

Multivector physics

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Maxwell's equations can be derived from the vector potential using only multivector calculations and duality in 4-dimension. Furthermore, by changing the dimension of the vector potential, three additional Maxwell's equations with different dimensions are derived. These four Maxwell's equations give us hints about charge, spin, quarks, and mass. The divergence of the electromagnetic field energy density expressed in multivectors is the electromagnetic force and Joule heat. In the same way, three more forces are also derived from three additional Maxwell's equations. One is gravity, another explains weak forces well, and the last one seems to be strong force. By defining the dot product of multivectors, the dual multivector, the curl, and the divergence of multivector fields are clearly defined. By defining the norm of a multivectors, the multiplication table of 16 multibases in 4-dimension is obtained. Multivectors are numbers that include direction and dimension, and an optimal language for physical quantities.

Keywords: multivector, Maxwell's equations, electromagnetic force

1 Introduction

A multivector is a series of vectors connected by a wedge product, and the order of the vectors is the sign of the multivector. Swapping two vectors results in the original multivector with the opposite sign. Therefore, the order of vectors is significant in the dot product of multivectors. Now, a vector means a multivector.

If an infinitesimal volume and its boundary are represented as vectors, the boundary vector can be obtained from the volume vector using the dot product. Rewriting the dot product of a vector field and a boundary vector defines the curl. This process itself is Stokes' theorem. The shape of a curl is similar to the exterior derivative of a differential form. Curl transforms a vector field into a one higher dimension that. This doesn't mean a new vector field is created; it just looks different depending on the dimension. The square of the curl is zero, like the square of the exterior derivative. Divergence is just curl for dual vectors. A crucial role of divergence is to determine the continuity of a vector field.

the product of a vector and its dual vector is pseudoscalar with the magnitude of the square of conventional vector norm. Then the multiplication table is created.

The vector field A and $\text{curl}(A)$ are one entity. And the dual vector of $\text{curl}(A)$ exists. The square of curl is 0. These propositions are Maxwell's equations, not experiments. The energy density E is expressed in the product of $\text{curl}(A)$ and dual vector of $\text{curl}(A)$, and the force F is $-\text{div}(E)$. The electromagnetic force and gravity matche this formula exactly.

2 Multivector and curl

A multivector is a product of several 1-vectors with a wedge (\wedge). If v_1, v_2, \dots, v_r are 1-vectors, then $(v_1 \wedge v_2)$ is a 2-vector and $(v_1 \wedge v_2 \wedge v_3)$ is a 3-vector. Multivector is anti-commutative in order.

$$(v_1 \wedge v_2) = -(v_2 \wedge v_1), \quad (v_1 \wedge v_1) = 0 \\ (\dots \wedge v_{i-1} \wedge v_i \wedge \dots) = -(\dots \wedge v_i \wedge v_{i-1} \wedge \dots)$$

2.1. Dot product

The dot product of vectors is commutative and defined recursively. $a, b, v_1, \dots, v_r, w_1, \dots, w_r$ are 1-vectors.

$$(v_1 \wedge v_2 \wedge \dots \wedge v_p) \cdot (w_1 \wedge w_2 \wedge \dots \wedge w_q) = (w_1 \wedge w_2 \wedge \dots \wedge w_q) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_p) \\ = (v_2 \wedge \dots \wedge v_p) \cdot \{v_1 \cdot (w_1 \wedge w_2 \wedge \dots \wedge w_q)\} \\ = (v_3 \wedge \dots \wedge v_p) \cdot [v_2 \cdot \{v_1 \cdot (w_1 \wedge w_2 \wedge \dots \wedge w_q)\}] = \dots, \quad (p \leq q)$$

$$a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = (-1)^{i-1} (a \cdot v_i) (v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_r), \quad \text{Einstein summation}$$

$$a \cdot (v_1 \wedge v_2 \wedge v_3) = (a \cdot v_1)(v_2 \wedge v_3) - (a \cdot v_2)(v_1 \wedge v_3) + (a \cdot v_3)(v_1 \wedge v_2)$$

$$\begin{aligned}
& (a \wedge b) \cdot (v_1 \wedge v_2) = b \cdot \{a \cdot (v_1 \wedge v_2)\} \\
& = b \cdot \{(a \cdot v_1)v_2 - (a \cdot v_2)v_1\} = (a \cdot v_1)(b \cdot v_2) - (a \cdot v_2)(b \cdot v_1) \\
& = -(b \wedge a) \cdot (v_1 \wedge v_2)
\end{aligned}$$

$$\begin{aligned}
& (a \wedge b) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) \\
& = (-1)^{i+j-1} \{(a \cdot v_i)(b \cdot v_j) - (a \cdot v_j)(b \cdot v_i)\} (v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_{j-1} \wedge v_{j+1} \wedge \dots \wedge v_r) \\
& = (-1)^{i+j} \{(b \cdot v_i)(a \cdot v_j) - (b \cdot v_j)(a \cdot v_i)\} (v_1 \wedge \dots \wedge v_{i-1} \wedge v_{i+1} \wedge \dots \wedge v_{j-1} \wedge v_{j+1} \wedge \dots \wedge v_r) \\
& = -(b \wedge a) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)
\end{aligned}$$

$$(a \wedge a) \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r) = a \cdot \{a \cdot (v_1 \wedge v_2 \wedge \dots \wedge v_r)\} = 0$$

2.2. Dual vector

In n-dimension, Cartesian coordinates basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$, unit pseudoscalar I .

$$I = (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \dots \wedge \mathbf{e}_n)$$

The dual vector of v , v^d is defined as follows. v and v^d are perpendicular to each other and are 1:1.

$$v^d = v \cdot I \quad \text{in particular} \quad I^d = 1, \quad 1^d = I$$

$$v \cdot v^d = v \cdot (v \cdot I) = 0$$

For example, in 3-dim,

$$I = (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3), \quad \mathbf{e}_1^d = \mathbf{e}_1 \cdot I = \mathbf{e}_1 \cdot (\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3) = (\mathbf{e}_2 \wedge \mathbf{e}_3)$$

2.3. Basis and reciprocal basis

In n-dimension, differentiable spatial coordinate system, the coordinate of a point is $P(p^1, p^2, \dots, p^n)$, which means the vector from the origin O to P . At P , the basis u_i is defined as follows.

$$u_i = \frac{\partial P}{\partial p^i}$$

When $u^i \cdot u_j = \delta_j^i$, u^1, u^2, \dots, u^n are reciprocal basis at P . Assuming matrices g^{ij} and g_{jk} , u^i at that point can be expressed as u_j , and u_i can be expressed as u^j .

$$u^i = g^{ij} u_j, \quad u^i \cdot u^k = g^{ij} u_j \cdot u^k = g^{ik}$$

$$u_i = g_{ij} u^j, \quad u_i \cdot u_k = g_{ij} u^j \cdot u_k = g_{ik}$$

Here, g_{jk} and g^{ij} are inverse matrices relationships.

$$u^i \cdot u_k = g^{ij} u_j \cdot u_k = g^{ij} g_{jk} = \delta_k^i$$

Therefore, by finding the basis at P , finding the matrix $g_{ij} = u_i \cdot u_j$, and finding the inverse matrix g^{ij} , we can find u^i of that point.

2.4. Infinitesimal volume $[dV]_r$ and its boundary $\partial[dV]_r$

$[dV]_r$ is an r-vector. It means something like a very small r-dimensional parallelepiped.

The reference vertex is $P(p^1, p^2, \dots, p^n)$. The r edges are $u_{\sigma_1} dp^{\sigma_1}, u_{\sigma_2} dp^{\sigma_2}, \dots, u_{\sigma_r} dp^{\sigma_r}$.

$$[dV]_r = (u_{\sigma_1} \wedge u_{\sigma_2} \wedge \dots \wedge u_{\sigma_r}) dp^{\sigma_1} dp^{\sigma_2} \dots dp^{\sigma_r}$$

$$\{\sigma_1, \sigma_2, \dots, \sigma_r\} \subset \{1, 2, \dots, n\}, \quad (\sigma_1 < \sigma_2 < \dots < \sigma_r)$$

ex) $[dV]_3 = (u_1 \wedge u_3 \wedge u_4) dp^1 dp^3 dp^4$

The 2r (r-1)-vector boundaries on the surface of $[dV]_r$ are $\partial[dV]_r$.

$$\partial[dV]_r = \frac{\pm u^i}{dp^i} \cdot [dV]_r = \begin{cases} \frac{+u^i}{dp^i} \cdot [dV]_r & \text{at } P(\dots, p^i + dp^i, \dots) \\ \frac{-u^i}{dp^i} \cdot [dV]_r & \text{at } P(\dots, p^i, \dots) \end{cases}$$

ex) $[dV]_1 = u_2 dp^2$

$$\partial[dV]_1 = \frac{\pm u^i}{dp^i} \cdot [dV]_1 = \begin{cases} \frac{+u^2}{dp^2} \cdot u_2 dp^2 = 1 & \text{at } P(\dots, p^2 + dp^2, \dots) \\ \frac{-u^2}{dp^2} \cdot u_2 dp^2 = -1 & \text{at } P(\dots, p^2, \dots) \end{cases}$$

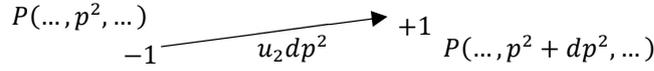


Fig.1 $\partial[dV]_1$ is ± 1 . It has opposite signs to adjacent $\partial[dV]_1$

ex) $[dV]_2 = (u_1 \wedge u_2) dp^1 dp^2$

$$\partial[dV]_2 = \frac{\pm u^i}{dp^i} \cdot [dV]_2 = \begin{cases} \frac{+u^1}{dp^1} \cdot (u_1 \wedge u_2) dp^1 dp^2 = +u_2 dp^2 & \text{at } P(\dots, p^1 + dp^1, \dots) \\ \frac{-u^1}{dp^1} \cdot (u_1 \wedge u_2) dp^1 dp^2 = -u_2 dp^2 & \text{at } P(\dots, p^1, \dots) \\ \frac{+u^2}{dp^2} \cdot (u_1 \wedge u_2) dp^1 dp^2 = -u_1 dp^1 & \text{at } P(\dots, p^2 + dp^2, \dots) \\ \frac{-u^2}{dp^2} \cdot (u_1 \wedge u_2) dp^1 dp^2 = +u_1 dp^1 & \text{at } P(\dots, p^2, \dots) \end{cases}$$

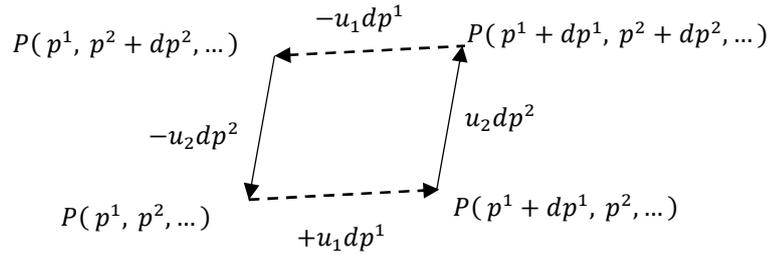


Fig.2 $\partial[dV]_2$ rotates in one direction. It rotates in the opposite direction to adjacent $\partial[dV]_2$

2.5. Curl

The definition of $curl(A)$ is the formula itself. The dot product of a $(r-1)$ -vector field and the boundary vector.

$$\begin{aligned} A \cdot \partial[dV]_r &= A \cdot \left(\frac{\pm u^i}{dp^i} \cdot [dV]_r \right) = \left(\frac{\pm u^i}{dp^i} \wedge A \right) \cdot [dV]_r = \left(u^i \wedge \frac{\pm A}{dp^i} \right) \cdot [dV]_r \\ &= \left(u^i \wedge \frac{\partial A}{\partial p^i} \right) \cdot [dV]_r = curl(A) \cdot [dV]_r, \quad curl(A) = u^i \wedge \frac{\partial A}{\partial p^i} \end{aligned}$$

The definition itself is Stokes' theorem. This does not mean that a new vector field $curl(A)$ is created, but rather that it is a view of A from one dimension higher.

$$\oint A \cdot \partial[dV]_r = \oint curl(A) \cdot [dV]_r$$

In calculus, a vector field must be represented by an reciprocal basis so that the basis cancels out with dot product. Then, differentiate only the coefficients of A to find $curl(A)$. Cartesian coordinates already use a reciprocal basis. The covariant derivative is superfluous.

$$curl^2(A) = curl\{curl(A)\} = \left(u^j \wedge u^i \wedge \frac{\partial^2 A}{\partial p^j \partial p^i} \right) = \left(u^i \wedge u^j \wedge \frac{\partial^2 A}{\partial p^i \partial p^j} \right) = 0$$

That is, only A and $curl(A)$ are one entity.

2.6. Comparison to conventional curl

Comparison to conventional curl using the well-known vector field $A = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi$ in spherical coordinates. For this, first convert $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ to curvilinear coordinates basis u^r, u^θ, u^ϕ , and calculate it, and then convert the result back to $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$ and check it.

$$\begin{aligned} p^1 &= r, & p^2 &= \theta, & p^3 &= \phi \\ u_r &= \frac{\partial P}{\partial r} = \mathbf{e}_r, & u_\theta &= \frac{\partial P}{\partial \theta} = r \mathbf{e}_\theta, & u_\phi &= \frac{\partial P}{\partial \phi} = r \sin \theta \mathbf{e}_\phi \\ u^i \cdot u_k &= \delta_k^i, & u^r &= \mathbf{e}_r, & u^\theta &= \frac{\mathbf{e}_\theta}{r}, & u^\phi &= \frac{\mathbf{e}_\phi}{r \sin \theta} \\ A &= A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi = A_r u^r + A_\theta r u^\theta + A_\phi r \sin \theta u^\phi \end{aligned}$$

Among $\text{curl}(A)$, only $(u^\theta \wedge u^\phi)$ term is compared.

$$\begin{aligned} \text{curl}(A) &= u^i \wedge \frac{\partial A}{\partial p^i} \\ u^\theta \wedge \frac{\partial(A_\phi r \sin \theta)}{\partial \theta} u^\phi + u^\phi \wedge \frac{\partial(A_\theta r)}{\partial \phi} u^\theta &= \left\{ \frac{\partial(A_\phi r \sin \theta)}{\partial \theta} - \frac{\partial(A_\theta r)}{\partial \phi} \right\} (u^\theta \wedge u^\phi) \\ &= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial(A_\phi r \sin \theta)}{\partial \theta} - \frac{\partial(A_\theta r)}{\partial \phi} \right\} (\mathbf{e}_\theta \wedge \mathbf{e}_\phi) \end{aligned}$$

Compared to the conventional results, the coefficient is the same. However, it is not the \mathbf{e}_r term, but the $(\mathbf{e}_\theta \wedge \mathbf{e}_\phi)$ term. These two are dual basis each other.

2.7. Divergence

$\text{divergence}(A)$ is curl with respect to A^d .

$$\text{div}(A) = \text{curl}(A^d), \quad A^d = A \cdot I$$

In 3-dim, $A = A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi$.

$$\begin{aligned} A &= A_r \mathbf{e}_r + A_\theta \mathbf{e}_\theta + A_\phi \mathbf{e}_\phi = A_r u^r + \frac{A_\theta}{r} u_\theta + \frac{A_\phi}{r \sin \theta} u_\phi \\ I &= \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_r \wedge \mathbf{e}_\theta \wedge \mathbf{e}_\phi = r^2 \sin \theta (u^r \wedge u^\theta \wedge u^\phi) \\ \mathbf{e}_r &= \sin \theta \cos \phi \mathbf{e}_1 + \sin \theta \sin \phi \mathbf{e}_2 + \cos \theta \mathbf{e}_3 \\ \mathbf{e}_\theta &= \cos \theta \cos \phi \mathbf{e}_1 + \cos \theta \sin \phi \mathbf{e}_2 - \sin \theta \mathbf{e}_3 \\ \mathbf{e}_\phi &= -\sin \phi \mathbf{e}_1 + \cos \phi \mathbf{e}_2 \dots \end{aligned}$$

$$\begin{aligned} A^d &= A \cdot I = \left(A_r u^r + \frac{A_\theta}{r} u_\theta + \frac{A_\phi}{r \sin \theta} u_\phi \right) \cdot r^2 \sin \theta (u^r \wedge u^\theta \wedge u^\phi) \\ &= A_r r^2 \sin \theta (u^\theta \wedge u^\phi) - A_\theta r \sin \theta (u^r \wedge u^\phi) + A_\phi r (u^r \wedge u^\theta) \\ \text{div}(A) &= \text{curl}(A^d) = u^i \wedge \frac{\partial A^d}{\partial p^i} \\ &= \left\{ \frac{\partial(A_r r^2 \sin \theta)}{\partial r} + \frac{\partial(A_\theta r \sin \theta)}{\partial \theta} + \frac{\partial(A_\phi r)}{\partial \phi} \right\} (u^r \wedge u^\theta \wedge u^\phi) \\ &= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial(A_r r^2 \sin \theta)}{\partial r} + \frac{\partial(A_\theta r \sin \theta)}{\partial \theta} + \frac{\partial(A_\phi r)}{\partial \phi} \right\} (\mathbf{e}_r \wedge \mathbf{e}_\theta \wedge \mathbf{e}_\phi) \end{aligned}$$

Compared to the conventional results, the coefficient is the same. However, it is not a scalar term but a $I = (\mathbf{e}_r \wedge \mathbf{e}_\theta \wedge \mathbf{e}_\phi)$ term. These two are also dual basis each other.

2.8. Continuity equation, $\text{div}(A) = 0$

$$\text{div}(A) \cdot [dV]_r = \text{curl}(A^d) \cdot [dV]_r = A^d \cdot \partial[dV]_r = (A \cdot I) \cdot \partial[dV]_r = (A \wedge \partial[dV]_r) \cdot I$$

$(A \wedge \partial[dV]_r) \cdot I$ is a physical quantity that moves in and out at speed A through the boundary $\partial[dV]_r$ of $[dV]_r$. $(A \wedge \partial[dV]_r)$ is the product $\partial[dV]_r$ and only the components of A perpendicular to $\partial[dV]_r$. The parallel components cancel out in the wedge product. If $\text{div}(A) = 0$ at all points, this means that a physical quantity flows without accumulation or leakage. In this case, A is said to be continuous.

3. Multiplication table of 16 bases

In Minkowski space, the relationship between \mathbb{t} $\mathbb{e}_1, \mathbb{e}_2, \mathbb{e}_3, \mathbb{e}_t$ and $\mathbb{e}^1, \mathbb{e}^2, \mathbb{e}^3, \mathbb{e}^t$ is as follows.

$$\begin{aligned} \mathbb{e}_t \cdot \mathbb{e}_t &= \mathbb{e}^t \cdot \mathbb{e}^t = -1, & \mathbb{e}^t \cdot \mathbb{e}_t &= 1, & \mathbb{e}^t &= -\mathbb{e}_t, & \mathbb{e}^1 &= \mathbb{e}_1, & \mathbb{e}^2 &= \mathbb{e}_2, & \mathbb{e}^3 &= \mathbb{e}_3 \\ I &= \mathbb{e}_1 \wedge \mathbb{e}_2 \wedge \mathbb{e}_3 \wedge \mathbb{e}_t = -\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t \end{aligned}$$

For any arbitrary vector v , assume $vv^d = \|v\|^2 I$, $\|v\|$ is vector norm, then the multiplication table is obtained.

3.1. Basic multiplication

$$v = 2\mathbb{e}^1, \quad v^d = v \cdot I = 2(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$vv^d = \|v\|^2 I = (2)^2 I = (2)^2 \mathbb{e}^1 (-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t), \quad \mathbb{e}^1 (-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = I$$

$$\begin{aligned} \mathbb{e}^1 (-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) &= \mathbb{e}^2 (-\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) = \mathbb{e}^3 (-\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) = \mathbb{e}^t (-\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) \\ &= (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) (-\mathbb{e}^1) = (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) (-\mathbb{e}^2) = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) (-\mathbb{e}^3) = (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) (-\mathbb{e}^t) = I \end{aligned}$$

$$\begin{aligned} (\mathbb{e}^1 \wedge \mathbb{e}^2) (-\mathbb{e}^3 \wedge \mathbb{e}^t) &= (\mathbb{e}^2 \wedge \mathbb{e}^3) (-\mathbb{e}^1 \wedge \mathbb{e}^t) = (\mathbb{e}^3 \wedge \mathbb{e}^1) (-\mathbb{e}^2 \wedge \mathbb{e}^t) \\ &= (\mathbb{e}^1 \wedge \mathbb{e}^t) (\mathbb{e}^2 \wedge \mathbb{e}^3) = (\mathbb{e}^2 \wedge \mathbb{e}^t) (\mathbb{e}^3 \wedge \mathbb{e}^1) = (\mathbb{e}^3 \wedge \mathbb{e}^t) (\mathbb{e}^1 \wedge \mathbb{e}^2) = I \end{aligned}$$

$$11^d = II^d = I, \quad 1^d = I, \quad I^d = 1$$

$$\mathbb{b}_1: \mathbb{e}^1, \mathbb{e}^2, \mathbb{e}^3, \mathbb{e}^t$$

$$\mathbb{b}_2: (\mathbb{e}^1 \wedge \mathbb{e}^2), (\mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^3 \wedge \mathbb{e}^1), (\mathbb{e}^1 \wedge \mathbb{e}^t), (\mathbb{e}^2 \wedge \mathbb{e}^t), (\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$\mathbb{b}_3: (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t), (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t), (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)$$

3.2. $\mathbb{b}_i I, I \mathbb{b}_i$

$$v = (\mathbb{e}^1 + 2), \quad v^d = \{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\}, \quad vv^d = ((1)^2 + (2)^2)I$$

$$vv^d = (\mathbb{e}^1 + 2)\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\} = ((1)^2 + (2)^2)I + 2(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\mathbb{e}^1 I$$

$$\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\}(\mathbb{e}^1 + 2) = ((1)^2 + (2)^2)I + 2(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2I\mathbb{e}^1$$

$$\mathbb{e}^1 I = (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t), \quad I \mathbb{e}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$vv^d = \{(-\mathbb{e}^3 \wedge \mathbb{e}^t) + 2\}\{(-\mathbb{e}^1 \wedge \mathbb{e}^2) + 2I\} = ((1)^2 + (2)^2)I$$

$$vv^d = \{(\mathbb{e}^1 \wedge \mathbb{e}^2) + 2I\}\{(-\mathbb{e}^3 \wedge \mathbb{e}^t) + 2\} = ((1)^2 + (2)^2)I$$

$$(\mathbb{e}^3 \wedge \mathbb{e}^t)I = -(\mathbb{e}^1 \wedge \mathbb{e}^2), \quad I(\mathbb{e}^3 \wedge \mathbb{e}^t) = (\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$\{(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\}\{-\mathbb{e}^1 + 2I\} = ((1)^2 + (2)^2)I$$

$$\{-\mathbb{e}^1 + 2I\}\{(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2\} = ((1)^2 + (2)^2)I$$

$$(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)I = \mathbb{e}^1 \quad I(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = \mathbb{e}^1$$

3.3. $\mathbb{b}_1 \mathbb{b}_1, \mathbb{b}_3 \mathbb{b}_3$ anti-commutative

$$vv^d = \{\mathbb{e}^1 + 2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)\}\{(-\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + 2(-\mathbb{e}^2)\} = ((1)^2 + (2)^2)I$$

$$\mathbb{e}^1 \mathbb{e}^2 + (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

These multiplications are interpreted as follows: the common basis is eliminated in the following way.

$$\mathbb{e}^1 \mathbb{e}^2 = (\mathbb{e}^1 \wedge \mathbb{e}^2), \quad \mathbb{e}^2 \mathbb{e}^1 = (\mathbb{e}^2 \wedge \mathbb{e}^1) = -(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$\begin{aligned} (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) &= (\mathbb{e}^1 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^t \wedge \mathbb{e}^3 \wedge \mathbb{e}^2) = (\mathbb{e}^t \cdot \mathbb{e}^t)(\mathbb{e}^1 \wedge \mathbb{e}^3)(\mathbb{e}^3 \wedge \mathbb{e}^2) \\ &= -(\mathbb{e}^3 \cdot \mathbb{e}^3)\mathbb{e}^1 \mathbb{e}^2 = -(\mathbb{e}^1 \wedge \mathbb{e}^2) = -(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) \end{aligned}$$

$$\begin{aligned}
vv^d &= \{e^1 + 2(e^1 \wedge e^2 \wedge e^3)\}\{-e^2 \wedge e^3 \wedge e^t + 2(-e^t)\} = ((1)^2 + (2)^2)I \\
e^1 e^t + (e^1 \wedge e^2 \wedge e^3)(e^2 \wedge e^3 \wedge e^t) &= 0 \\
e^1 e^t &= (e^1 \wedge e^t) \\
(e^1 \wedge e^2 \wedge e^3)(e^2 \wedge e^3 \wedge e^t) &= -(e^1 \wedge e^2 \wedge e^3)(e^3 \wedge e^2 \wedge e^t) = -(e^3 \cdot e^3)(e^2 \cdot e^2)e^1 e^t = \\
&= -(e^1 \wedge e^t)
\end{aligned}$$

3.4. $\mathbb{b}_2 \mathbb{b}_2$ anti-commutative

$$\begin{aligned}
vv^d &= \{(e^1 \wedge e^2) + 2(e^2 \wedge e^t)\}\{-e^3 \wedge e^t + 2(e^3 \wedge e^1)\} = ((1)^2 + (2)^2)I \\
(e^1 \wedge e^2)(e^3 \wedge e^1) - (e^2 \wedge e^t)(e^3 \wedge e^t) &= 0 \\
(e^1 \wedge e^2)(e^3 \wedge e^1) &= (e^2 \wedge e^1)(e^1 \wedge e^3) = (e^1 \cdot e^1)e^2 e^3 = (e^2 \wedge e^3) \\
(e^2 \wedge e^t)(e^3 \wedge e^t) &= -(e^2 \wedge e^t)(e^t \wedge e^3) = -(e^t \cdot e^t)e^2 e^3 = (e^2 \wedge e^3)
\end{aligned}$$

3.5. $\mathbb{b}_1 \mathbb{b}_3, \mathbb{b}_3 \mathbb{b}_1$ commutative

$$\begin{aligned}
vv^d &= (e^1 + 2e^2)\{-e^2 \wedge e^3 \wedge e^t + 2(-e^3 \wedge e^1 \wedge e^t)\} = ((1)^2 + (2)^2)I \\
e^1(e^3 \wedge e^1 \wedge e^t) + e^2(e^2 \wedge e^3 \wedge e^t) &= 0 \\
e^1(e^3 \wedge e^1 \wedge e^t) &= e^1(e^1 \wedge e^t \wedge e^3) = (e^1 \cdot e^1)(e^t \wedge e^3) = -(e^3 \wedge e^t) \\
e^2(e^2 \wedge e^3 \wedge e^t) &= (e^2 \cdot e^2)(e^3 \wedge e^t) = (e^3 \wedge e^t) \\
\{(-e^1 \wedge e^2 \wedge e^3) + 2(-e^3 \wedge e^1 \wedge e^t)\}(e^t + 2e^2) &= ((1)^2 + (2)^2)I \\
(e^1 \wedge e^2 \wedge e^3)e^2 + (e^3 \wedge e^1 \wedge e^t)e^t &= 0 \\
(e^1 \wedge e^2 \wedge e^3)e^2 &= (e^3 \wedge e^1 \wedge e^2)e^2 = (e^2 \cdot e^2)(e^3 \wedge e^1) = (e^3 \wedge e^1) \\
(e^3 \wedge e^1 \wedge e^t)e^t &= (e^3 \wedge e^1 \wedge e^t)e^t = (e^t \cdot e^t)(e^3 \wedge e^1) = -(e^3 \wedge e^1)
\end{aligned}$$

3.6. $\mathbb{b}_1 \mathbb{b}_2, \mathbb{b}_2 \mathbb{b}_3$

case1: $\mathbb{b}_1 \mathbb{b}_2$ has one common basis, and $\mathbb{b}_2 \mathbb{b}_3$ has two common bases.

$$\begin{aligned}
vv^d &= \{e^2 + 2(e^3 \wedge e^t)\}\{-e^3 \wedge e^1 \wedge e^t + 2(e^1 \wedge e^2)\} = ((1)^2 + (2)^2)I \\
vv^d &= \{(-e^3 \wedge e^1 \wedge e^t) + 2(e^1 \wedge e^2)\}\{e^2 + 2(-e^3 \wedge e^t)\} = ((1)^2 + (2)^2)I \\
e^2(e^1 \wedge e^2) - (e^3 \wedge e^t)(e^3 \wedge e^1 \wedge e^t) &= 0 \\
(e^1 \wedge e^2)e^2 + (e^3 \wedge e^1 \wedge e^t)(e^3 \wedge e^t) &= 0 \\
e^2(e^1 \wedge e^2) &= -e^2(e^2 \wedge e^1) = -(e^2 \cdot e^2)e^1 = -e^1 \\
(e^3 \wedge e^t)(e^3 \wedge e^1 \wedge e^t) &= (e^3 \wedge e^t)(e^t \wedge e^3 \wedge e^1) = (e^t \cdot e^t)(e^3 \cdot e^3)e^1 = -e^1 \\
(e^1 \wedge e^2)e^2 &= (e^2 \cdot e^2)e^1 = e^1 \\
(e^3 \wedge e^1 \wedge e^t)(e^3 \wedge e^t) &= (e^1 \wedge e^t \wedge e^3)(e^3 \wedge e^t) = (e^3 \cdot e^3)(e^t \cdot e^t)e^1 = -e^1
\end{aligned}$$

case2: $\mathbb{b}_1 \mathbb{b}_2$ has no common basis, and $\mathbb{b}_2 \mathbb{b}_3$ has one common basis. In particular, this multiplication changes the sign while removing the common basis.

$$\begin{aligned}
vv^d &= \{e^1 + 2(e^1 \wedge e^t)\}\{-e^2 \wedge e^3 \wedge e^t + 2(e^2 \wedge e^3)\} = ((1)^2 + (2)^2)I \\
vv^d &= \{(-e^2 \wedge e^3 \wedge e^t) + 2(e^2 \wedge e^3)\}\{e^1 + 2(-e^1 \wedge e^t)\} = ((1)^2 + (2)^2)I \\
e^1(e^2 \wedge e^3) - (e^1 \wedge e^t)(e^2 \wedge e^3 \wedge e^t) &= 0 \\
(e^2 \wedge e^3)e^1 + (e^2 \wedge e^3 \wedge e^t)(e^1 \wedge e^t) &= 0 \\
e^1(e^2 \wedge e^3) &= (e^1 \wedge e^2 \wedge e^3) \\
(e^1 \wedge e^t)(e^2 \wedge e^3 \wedge e^t) &= (e^1 \wedge e^t)(e^t \wedge e^2 \wedge e^3) = -(e^t \cdot e^t)e^1(e^2 \wedge e^3) = e^1(e^2 \wedge e^3)
\end{aligned}$$

$$\begin{aligned}
(e^2 \wedge e^3)e^1 &= (e^2 \wedge e^3 \wedge e^1) = (e^1 \wedge e^2 \wedge e^3) \\
(e^2 \wedge e^3 \wedge e^t)(e^1 \wedge e^t) &= -(e^2 \wedge e^3 \wedge e^t)(e^t \wedge e^1) = (e^t \cdot e^t)(e^2 \wedge e^3)e^1 = -(e^2 \wedge e^3)e^1
\end{aligned}$$

3.7. Square of basis

$$\begin{aligned}
vv^d &= (I + 2)(1 + 2I) = ((1)^2 + (2)^2)I + 2(1 + I^2) \\
1^2 + I^2 &= 0, \quad I^2 = -1
\end{aligned}$$

The square of basis is chosen in ± 1 . The value is determined by considering the energy density in chapter 5.

$$\begin{aligned}
vv^d &= \{(e^1 \wedge e^t) + 2(e^2 \wedge e^3)\}\{(e^2 \wedge e^3) - 2(e^1 \wedge e^t)\} = ((1)^2 + (2)^2)I \\
(e^1 \wedge e^t)(e^1 \wedge e^t) - (e^2 \wedge e^3)(e^2 \wedge e^3) &= 0 \\
(e^1 \wedge e^t)^2 = (e^2 \wedge e^3)^2 = (e^3 \wedge e^t)^2 = (e^1 \wedge e^2)^2 = (e^2 \wedge e^3)^2 = (e^3 \wedge e^1)^2 &= 1 \\
vv^d &= \{e^1 + 2(e^2 \wedge e^3 \wedge e^t)\}\{-e^2 \wedge e^3 \wedge e^t + 2(-e^1)\} = ((1)^2 + (2)^2)I \\
e^1 e^1 + (e^2 \wedge e^3 \wedge e^t)^2 &= 0 \\
(e^1)^2 = (e^2)^2 = (e^3)^2 = (e^t)^2 &= -1 \\
(e^2 \wedge e^3 \wedge e^t)^2 = (e^1 \wedge e^2 \wedge e^t)^2 = (e^3 \wedge e^1 \wedge e^t)^2 = (e^1 \wedge e^2 \wedge e^3)^2 &= 1
\end{aligned}$$

3.8. Square of vector formed by bases in a dimension

Multiplication, $\mathbb{b}_1\mathbb{b}_1$, $\mathbb{b}_2\mathbb{b}_2$, $\mathbb{b}_3\mathbb{b}_3$ are all anticommutative. Therefore.

$$\begin{aligned}
\text{ex) } v &= e^1 + 2e^2 + 3e^t \\
v^2 &= (e^1 + 2e^2 + 3e^t)(e^1 + 2e^2 + 3e^t) \\
&= (e^1)^2 + (2e^2)^2 + (3e^t)^2 + 2(e^1e^2 + e^2e^1) + 6(e^2e^t + e^te^2) + 3(e^1e^2 + e^2e^1) \\
&= -\{(1)^2 + (2)^2 + (3)^2\} = -(\|v\|)^2
\end{aligned}$$

Dual vectors also have the same coefficients.

$$\begin{aligned}
v^d &= v \cdot I = (-e^2 \wedge e^3 \wedge e^t) + 2(-e^3 \wedge e^1 \wedge e^t) + 3(-e^1 \wedge e^2 \wedge e^3) \\
\{v^d\}^2 &= \{(1)^2 + (2)^2 + (3)^2\} = (\|v\|)^2
\end{aligned}$$

4. Maxwell's equations

- 1) If vector potential A and $\text{curl}(A)$ exists. They are one entity.
- 2) Then, $\text{curl}^d(A)$ and $\text{curl}\{\text{curl}^d(A)\} = J$ also exist. It's duality.
 $\text{curl}^d(A) = \text{curl}(A) \cdot I$, $\text{curl}\{\text{curl}^d(A)\} = \text{div}\{\text{curl}(A)\} = \text{Laplacian}(A) = J$
This proposition is Gauss's law and Ampere-Maxwell's law. These are not experimental laws.
- 3) $\text{curl}^2(A) = 0$ and $\text{curl}(J) = \text{curl}^2\{\text{curl}^d(A)\} = 0$.
These are Gauss's law for magnetism, and Faraday's law, and charge conservative law.
- 4) If enough time passes and equilibrium is reached, vector potential A becomes continuous, $\text{div}(A) = 0$.
Then the wave equation for A and J is obtained.

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial_{\mu\nu} = \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \quad (\mu, \nu = 1, 2, 3, 4), \quad \square = \partial_{tt} - \partial_{11} - \partial_{22} - \partial_{33}$$

$$\begin{aligned}
1) \quad A &= A_1 e^1 + A_2 e^2 + A_3 e^3 + A_t e^t = A_1 e_1 + A_2 e_2 + A_3 e_3 - A_t e_t, \quad (A_t = -\varphi) \\
\text{curl}(A) &= e^\mu \wedge \partial_\mu A \\
&= (\partial_2 A_3 - \partial_3 A_2)(e^2 \wedge e^3) + (\partial_3 A_1 - \partial_1 A_3)(e^3 \wedge e^1) + (\partial_1 A_2 - \partial_2 A_1)(e^1 \wedge e^2) \\
&\quad + (\partial_1 A_t - \partial_t A_1)(e^1 \wedge e^t) + (\partial_2 A_t - \partial_t A_2)(e^2 \wedge e^t) + (\partial_3 A_t - \partial_t A_3)(e^3 \wedge e^t) \\
&= B + E = B_1(e^2 \wedge e^3) + B_2(e^3 \wedge e^1) + B_3(e^1 \wedge e^2) + E_1(e^1 \wedge e^t) + E_2(e^2 \wedge e^t) + E_3(e^3 \wedge e^t)
\end{aligned}$$

B : magnetic field, E : electric field

$$2) \quad \text{curl}^d(A) = \text{curl}(A) \cdot I$$

$$\begin{aligned}
&= E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + E_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + E_3(\mathbb{e}^1 \wedge \mathbb{e}^2) - B_1(\mathbb{e}^1 \wedge \mathbb{e}^t) - B_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - B_3(\mathbb{e}^3 \wedge \mathbb{e}^t) \\
&\quad J = -J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) \\
&= \text{curl}\{\text{curl}^d(A)\} = \mathbb{e}^\mu \wedge \partial_\mu \{\text{curl}^d(A)\} \\
&= (\partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - (\partial_2 B_3 - \partial_3 B_2 - \partial_t E_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) \\
&\quad - (\partial_3 B_1 - \partial_1 B_3 - \partial_t E_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - (\partial_1 B_2 - \partial_2 B_1 - \partial_t E_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)
\end{aligned}$$

$J_t = -\rho$: charge density, J_i : current density.

Conventional coefficient representation. Gauss's law, and Ampère–Maxwell law.

$$\nabla \cdot E = -J_t = \rho, \quad \nabla \times B = J + \partial_t E$$

$$3) \quad \text{curl}^2(A) = \mathbb{e}^\mu \wedge \partial_\mu \{\text{curl}(A)\}$$

$$\begin{aligned}
&= (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_2 E_3 - \partial_3 E_2 + \partial_t B_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) \\
&\quad + (\partial_3 E_1 - \partial_1 E_3 + \partial_t B_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_1 E_2 - \partial_2 E_1 + \partial_t B_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) = 0
\end{aligned}$$

Conventional coefficient representation. Gauss's law for magnetism and Faraday's law.

$$\nabla \cdot B = 0, \quad \nabla \times E + \partial_t B = 0$$

$$\text{curl}(J) = \mathbb{e}^\mu \wedge \partial_\mu J = (\partial_1 J_1 + \partial_2 J_2 + \partial_3 J_3 - \partial_t J_t)I = 0$$

Conventional coefficient representation. Charge conservative law.

$$\nabla \cdot J + \partial_t \rho = 0$$

$$4) \quad A^d = A \cdot I = -A_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - A_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - A_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) - A_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$\text{div}(A) = \text{curl}\{A^d\} = \mathbb{e}^\mu \wedge \partial_\mu A^d = (\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t)I = 0$$

Conventional coefficient representation. Lorentz condition.

$$\nabla \cdot A + \partial_t \varphi = 0$$

$\text{curl}^d(A)$ is also continuous. $\text{div}\{\text{curl}^d(A)\} = \text{curl}\{\text{curl}^{dd}(A)\} = -\text{curl}^2(A) = 0$.

Substituting A_μ for E_i, B_j in the 2), it becomes the wave equation.

$$\begin{aligned}
-J_t &= \partial_1 E_1 + \partial_2 E_2 + \partial_3 E_3 = \partial_{11} A_t - \partial_{1t} A_1 + \partial_{22} A_t - \partial_{2t} A_2 + \partial_{33} A_t - \partial_{3t} A_3 \\
&= \partial_{11} A_t + \partial_{22} A_t + \partial_{33} A_t - \partial_{tt} A_t - \partial_t(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t) \\
&= \partial_{11} A_t + \partial_{22} A_t + \partial_{33} A_t - \partial_{tt} A_t = -\square A_t
\end{aligned}$$

$$\begin{aligned}
J_1 &= \partial_2 B_3 - \partial_3 B_2 - \partial_t E_1 = \partial_{21} A_2 - \partial_{22} A_1 - \partial_{33} A_1 + \partial_{31} A_3 + \partial_{tt} A_1 - \partial_{t1} A_t \\
&= \partial_{tt} A_1 - \partial_{11} A_1 - \partial_{22} A_1 - \partial_{33} A_1 + \partial_1(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3 - \partial_t A_t) = \square A_1
\end{aligned}$$

$$J_t = \square A_t, \quad J_1 = \square A_1, \quad J_2 = \square A_2, \quad J_3 = \square A_3$$

4.1. 2-vector field

There is no problem in changing the dimension of vector potential in Maxwell's equations propositions.

$$1) \quad B + \mathcal{E} = B_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + B_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + B_3(\mathbb{e}^1 \wedge \mathbb{e}^2) + \mathcal{E}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{E}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + \mathcal{E}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$\text{curl}(B + \mathcal{E}) = \mathbb{e}^\mu \wedge \partial_\mu (B + \mathcal{E}) = \mathcal{A}$$

$$= \mathcal{A}_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) + \mathcal{A}_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) + \mathcal{A}_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{A}_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$\mathcal{A}_t = \partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3, \quad \mathcal{A}_1 = \partial_2 \mathcal{E}_3 - \partial_3 \mathcal{E}_2 + \partial_t B_1$$

$$\mathcal{A}_2 = \partial_3 \mathcal{E}_1 - \partial_1 \mathcal{E}_3 + \partial_t B_2, \quad \mathcal{A}_3 = \partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1 + \partial_t B_3$$

Conventional expression.

$$\mathcal{A}_t = \nabla \cdot B, \quad \mathcal{A} = \nabla \times \mathcal{E} + \partial_t B$$

$$2) \quad \text{curl}^d(B + \mathcal{E}) = \text{curl}(B + \mathcal{E}) \cdot I = -\mathcal{A}_1 \mathbb{e}^1 - \mathcal{A}_2 \mathbb{e}^2 - \mathcal{A}_3 \mathbb{e}^3 - \mathcal{A}_t \mathbb{e}^t$$

$$J = -\mathcal{K}_1(\mathbb{e}^2 \wedge \mathbb{e}^3) - \mathcal{K}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) - \mathcal{K}_3(\mathbb{e}^1 \wedge \mathbb{e}^2) + \mathcal{L}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{L}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + \mathcal{L}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$= \mathbb{e}^\mu \wedge \partial_\mu \{\text{curl}^d(B + \mathcal{E})\}$$

$$= (\partial_3 \mathcal{A}_2 - \partial_2 \mathcal{A}_3)(\mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_1 \mathcal{A}_3 - \partial_3 \mathcal{A}_1)(\mathbb{e}^3 \wedge \mathbb{e}^1) + (\partial_2 \mathcal{A}_1 - \partial_1 \mathcal{A}_2)(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$+ (\partial_t \mathcal{A}_1 - \partial_1 \mathcal{A}_t)(\mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_t \mathcal{A}_2 - \partial_2 \mathcal{A}_t)(\mathbb{e}^2 \wedge \mathbb{e}^t) + (\partial_t \mathcal{A}_3 - \partial_3 \mathcal{A}_t)(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

\mathcal{K}_i : spin density, \mathcal{L}_i : spin current density.

Conventional expression.

$$-\mathcal{K} = \nabla \times \mathcal{A}, \quad -\mathcal{L} = \nabla \mathcal{A}_t - \partial_t \mathcal{A}$$

$$3) \quad \text{curl}^2(\mathcal{B} + \mathcal{E}) = \mathbb{e}^\mu \wedge \partial_\mu \mathcal{A} = (\partial_1 \mathcal{A}_1 + \partial_2 \mathcal{A}_2 + \partial_3 \mathcal{A}_3 - \partial_t \mathcal{A}_t)(-I) = 0$$

Conventional expression.

$$\nabla \cdot \mathcal{A} - \partial_t \mathcal{A}_t = 0$$

$$\begin{aligned} \text{curl}(\mathcal{J}) &= \text{curl}(\mathcal{K} + \mathcal{L}) = \mathbb{e}^\mu \wedge \partial_\mu (\mathcal{K} + \mathcal{L}) \\ &= -(\partial_1 \mathcal{K}_1 + \partial_2 \mathcal{K}_2 + \partial_3 \mathcal{K}_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_2 \mathcal{L}_3 - \partial_3 \mathcal{L}_2 - \partial_t \mathcal{K}_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) \\ &\quad + (\partial_3 \mathcal{L}_1 - \partial_1 \mathcal{L}_3 - \partial_t \mathcal{K}_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_1 \mathcal{L}_2 - \partial_2 \mathcal{L}_1 - \partial_t \mathcal{K}_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) = 0 \end{aligned}$$

Conventional expression

$$\nabla \cdot \mathcal{K} = 0, \quad \nabla \times \mathcal{L} - \partial_t \mathcal{K} = 0$$

$$4) \quad (\mathcal{B} + \mathcal{E})^d = (\mathcal{B} + \mathcal{E}) \cdot I$$

$$= \mathcal{E}_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + \mathcal{E}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + \mathcal{E}_3(\mathbb{e}^1 \wedge \mathbb{e}^2) - \mathcal{B}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) - \mathcal{B}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - \mathcal{B}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$\text{div}(\mathcal{B} + \mathcal{E}) = \text{curl}\{\mathcal{B}^d + \mathcal{E}^d\} = \mathbb{e}^\mu \wedge \{\mathcal{B}^d + \mathcal{E}^d\}$$

$$= (\partial_1 \mathcal{E}_1 + \partial_2 \mathcal{E}_2 + \partial_3 \mathcal{E}_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - (\partial_2 \mathcal{B}_3 - \partial_3 \mathcal{B}_2 - \partial_t \mathcal{E}_1)(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$- (\partial_3 \mathcal{B}_1 - \partial_1 \mathcal{B}_3 - \partial_t \mathcal{E}_2)(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - (\partial_1 \mathcal{B}_2 - \partial_2 \mathcal{B}_1 - \partial_t \mathcal{E}_3)(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) = 0$$

Conventional expression

$$\nabla \cdot \mathcal{E} = 0, \quad \nabla \times \mathcal{B} - \partial_t \mathcal{E} = 0$$

Substituting $\mathcal{B}_i, \mathcal{E}_j$ for \mathcal{A}_μ in the 2), it becomes the wave equation.

$$\mathcal{K}_1 = \partial_2 \mathcal{A}_3 - \partial_3 \mathcal{A}_2 = \partial_2(\partial_1 \mathcal{E}_2 - \partial_2 \mathcal{E}_1 + \partial_t \mathcal{B}_3) - \partial_3(\partial_3 \mathcal{E}_1 - \partial_1 \mathcal{E}_3 + \partial_t \mathcal{B}_2)$$

$$= -\partial_{22} \mathcal{E}_1 - \partial_{33} \mathcal{E}_1 + \partial_t(\partial_2 \mathcal{B}_3 - \partial_3 \mathcal{B}_2) + \partial_1(\partial_2 \mathcal{E}_2 + \partial_3 \mathcal{E}_3)$$

$$= -\partial_{22} \mathcal{E}_1 - \partial_{33} \mathcal{E}_1 + \partial_t(\partial_t \mathcal{E}_1) - \partial_1(\partial_1 \mathcal{E}_1) = \square \mathcal{E}_1$$

$$\mathcal{K}_1 = \square \mathcal{E}_1, \quad \mathcal{K}_2 = \square \mathcal{E}_2, \quad \mathcal{K}_3 = \square \mathcal{E}_3$$

$$\mathcal{L}_1 = -(\partial_1 \mathcal{A}_t - \partial_t \mathcal{A}_1) = -\partial_1(\partial_1 \mathcal{B}_1 + \partial_2 \mathcal{B}_2 + \partial_3 \mathcal{B}_3) + \partial_t(\partial_2 \mathcal{E}_3 - \partial_3 \mathcal{E}_2 + \partial_t \mathcal{B}_1)$$

$$= -\partial_{11} \mathcal{B}_1 + \partial_{tt} \mathcal{B}_1 - \partial_2(\partial_1 \mathcal{B}_2 - \partial_t \mathcal{E}_3) - \partial_3(\partial_1 \mathcal{B}_3 + \partial_t \mathcal{E}_2)$$

$$= -\partial_{11} \mathcal{B}_1 + \partial_{tt} \mathcal{B}_1 - \partial_2(\partial_2 \mathcal{B}_1) - \partial_3(\partial_3 \mathcal{B}_1) = \square \mathcal{B}_1$$

$$\mathcal{L}_1 = \square \mathcal{B}_1, \quad \mathcal{L}_2 = \square \mathcal{B}_2, \quad \mathcal{L}_3 = \square \mathcal{B}_3$$

4.2. 3-vector field

$$1) \quad \mathbb{A} = -\mathbb{A}_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - \mathbb{A}_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - \mathbb{A}_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) - \mathbb{A}_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$\text{curl}(\mathbb{A}) = \mathbb{e}^\mu \wedge \partial_\mu \mathbb{A} = \mathbb{V} I \quad \mathbb{V} = -\partial_t \mathbb{A}_t + \partial_1 \mathbb{A}_1 + \partial_2 \mathbb{A}_2 + \partial_3 \mathbb{A}_3$$

$$2) \quad \text{curl}^d(\mathbb{A}) = \mathbb{V} I^d = \mathbb{V}$$

$$\mathbb{J} = \mathbb{J}_1 \mathbb{e}^1 + \mathbb{J}_2 \mathbb{e}^2 + \mathbb{J}_3 \mathbb{e}^3 + \mathbb{J}_t \mathbb{e}^t$$

$$= \mathbb{e}^\mu \wedge \partial_\mu \{\text{curl}^d(\mathbb{A})\}$$

$$= \partial_1 \mathbb{V} \mathbb{e}^1 + \partial_2 \mathbb{V} \mathbb{e}^2 + \partial_3 \mathbb{V} \mathbb{e}^3 + \partial_t \mathbb{V} \mathbb{e}^t$$

\mathbb{J}_i : color density, \mathbb{J}_t : Higgs density.

Conventional expression.

$$\mathbb{J} = \nabla \mathbb{V}, \quad \mathbb{J}_t = \partial_t \mathbb{V}$$

$$3) \quad \text{curl}^2(\mathbb{A}) = \mathbb{e}^\mu \wedge \partial_\mu \mathbb{V} = \text{null}$$

$$\text{curl}(\mathbb{J}) = \mathbb{e}^\mu \wedge \partial_\mu (\mathbb{J}) = 0$$

$$(\partial_2 \mathbb{J}_3 - \partial_3 \mathbb{J}_2)(\mathbb{e}^2 \wedge \mathbb{e}^3) = (\partial_3 \mathbb{J}_1 - \partial_1 \mathbb{J}_3)(\mathbb{e}^3 \wedge \mathbb{e}^1) = (\partial_1 \mathbb{J}_2 - \partial_2 \mathbb{J}_1)(\mathbb{e}^1 \wedge \mathbb{e}^2) = 0$$

$$(\partial_1 \mathbb{J}_t - \partial_t \mathbb{J}_1)(\mathbb{e}^1 \wedge \mathbb{e}^t) = (\partial_2 \mathbb{J}_t - \partial_t \mathbb{J}_2)(\mathbb{e}^2 \wedge \mathbb{e}^t) = (\partial_3 \mathbb{J}_t - \partial_t \mathbb{J}_3)(\mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

Conventional expression

$$\nabla \times \mathbb{J} = 0, \quad \nabla \mathbb{J}_t - \partial_t \mathbb{J} = 0$$

$$4) \quad \mathbb{A}^d = \mathbb{A} \cdot I = \mathbb{A}_1 \mathbb{e}^1 + \mathbb{A}_2 \mathbb{e}^2 + \mathbb{A}_3 \mathbb{e}^3 + \mathbb{A}_t \mathbb{e}^t$$

$$\text{div}(\mathbb{A}) = \text{curl}(\mathbb{A}^d) = \mathbb{e}^\mu \wedge \partial_\mu \mathbb{A}^d$$

$$= (\partial_2 \mathbb{A}_3 - \partial_3 \mathbb{A}_2)(\mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_3 \mathbb{A}_1 - \partial_1 \mathbb{A}_3)(\mathbb{e}^3 \wedge \mathbb{e}^1) + (\partial_1 \mathbb{A}_2 - \partial_2 \mathbb{A}_1)(\mathbb{e}^1 \wedge \mathbb{e}^2) \\ + (\partial_1 \mathbb{A}_t - \partial_t \mathbb{A}_1)(\mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_2 \mathbb{A}_t - \partial_t \mathbb{A}_2)(\mathbb{e}^2 \wedge \mathbb{e}^t) + (\partial_3 \mathbb{A}_t - \partial_t \mathbb{A}_3)(\mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

Conventional expression

$$\nabla \times \mathbb{A} = 0, \quad \nabla \mathbb{A}_t - \partial_t \mathbb{A} = 0$$

Substituting \mathbb{A}_μ for \mathbb{V} in the 2), it becomes the wave equation.

$$\mathbb{J}_1 = \partial_1 \mathbb{V} = \partial_1 (\partial_t \mathbb{A}_t - \partial_1 \mathbb{A}_1 - \partial_2 \mathbb{A}_2 - \partial_3 \mathbb{A}_3) = \partial_{t1} \mathbb{A}_t - \partial_{11} \mathbb{A}_1 - \partial_{21} \mathbb{A}_2 - \partial_{31} \mathbb{A}_3 \\ = \partial_{tt} \mathbb{A}_1 - \partial_{11} \mathbb{A}_1 - \partial_{22} \mathbb{A}_1 - \partial_{33} \mathbb{A}_1 = \square \mathbb{A}_1$$

$$\mathbb{J}_t = \partial_t \mathbb{V} = \partial_t (\partial_t \mathbb{A}_t - \partial_1 \mathbb{A}_1 - \partial_2 \mathbb{A}_2 - \partial_3 \mathbb{A}_3) = \partial_{tt} \mathbb{A}_t - \partial_{1t} \mathbb{A}_1 - \partial_{2t} \mathbb{A}_2 - \partial_{3t} \mathbb{A}_3 \\ = \partial_{tt} \mathbb{A}_t - \partial_{11} \mathbb{A}_t - \partial_{22} \mathbb{A}_t - \partial_{33} \mathbb{A}_t = \square \mathbb{A}_t$$

$$\mathbb{J}_1 = \square \mathbb{A}_1, \quad \mathbb{J}_2 = \square \mathbb{A}_2, \quad \mathbb{J}_3 = \square \mathbb{A}_3, \quad \mathbb{J}_t = \square \mathbb{A}_t$$

4.3. 0-vector field

$$1) \quad \boldsymbol{v}$$

$$\text{curl}(\boldsymbol{v}) = \mathbb{e}^\mu \wedge \partial_\mu \boldsymbol{v} = \boldsymbol{a} = a_1 \mathbb{e}^1 + a_2 \mathbb{e}^2 + a_3 \mathbb{e}^3 + a_t \mathbb{e}^t$$

$$a_1 = \partial_1 \boldsymbol{v}, \quad a_2 = \partial_2 \boldsymbol{v}, \quad a_3 = \partial_3 \boldsymbol{v}, \quad a_t = \partial_t \boldsymbol{v}$$

$$2) \quad \text{curl}^d(\boldsymbol{v}) = \text{curl}(\boldsymbol{v}) \cdot I$$

$$= -a_t (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - a_1 (\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - a_2 (\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - a_3 (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$\boldsymbol{j} = \mathbb{e}^\mu \wedge \partial_\mu \{\text{curl}^d(\boldsymbol{v})\} = (-\partial_t a_t + \partial_1 a_1 + \partial_2 a_2 + \partial_3 a_3) I = -m I$$

m : mass density.

Conventional expression.

$$m = \partial_t a_t - \nabla \cdot \boldsymbol{a}$$

$$3) \quad \text{curl}^2(\boldsymbol{v}) = \mathbb{e}^\mu \wedge \partial_\mu \boldsymbol{a}$$

$$= (\partial_2 a_3 - \partial_3 a_2)(\mathbb{e}^2 \wedge \mathbb{e}^3) + (\partial_3 a_1 - \partial_1 a_3)(\mathbb{e}^3 \wedge \mathbb{e}^1) + (\partial_1 a_2 - \partial_2 a_1)(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$+ (\partial_1 a_t - \partial_t a_1)(\mathbb{e}^1 \wedge \mathbb{e}^t) + (\partial_2 a_t - \partial_t a_2)(\mathbb{e}^2 \wedge \mathbb{e}^t) + (\partial_3 a_t - \partial_t a_3)(\mathbb{e}^3 \wedge \mathbb{e}^t) = 0$$

Conventional expression

$$\nabla \times \boldsymbol{a} = 0, \quad \nabla a_t - \partial_t \boldsymbol{a} = 0$$

$$\text{curl}(\boldsymbol{j}) = \mathbb{e}^\mu \wedge \partial_\mu \boldsymbol{j} = \mathbb{e}^\mu \wedge \partial_\mu (m I) = \text{null}$$

$$4) \quad \boldsymbol{v}^d = \boldsymbol{v} 1^d = \boldsymbol{v} I$$

$$\text{div}(\boldsymbol{v}) = \text{curl}(\boldsymbol{v}^d) = \mathbb{e}^\mu \wedge \partial_\mu \boldsymbol{v} I = \text{null}$$

Substituting a_μ for \boldsymbol{v} in the 2), it becomes the wave equation.

$$m = \partial_t a_t - \nabla \cdot \boldsymbol{a} = \partial_{tt} \boldsymbol{v} - \partial_{11} \boldsymbol{v} - \partial_{22} \boldsymbol{v} - \partial_{33} \boldsymbol{v} = \square \boldsymbol{v}$$

$$m = \square \boldsymbol{v}$$

4.4. Particles

The vector potential A has wave properties, and J has particle properties. They are two aspects of a single entity and dual-dimensional. Duality. If $J = \text{div}\{\text{curl}(A)\} \neq 0$, then $\text{curl}(A)$ is not continuous. Particles are a phenomenon in which $\text{curl}(A)$ accumulate or leak out.

$$1) \quad \text{Color is like a string.}$$

$$\text{color density: } \mathbb{e}^1, \mathbb{e}^2, \mathbb{e}^3$$

$$2) \quad \text{Spin is like a membrane.}$$

$$\text{spin density: } (\mathbb{e}^2 \wedge \mathbb{e}^3), (\mathbb{e}^3 \wedge \mathbb{e}^1), (\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$3) \quad \text{Charge is like a particle.}$$

$$\text{charge density: } (\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)$$

$$4) \quad \text{Higgs is essential for creating mass.}$$

$$\text{Higgs and mass density: } \mathbb{e}^t, (-\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$5) \quad \text{From a charge perspective, it's no coincidence that protons contain three quarks.}$$

- 6) There is a geometric difference between a positron and an electron. $(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) = -(\mathbb{e}^2 \wedge \mathbb{e}^1 \wedge \mathbb{e}^3)$
 7) Current density $(\mathbb{e}^i \wedge \mathbb{e}^j \wedge \mathbb{e}^t)$, superconductor is in a state where the spins are well aligned.

5. Energy density and force

The conventional energy density $eng(A)$ in electromagnetism is

$$eng(A) = \frac{1}{2}(B^2 + E^2) = \frac{1}{2}\{(B_i)^2 + (E_i)^2\}, \quad (i = 1,2,3)$$

$curl(A) = B + E$. The representation of multivectors is

$$eng(A) = \frac{1}{2}curl(A)curl^d(A) = \frac{1}{2}\|curl(A)\|^2 I$$

And force is a divergence of energy density.

$$F = -div\{eng(A)\} = -curl\{eng^d(A)\}$$

$curl(A)$ is a vector formed by bases in a dimension. chapter 3.8.

$$eng^d(A) = \frac{1}{2}\{curl(A)\}^2 \quad \text{or} \quad \frac{1}{2}\{curl^d(A)\}^2$$

The following assumption about vector fields X and Y are obvious. multivectors are numbers.

$$curl(XY) = \{curl(X)\}Y + X\{curl(Y)\}$$

The first case of $eng^d(A)$ cannot define the force, because $curl^2(A) = 0$.

$$\begin{aligned} F &= -\frac{1}{2}curl(\{curl^d(A)\}^2) = -\frac{1}{2}\{curl\{curl^d(A)\}curl^d(A) + curl^d(A)curl\{curl^d(A)\}\} \\ &= -\frac{1}{2}\{Jcurl^d(A) + curl^d(A)J\} \end{aligned}$$

Only 1-vector term of $\{Jcurl^d(A) + curl^d(A)J\}$ is force because the force is the $curl$ of scalar.

$curl^d(A)$ is a 2-vector and J is a 3-vector, this multiplication is commutative. 3.6. case1.

$$\begin{aligned} J &= -J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t) \\ curl^d(A) &= E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + E_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + E_3(\mathbb{e}^1 \wedge \mathbb{e}^2) \\ &\quad - B_1(\mathbb{e}^1 \wedge \mathbb{e}^t) - B_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - B_3(\mathbb{e}^3 \wedge \mathbb{e}^t) \end{aligned}$$

$$\begin{aligned} F_1\mathbb{e}^1 &= J_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3)E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) - J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)B_2(\mathbb{e}^2 \wedge \mathbb{e}^t) - J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)B_3(\mathbb{e}^3 \wedge \mathbb{e}^t) \\ &= (-J_tE_1 + J_2B_3 - J_3B_2)\mathbb{e}^1 \end{aligned}$$

$$\begin{aligned} F_t\mathbb{e}^t &= J_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t)E_1(\mathbb{e}^2 \wedge \mathbb{e}^3) + J_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t)E_2(\mathbb{e}^3 \wedge \mathbb{e}^1) + J_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)E_3(\mathbb{e}^1 \wedge \mathbb{e}^2) \\ &= (-J_1E_1 - J_2E_2 - J_3E_3)\mathbb{e}^t \end{aligned}$$

Conventional expression.

$$F = -J_tE + J \times B = \rho E + J \times B, \quad F_t = -J \cdot E$$

It is consistent with the experimental law. $F_t\mathbb{e}^t$ is the electromagnetic energy lost per unit time. Joule heat.

5.1. Weak force

$curl^d(B + \mathcal{E})$ is a 1-vector and J is a 2-vector, this multiplication is anti-commutative. 3.6. case1. Weak force is not a force itself that is weak rather force that cancel each other out. And since $F_t\mathbb{e}^t$ is also 0, there is no energy lost. Perhaps weak force is like pushing or pulling a spring. The phenomenon observed later when the spring is released may be a known weak force.

pushing or pulling force.

$$eng^d(A) = -\frac{1}{2}\{curl^d(B + \mathcal{E})\}^2, \quad F = Jcurl^d(B + \mathcal{E})$$

$curl^d(B + \mathcal{E})$ is a 1-vector, so it has different signs.

$$J = \mathcal{K} + \mathcal{L}$$

$$\mathcal{K} = -\mathcal{K}_1(\mathbb{e}^2 \wedge \mathbb{e}^3) - \mathcal{K}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) - \mathcal{K}_3(\mathbb{e}^1 \wedge \mathbb{e}^2)$$

$$\mathcal{L} = \mathcal{L}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) + \mathcal{L}_2(\mathbb{e}^2 \wedge \mathbb{e}^t) + \mathcal{L}_3(\mathbb{e}^3 \wedge \mathbb{e}^t)$$

$$\text{curl}^d(\mathcal{B} + \mathcal{E}) = -\mathcal{A}_1 \mathbb{e}^1 - \mathcal{A}_2 \mathbb{e}^2 - \mathcal{A}_3 \mathbb{e}^3 - \mathcal{A}_t \mathbb{e}^t$$

$$F_1 \mathbb{e}^1 = \mathcal{L}_1(\mathbb{e}^1 \wedge \mathbb{e}^t) \mathcal{A}_t \mathbb{e}^t - \mathcal{K}_2(\mathbb{e}^3 \wedge \mathbb{e}^1) \mathcal{A}_3 \mathbb{e}^3 - \mathcal{K}_3(\mathbb{e}^1 \wedge \mathbb{e}^2) \mathcal{A}_2 \mathbb{e}^2$$

$$= (-\mathcal{L}_1 \mathcal{A}_t + \mathcal{K}_2 \mathcal{A}_3 - \mathcal{K}_3 \mathcal{A}_2) \mathbb{e}^1$$

Conventional expression

$$F = -\mathcal{L} \mathcal{A}_t + \mathcal{K} \times \mathcal{A}$$

5.3. Strong force

$\text{curl}^d(\mathbb{A})$ is a scalar and \mathbb{J} is a 1-vector, this multiplication is commutative.

$$\text{eng}^d(\mathbb{A}) = \frac{1}{2} \{\text{curl}^d(\mathbb{A})\}^2, \quad F = -\mathbb{J} \text{curl}^d(\mathbb{A})$$

$$\mathbb{J} = \mathbb{J}_1 \mathbb{e}^1 + \mathbb{J}_2 \mathbb{e}^2 + \mathbb{J}_3 \mathbb{e}^3 + \mathbb{J}_t \mathbb{e}^t, \quad \text{curl}^d(\mathbb{A}) = \mathbb{V}$$

$$F_1 \mathbb{e}^1 = -\mathbb{V} \mathbb{J}_1 \mathbb{e}^1, \quad F_t \mathbb{e}^t = -\mathbb{V} \mathbb{J}_t \mathbb{e}^t$$

Conventional expression

$$F = -\mathbb{V} \mathbb{J}, \quad F_t = -\mathbb{V} \mathbb{J}_t$$

In the formula, the color density itself is the force, and the Higgs density is the energy lost per time.

5.4. Gravity

$\text{curl}^d(\mathcal{v})$ is a 3-vector and j is a pseudoscalar, this multiplication is commutative. 3.2.

$$\text{eng}^d(\mathcal{v}) = \frac{1}{2} \{\text{curl}^d(\mathcal{v})\}^2, \quad F = -j \text{curl}^d(\mathcal{v})$$

$$j = -mI$$

$$\text{curl}^d(\mathcal{v}) = -a_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) - a_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) - a_2(\mathbb{e}^3 \wedge \mathbb{e}^1 \wedge \mathbb{e}^t) - a_3(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^t)$$

$$F_1 \mathbb{e}^1 = -mI a_1(\mathbb{e}^2 \wedge \mathbb{e}^3 \wedge \mathbb{e}^t) = -m a_1 \mathbb{e}^1 = m(-\partial_1 \mathcal{v}) \mathbb{e}^1$$

$$F_t \mathbb{e}^t = -mI a_t(\mathbb{e}^1 \wedge \mathbb{e}^2 \wedge \mathbb{e}^3) = -m a_t \mathbb{e}^t = m(-\partial_t \mathcal{v}) \mathbb{e}^t$$

Conventional expression

$$F = -m a = -m(\nabla \mathcal{v}), \quad F_t = -m = -m(\partial_t \mathcal{v})$$