\lfloor a/10 \rfloor.$$

This equals $(a \bmod 10) - 2\lfloor a/10 \rfloor$ after manipulation, recovering the classical rule.

Example 5 (Divisibility by 13)**.** *For $b = -13$, $c = 10$, $\lfloor -13/10 \rfloor = -2 \neq \pm 1$. Choose instead $c = 100$ with $b = -13$: $\lfloor -13/100 \rfloor = -1$. Then:*

$$d(a,-13,100) = -13\lfloor a/100 \rfloor - a(-1) = a - 13\lfloor a/100 \rfloor.$$

This yields the rule: multiply the hundreds-and-above part by 13 and compare with the last two digits.

3 Extensions to Non-Integer Domains

3.1 Rational Numbers $\mathbb{Q}$

For $a = p_1/q_1$, $b = p_2/q_2$ with $p_i, q_i \in \mathbb{Z}$, define $\lfloor x \rfloor$ as the greatest integer $\leq x$, and $x \bmod y = x - y\lfloor x/y \rfloor$. Then the theorem applies directly.

Example 6. *Test if 2.5 divides 7.5. Set $a = 7.5$, $b = 2.5$, $c = 10$. Compute:*

$$d(7.5, 2.5, 10) = 2.5\lfloor 0.75 \rfloor - 7.5\lfloor 0.25 \rfloor = 0.$$

Since $0 \equiv 0 \pmod{2.5}$, we conclude $2.5 \mid 7.5$, and indeed $7.5/2.5 = 3 \in \mathbb{Z}$.

3.2 Real Numbers $\mathbb{R}$

For $a, b \in \mathbb{R}$, $b > 0$, define $a \bmod b = a - b\lfloor a/b \rfloor \in [0, b)$. The framework remains valid.

Example 7. *Test if $\sqrt{2} \mid 2\sqrt{2}$. With $a = 2\sqrt{2} \approx 2.828$, $b = \sqrt{2} \approx 1.414$, $c = 10$:*

$$d(a,b,10) = \sqrt{2}\lfloor 2\sqrt{2}/10 \rfloor - 2\sqrt{2}\lfloor \sqrt{2}/10 \rfloor = 0.$$

Thus $\sqrt{2} \mid 2\sqrt{2}$, as expected.

3.3 Complex Numbers and Gaussian Integers

For $z = x + iy \in \mathbb{C}$ and $m \in \mathbb{C} \setminus \{0\}$, define $\lfloor z \rfloor = \lfloor x \rfloor + i\lfloor y \rfloor$, and $z \bmod m = z - m\lfloor z/m \rfloor$.

Example 8. *Test if $1 + i \mid 3 + 5i$ in $\mathbb{Z}[i]$. Compute quotient:*

$$\frac{3+5i}{1+i} = \frac{(3+5i)(1-i)}{2} = 4 - i.$$

This is a Gaussian integer, so divisibility holds. UDF with $c = 10$ gives:

$$d(3+5i, 1+i, 10) = (1+i)\lfloor (3+5i)/10 \rfloor - (3+5i)\lfloor (1+i)/10 \rfloor = 0.$$

3.4 Matrix Algebras $M_n(\mathbb{R})$

For matrices A, M with M invertible, define $\lfloor A \rfloor_{ij} = \lfloor (AM^{-1})_{ij} \rfloor$, and $A \bmod M = A - M\lfloor AM^{-1} \rfloor$.

Example 9. *Let $A = \begin{pmatrix} 6 & 12 \\ 9 & 18 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Compute $AB^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, which has integer entries, indicating $B \mid A$ in the sense of matrix scaling.*

4 Unification of Classical Divisibility Rules

All known divisibility rules emerge as special cases of the UDF. Table 1 illustrates this for common divisors.

Table 1: Unification of classical divisibility rules via UDF

Divisor	**Classical Rule**	**UDF Parameters**	**UDF Expression**	**Condition**
2	Last digit even	$b = -2, c = 10$	$a \bmod 10$ even	$\lfloor -2/10 \rfloor = -1$
3	Digit sum divisible by 3	$b = -3, c = 10$	$\sum$ digits $\bmod 3 = 0$	$\lfloor -3/10 \rfloor = -1$
5	Last digit 0 or 5	$b = -5, c = 10$	$a \bmod 10 \in \{0, 5\}$	$\lfloor -5/10 \rfloor = -1$
7	Double last digit, subtract	$b = -7, c = 10$	$a - 7\lfloor a/10 \rfloor \equiv 0 \pmod 7$	$\lfloor -7/10 \rfloor = -1$
9	Digit sum divisible by 9	$b = -9, c = 10$	$\sum$ digits $\bmod 9 = 0$	$\lfloor -9/10 \rfloor = -1$
11	Alternating digit sum	$b = -11, c = 10$	$\sum (-1)^k d_k \equiv 0 \pmod{11}$	$\lfloor -11/10 \rfloor = -2$
13	Multiply last digit by 4, add	$b = -13, c = 10$	$\lfloor a/10 \rfloor + 4(a \bmod 10)$	$\lfloor -13/10 \rfloor = -2$

For divisors where $\lfloor b/c \rfloor \neq \pm 1$ with $c = 10$ (like 11, 13), we may either accept a slightly different form or choose a different c to satisfy the condition.

5 Elimination of False Positives

A major advantage of UDF over naive extensions is elimination of false positives. Consider the classical rule for 13: $\lfloor a/10 \rfloor + 4(a \bmod 10)$. For integer a, this works perfectly. But for $a = 3.25$:

$$\lfloor 3.25/10 \rfloor + 4(3.25 \bmod 10) = 0 + 4 \times 3.25 = 13,$$

which is divisible by 13, incorrectly suggesting $13 \mid 3.25$.

In UDF, with $b = -13$, $c = 10$, we compute:

$$d(3.25, -13, 10) = -13\lfloor 0.325 \rfloor - 3.25(-2) = 0 + 6.5 = 6.5.$$

Then $6.5 \bmod 13 = 6.5 \neq 0$, correctly showing non-divisibility. The condition $\lfloor b/c \rfloor = \pm 1$ ensures scaling preserves exactness.

6 Computational Methods and Applications

6.1 Parameter Optimization

For efficient computation, choose c such that:

1. $\lfloor b/c \rfloor = \pm 1$ for exactness.
2. c is a power of 10 for digit-based computation.
3. $|c|$ is close to $|b|$ for rapid convergence.

Algorithm for parameter selection:

1. If $|b| < 10$, try $c = 10$.
2. If $10 \leq |b| < 100$, try $c = 100$ or $c = -10$ to achieve $\lfloor b/c \rfloor = \pm 1$.
3. For larger b, choose $c = 10^k$ where $10^{k-1} \leq |b| < 10^k$.

6.2 Divisibility Formulas for Numbers 1–1000

The UDF generates two families of formulas:

1. **Negative parameterization**: For b negative, formula is $(rest) \times Q + (last\ N\ digits)$.
2. **Positive parameterization**: For b positive, formula is $(rest) \times Q - (last\ N\ digits) \times P$.

Table 2 shows examples.

Table 2: Sample divisibility formulas from UDF

Divisor	**b**	**c**	**Formula**	**Type**
7	-7	10	$\lfloor a/10 \rfloor \times 2 + (a \bmod 10)$	Negative
13	-13	100	$\lfloor a/100 \rfloor \times 13 + (a \bmod 100)$	Negative
19	19	20	$\lfloor a/20 \rfloor \times 1 - (a \bmod 20) \times 1$	Positive
23	-23	100	$\lfloor a/100 \rfloor \times 23 + (a \bmod 100)$	Negative
37	-37	100	$\lfloor a/100 \rfloor \times 37 + (a \bmod 100)$	Negative

Complete tables for 1–1000 are available in the supplementary material.

6.3 Applications to Diophantine Equations

Consider the linear Diophantine equation $ax + by = c$. The UDF provides an alternative approach to testing solvability. For integer solutions to exist, we need $d \mid c$ where $d = \gcd(a, b)$. Using UDF:

$$d \mid c \iff d(c, d, c_0) \equiv 0 \pmod{d}$$

for appropriate c_0. This avoids explicit GCD computation in some cases.

Example 10. *Test if* $15x + 21y = 39$ *has integer solutions. Here* $\gcd(15, 21) = 3$. *Check if* $3 \mid 39$ *using UDF with* $b = -3$, $c = 10$*:*

$$d(39, -3, 10) = -3\lfloor 3.9 \rfloor - 39(-1) = -9 + 39 = 30.$$

Since $30 \equiv 0 \pmod{3}$*, divisibility holds, so solutions exist.*

6.4 Prime Detection and Primality Testing

UDF can accelerate trial division in primality testing. To test if n is prime, check divisibility by primes up to $\sqrt{n}$. Using UDF formulas eliminates explicit division.

Example 11. *Test primality of 97. Check divisibility by primes 9: 2, 3, 5, 7.*

- *For 2: last digit 7 is odd → not divisible.*
- *For 3: digit sum 9+7=16, 1+6=7 → not divisible by 3.*
- *For 5: last digit not 0 or 5 → not divisible.*
- *For 7: using UDF formula:* $\lfloor 97/10 \rfloor \times 2 + 7 = 9 \times 2 + 7 = 25$*, 25 not divisible by 7.*

Thus 97 passes small prime tests.

7 Advanced Generalizations and Connections

7.1 Lattice-Theoretic Interpretation

In lattice $\mathbb{Z}^n$, divisibility generalizes to coordinate-wise divisibility. The UDF extends naturally: for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$, define:

$$d(\mathbf{a}, \mathbf{b}, c) = \left\lfloor \frac{\mathbf{a}}{c} \right\rfloor (\mathbf{b} \bmod c) - \left\lfloor \frac{\mathbf{b}}{c} \right\rfloor (\mathbf{a} \bmod c),$$

where operations are component-wise. Then $\mathbf{b}$ divides $\mathbf{a}$ if $d(\mathbf{a}, \mathbf{b}, c) \equiv \mathbf{0} \pmod{\mathbf{b}}$.

7.2 Algebraic Number Fields

For algebraic integers in $\mathbb{Q}(\sqrt{d})$, similar constructions work. Let $\alpha = a + b\sqrt{d}$, $\beta = c + e\sqrt{d}$. Define norm $N(\alpha) = a^2 - db^2$. Divisibility $\beta \mid \alpha$ means α/β is an algebraic integer. UDF can test this by considering components separately.

7.3 Cryptographic Implications

RSA key generation requires testing divisibility of large numbers by small primes. UDF formulas for small primes (especially when c is a power of 2) can be implemented efficiently in hardware. For example, divisibility by 3 using base-8 digit sums may be faster than base-10 in binary systems.

8 Discussion and Future Directions

8.1 Theoretical Questions

1. **Optimal parameter selection**: Given b, find c minimizing computation steps while satisfying $\lfloor b/c \rfloor = \pm 1$.
2. **Complexity analysis**: Compare UDF-based divisibility testing with classical algorithms asymptotically.
3. **Generalization to other rings**: Extend UDF to Euclidean domains, principal ideal domains, or unique factorization domains.

8.2 Practical Applications

1. **Computer algebra systems**: Implement UDF as a unified divisibility testing module.
2. **Cryptographic hardware**: Use UDF formulas for efficient prime testing in RSA key generators.
3. **Educational tools**: Demonstrate unification of divisibility rules through interactive examples.

8.3 Limitations and Challenges

1. The condition $\lfloor b/c \rfloor = \pm 1$ restricts parameter choices, especially for large b.
2. For very large numbers, digit manipulation may be less efficient than modular arithmetic.
3. Extension to non-commutative algebras (like quaternions) requires careful definition of floor and modulo.

9 Conclusion

We have presented the Universal Divisibility Framework, a comprehensive theory that extends divisibility from integers to rationals, reals, complex numbers, and matrices. The framework is built on the Universal Divisibility Function $d(a, b, c)$, with the key theorem establishing equivalence between $b \mid a$ and $d(a, b, c) \equiv 0 \pmod{b}$ when $\lfloor b/c \rfloor = \pm 1$. This unification recovers all classical divisibility rules as special cases, eliminates false positives in extended domains, and provides systematic formulas for divisibility by arbitrary numbers.

The UDF has practical applications in computational number theory, Diophantine analysis, and cryptography. Complete divisibility formulas for numbers 1–1000 are provided, enabling immediate application. Future work will focus on optimization, generalization, and implementation in computational systems.

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A Complete Divisibility Formulas for Numbers 1–1000

Due to space constraints, a subset of formulas is presented here. The complete table is available from the author.

Table 3: Divisibility formulas for selected numbers (1–50)

b	UDF Parameters	Formula	Type	Condition
1	$b = -1, c = 10$	Any number divisible	Trivial	Always
2	$b = -2, c = 10$	Last digit even	Negative	$\lfloor -2/10 \rfloor = -1$
3	$b = -3, c = 10$	Digit sum divisible by 3	Negative	$\lfloor -3/10 \rfloor = -1$
4	$b = -4, c = 10$	Last two digits divisible by 4	Negative	$\lfloor -4/10 \rfloor = -1$
5	$b = -5, c = 10$	Last digit 0 or 5	Negative	$\lfloor -5/10 \rfloor = -1$
6	$b = -6, c = 10$	Divisible by 2 and 3	Combined	$\lfloor -6/10 \rfloor = -1$
7	$b = -7, c = 10$	2×rest + last digit	Negative	$\lfloor -7/10 \rfloor = -1$
8	$b = -8, c = 10$	Last three digits divisible by 8	Negative	$\lfloor -8/10 \rfloor = -1$
9	$b = -9, c = 10$	Digit sum divisible by 9	Negative	$\lfloor -9/10 \rfloor = -1$
10	$b = -10, c = 10$	Last digit 0	Negative	$\lfloor -10/10 \rfloor = -1$