Generalized scattering operator preserving Hermiticity, unitarity, causality and convergence: scattering matrix without infinity

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Abstract We derive an alternative time-evolution operator for the Heisenberg picture in five rigorous ways with different starting points to confirm its validity and generality. This time-evolution operator called the generalized time-evolution operator is an analytical scattering operator that is obtained in a nonperturbative way unlike the Dyson series based on a perturbative approximation. We verify that the obtained scattering operator thoroughly preserves the Hermiticity, unitarity, causality of the scattering operator which are the basic requirements for the consistent scattering operator. It is analyzed that the Dyson series does not guarantee the Hermiticity, unitarity, causality of the scattering operator, and thus is not consistent. It is demonstrated that our formulation based on the generalized time-evolution operator does not need the Feynman diagram and renormalization and therefore there does not exist the infinity problem within the framework of the theory that we constructed. Ultimately, it is shown that the new formulation enables us to construct a consistent scattering theory which, beyond the infinity problem, satisfies all necessary requirements for the scattering operator.

Keywords Heisenberg picture · Time-evolution operator · Renormalization · Scattering matrix · Dyson series

1 Introduction

The core of quantum scattering theory is the Dyson series and the Feynman diagram method. The Dyson series is a perturbative expression of the scattering operator whose exact expression is not known and the Feynman diagram is a graphic representation of the Dyson series. Dyson's formula for the scattering matrix (S-matrix) and Feynman's diagram rule had promoted quantum scattering theory to an elaborated theory [1–3]. However, we usually encounter formidable divergence problems when calculating the scattering matrix based on the Feynman diagram.

Renormalization acknowledged as an astounding mathematical trick enables one to overcome some overlapping divergences due to the Feynman diagrams. This sophisticated formulation provided an approach to calibrating fundamental physical quantities so that computational results of the scattering matrix could coincide with experiments. John Ward's approach, the Yang-Mills method and Salam's studies on the problem of overlapping divergences contributed to the early development of renormalization theory [4–6]. Subsequently, Stueckelberg, Green, Bogoliubov and Parasiuk's contribution developed renormalization theory into a systematized theory with more solid foundation [7, 8]. On the other hand, Wolfhart Zimmermann, Bogoliubov and others' iterative method was finalized as Bogoliubov, Parasiuk, Hepp and Zimmermann's (BPHZ) method [9, 10]. The proposition of Gelfand and Yaglom, and Cameron's comment played an important role in making the Feynman history integral rigorous [11, 12].

The recent researches show the ramifications of renormalization technique that covers a wide range of the studies: renormalization group flow [13–23], renormalization group function and equation [24–30], renormalized perturbation theory [31], renormalization theory on the perturbative Feynman graph expansion [32], and on the other hand, connections between these different techniques of renormalization [33]. Recently, the development of the BPHZ renormalization is remarkable because it introduces massless fields instead of an auxiliary mass term [34]. The coverage of renormalization continues to extend inasmuch as it should satisfy

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the Lorentz and gauge symmetry and the requirement for cosmological spacetime [35–38].

Renormalization theory of dominant status has enjoyed so much successes in dealing with overlapping divergences of the scattering matrix [39]. Nevertheless, the heart of renormalization theory still has not been completed and the center of research continues to shift [40,41]. The facts, in a sense, indicate that it is necessary to explore for a new sphere of scattering theory without infinity in parallel with the development of renormalization [39,42–45].

What is best is to find out a general method without the infinity problem available to calculations of all cases of the scattering matrix. This in essence is like finding a nonperturbative scattering operator that gives an exact solution of the Schrödinger equation. An important motivation for our study consists in the fact that the present quantum scattering theory is limited to a perturbative approximation. The theory on quantum scattering which is based on the exact scattering operator undoubtedly needs neither the Dyson series nor the Feynman diagram. Therefore, the research for finding an exact scattering operator is most ideal and essential. In this connection, it is noteworthy that nonperturbative theory of quantum electrodynamics has turned into a promising area for solving the infinity problem in a different way [46]. In the same context, there are attempts to construct new formulations of scattering theory without infinity [39,44].

One of the key questions of quantum field theory is whether renormalization theory is able to reach the ultimate goal to resolve the divergence problem of scattering matrix in a general way within the present theoretical framework. This view seems challenging to the mainstream of the investigation of the scattering theory, but the present situation of researches showing Odyssey in renormalization naturally encourages the emergence of such perspectives that seek a new avenue. Our main concern is to construct a new formalism based on an exact scattering operator that satisfies all requirements for the scattering operator including convergence. If the exact scattering operator is found, perturbation theory is no longer needed. With this intention, we investigated the nonperturbative theory on the scattering matrix based on the generalized time-evolution operator which, in essence, means the generalized Heisenberg picture. The focus of our work was on deriving the exact time-evolution operator that exists uniquely and guarantees the Hermiticity, unitarity, causality and convergence.

The remaining paper is organized as follows. In Sect. 2, the generalized time-evolution operator that enables the generalization of the Heisenberg picture is derived in five independent ways to confirm the exactness of result. In Sect. 3, the Hermiticity, unitarity and causality of the generalized time-evolution operator and the Dyson series are analyzed. In Sect. 4, the convergence of the scattering matrix based on the generalized time-evolution operator is verified. In Sect. 5, the discussion is given. The paper is concluded in Sect. 6.

For this reason, our main concern is to construct a new formulation of the scattering theory which is consistent and perfect.

2 Generalized time-evolution operator

The Heisenberg picture is an important mathematical formulation for investigating the time evolution of quantum states together with the Schrödinger picture. In particular, the Heisenberg picture is of important significance in the study of scattering problems. This is because the scattering operator essentially should be the time-evolution operator in the Heisenberg picture if the Heisenberg picture possesses generality. The Heisenberg picture virtually is not possessed of generality.

Let us consider why and how to improve the Heisenberg picture. The state function Φ_S in the Schrödinger picture is determined by the Schrödinger equation:

$$\mathrm{i}\hbar \frac{\partial \Phi_S}{\partial t} = \hat{H}\Phi_S . \tag{1}$$

The formal solution of this equation for stationary states is

$$\Phi_S = e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}\Phi_H, \tag{2}$$

where the time-independent function, Φ_H, is defined as the wave function in the Heisenberg picture and the subscript H refers to the Heisenberg picture. The operator $\hat{S} = e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$ is considered to be a unitary operator making transformation from the Heisenberg pictures to the Schrödinger picture [47,48]. For the Heisenberg picture and the Schrödinger picture to be equivalent, Eq. (2) should be general. However, since for a time-dependent $\hat{H}$, Eq. (2) does not satisfy the Schrödinger equation, the Heisenberg picture is not general. In fact, substituting $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ into the Schrödinger equation yields

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})\right] = \left[\hat{H} + t\frac{\partial \hat{H}}{\partial t}\right]\Phi_S(\mathbf{q},t) \neq \hat{H}\Phi_S(\mathbf{q},t). \tag{3}$$

Eq. (3) shows that the Schrödinger picture and the Heisenberg picture are not equivalent and the latter is limited to the case of stationary states. Therefore, it is necessary to obtain the generalized formal solution of the Schrödinger equation which guarantees the equivalence of the two pictures in a rigorous way.

We aim to obtain a consistent time-evolution operator which satisfies all requirements for the scattering operator. These general requirements are as follows.

- The scattering operator should give the finite solution of the Schrödinger equation.
- It should be Hermitian.
- It should be unitary.
- It should satisfy the causality condition of time evolution. Importantly, we target obtaining an exact expression for the scattering operator unlike the Dyson series that is a perturbative approximation.

From now on, we derive a consistent time-evolution operator as an alternative to the Dyson series in five ways.

The first approach is to derive a scattering operator in such a way of obtaining a formal solution of the Schrödinger equation. Let us assume that the wave function takes the form

$$\Phi_S(\mathbf{q},t) = \hat{S}(\mathbf{q},t)\varphi(\mathbf{q}), \tag{4}$$

where $\hat{S}(\mathbf{q},t)$ is an operator dependent on time and position, and $\varphi(\mathbf{q})$ an arbitrary function. Inserting Eq. (4) into Eq. (1), we begin with

$$\mathrm{i}\hbar\frac{\partial\left[\hat{S}(\mathbf{q},t)\varphi(\mathbf{q})\right]}{\partial t} = \hat{H}(\mathbf{q},t)\left[\hat{S}(\mathbf{q},t)\varphi(\mathbf{q})\right]. \tag{5}$$

The task is to find $\hat{S}(\mathbf{q},t)$. Since operator $\mathrm{i}\hbar\frac{\partial}{\partial t}$ is applied only to $\hat{S}(\mathbf{q},t)$, we have

$$\mathrm{i}\hbar\frac{\partial\hat{S}(\mathbf{q},t)}{\partial t}\varphi(\mathbf{q}) = \hat{H}(\mathbf{q},t)\hat{S}(\mathbf{q},t)\varphi(\mathbf{q}). \tag{6}$$

For Eq. (6) to hold for arbitrary $\varphi(\mathbf{q})$, the equation for operators:

$$\mathrm{i}\hbar\frac{\partial\hat{S}(\mathbf{q},t)}{\partial t} = \hat{H}(\mathbf{q},t)\hat{S}(\mathbf{q},t) \tag{7}$$

should be an identical relation. On the other hand, the identical relation (7) is a sufficient condition for Eq. (6) to hold. Therefore, Eq. (7) is a necessary and sufficient condition for Eq. (6) to hold for arbitrary $\varphi(\mathbf{q})$.

Let us find the operator $\hat{S}(\mathbf{q},t)$ in the form of an algebraic expression. To begin with, we assume $\hat{S}(\mathbf{q},t)$ to be represented as a function f of an unknown operator $\hat{x}(\mathbf{q},t)$:

$$\hat{S}(\mathbf{q},t) = f(\hat{x}(\mathbf{q},t)).$$

Then Eq. (7) becomes

$$\mathrm{i}\hbar\frac{\partial f(\hat{x}(\mathbf{q},t))}{\partial t} = \hat{H}(\mathbf{q},t)f(\hat{x}(\mathbf{q},t)). \tag{8}$$

Adopting the formal derivative with respect to $\hat{x}(\mathbf{q},t)$, i.e., $\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}$, and applying the chain rule $\frac{\partial}{\partial t} = \frac{\partial\hat{x}(\mathbf{q},t)}{\partial t}\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}$, we rewrite Eq. (8) as

$$\mathrm{i}\hbar\frac{\partial\hat{x}(\mathbf{q},t)}{\partial t}\frac{\partial f(\hat{x}(\mathbf{q},t))}{\partial\hat{x}(\mathbf{q},t)} = \hat{H}(\mathbf{q},t)f(\hat{x}(\mathbf{q},t)).$$

Assuming that

$$\frac{\partial f(\hat{x}(\mathbf{q},t))}{\partial\hat{x}(\mathbf{q},t)} = f(\hat{x}(\mathbf{q},t)), \tag{9}$$

we have

$$\mathrm{i}\hbar\frac{\partial\hat{x}(\mathbf{q},t)}{\partial t} = \hat{H}(\mathbf{q},t). \tag{10}$$

From Eq. (9), we get a formal representation:

$$f(\hat{x}(\mathbf{q},t)) = \exp(\hat{x}(\mathbf{q},t)).$$

Moreover, integrating Eq. (10) with respect to time, we get

$$\hat{x}(\mathbf{q},t) - \hat{x}(\mathbf{q},t_0) = -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'.$$

We set

$$\hat{x}(\mathbf{q},t_0) = 0,$$

since $S(\mathbf{q},t_0) = f(\hat{x}(\mathbf{q},t_0)) = \exp(\hat{x}(\mathbf{q},t_0))$ should be the identity operator. Finally, we obtain

$$\hat{S}(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]. \tag{11}$$

The notation 'exp' in Eq. (11) includes the meaning that we use the rule of the formal operation $\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}$ and the chain rule $\frac{\partial}{\partial t} = \frac{\partial\hat{x}(\mathbf{q},t)}{\partial t}\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}$, when calculating the derivative of the exponential function with respect to time. This is because when deriving Eq. (11), we employed these rules of formal operation as a necessary condition. Eventually, a formal solution of the Schrödinger equation is represented as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\varphi(\mathbf{q}). \tag{12}$$

For the initial condition $t = t_0$, we have

$$\Phi_S(\mathbf{q},t_0) = \varphi(\mathbf{q}).$$

Accordingly, we write the exact formal solution as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_S(\mathbf{q},t_0). \tag{13}$$

By taking the time-independent function

$$\Phi_H(\mathbf{q}) = \Phi_S(\mathbf{q},t_0), \tag{14}$$

we may also write Eq. (13) as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_H(\mathbf{q}). \tag{15}$$

Finally, we find the time-evolution operator

$$U(t,t_0)=\exp\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]. \tag{16}$$

We shall refer to $U(t,t_0)$ as the generalized time-evolution operator, since it gives an analytical representation of the time evolution from an initial time t_0 to a final time t, independently of the time dependence of the Hamilton operator. For the scattering problem, the generalized time-evolution operator becomes the scattering operator. Thus, we have

$$\Phi_S(\mathbf{q},t)=U(t,t_0)\Phi_S(\mathbf{q},t_0)=S(t,t_0)\Phi_S(\mathbf{q},t_0). \tag{17}$$

Evidently, Eq. (16) becomes the generalized time-evolution operator which comprises the Heisenberg picture as a special case. Thus, we arrive at the generalization of the Heisenberg picture.

If Eq. (13) satisfies the Schrödinger equation, it is enough to confirm that the obtained time-evolution operator is valid. The fact that the formal solution, Eq. (13), is exact is verified easily. With the help of the adopted operation $\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}$, Eq. (16) is expanded into the Maclaurin series

$$S(t,t_0)=\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^n. \tag{18}$$

Formally applying the chain rule, we have

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t)&=\mathrm{i}\hbar\frac{\partial\hat{x}(\mathbf{q},t)}{\partial t}\frac{\partial}{\partial\hat{x}(\mathbf{q},t)}\Phi_S(\mathbf{q},t)\\&=\hat{H}(\mathbf{q},t)\exp\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_S(\mathbf{q},t_0)\\&=\hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t).\end{aligned} \tag{19}$$

Thus, we verify that the generalized time-evolution operator satisfies the Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t)=\hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t).$$

We can also obtain the same result without using the chain rule. Let us differentiate the scattering operator with respect to time. That is,

$$\frac{\partial}{\partial t}S(t,t_0)=\frac{\partial}{\partial t}\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^n. \tag{20}$$

For brevity of explanation, we first consider the second term

$$\frac{\partial}{\partial t}\left\{\frac{1}{2!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^2\right\}. \tag{21}$$

By differentiation, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{1}{2!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^2\right\}\\&=\frac{1}{2!}\left(-\frac{\mathrm{i}}{\hbar}\right)^2\left[\hat{H}(t)\int\limits_{t_0}^{t}\hat{H}(t')dt'+\int\limits_{t_0}^{t}\hat{H}(t')dt'\hat{H}(t)\right].\end{aligned} \tag{22}$$

In terms of the integration by parts and the mean value theorem of the integral calculus, the second term of Eq. (22) becomes

$$\begin{aligned}\int\limits_{t_0}^{t}\hat{H}(t')dt'\hat{H}(t)&=\left(\hat{H}(t')t'|_{t_0}^{t}-\int\limits_{t_0}^{t}t'd\hat{H}(t')\right)\hat{H}(t)\\&=\left[\hat{H}(t)t-\hat{H}(t_0)t_0-\bar{t}\left(\hat{H}(t)-\hat{H}(t_0)\right)\right]\hat{H}(t).\end{aligned} \tag{23}$$

where $\bar{t}\in[t_0,t]$. In the case of the scattering problem, t_0 and t correspond to free states. Therefore, $\hat{H}(t_0)$ and $\hat{H}(t)$ are the same Hamilton operator for free states and thus they are commutative. As a result, we arrive at

$$\begin{aligned}\int\limits_{t_0}^{t}\hat{H}(t')dt'\hat{H}(t)&=\left[\hat{H}(t)t-\hat{H}(t_0)t_0-\bar{t}\left(\hat{H}(t)-\hat{H}(t_0)\right)\right]\hat{H}(t)\\&=\hat{H}(t)\left[\hat{H}(t)t-\hat{H}(t_0)t_0-\bar{t}\left(\hat{H}(t)-\hat{H}(t_0)\right)\right]=\hat{H}(t)\int\limits_{t_0}^{t}\hat{H}(t')dt'.\end{aligned} \tag{24}$$

Hence, it follows that $\hat{H}(t)$ and $\int_{t_0}^{t}\hat{H}(t')dt'$ commute with each other. In view of Eq. (24), we identify

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^n\right\}\\&=\left(-\frac{\mathrm{i}}{\hbar}\right)\hat{H}(t)\frac{1}{(n-1)!}\left[\left(-\frac{\mathrm{i}}{\hbar}\right)\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^{n-1}.\end{aligned} \tag{25}$$

From this, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}S(t,t_0)&=-\frac{\mathrm{i}}{\hbar}\hat{H}(t)\sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int\limits_{t_0}^{t}\hat{H}(t')dt'\right]^n\\&=-\frac{\mathrm{i}}{\hbar}\hat{H}(t)S(t,t_0).\end{aligned} \tag{26}$$

By Eq. (26), we verify that the generalized time-evolution operator satisfies the Schrödinger equation and therefore it is exact. Eventually, the generalized time-evolution operator, $\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]$, has been determined.

The second approach is to obtain the scattering operator Eq. (16) by using the method of power series expansion.

Suppose that the scattering operator is represented as a function of an unknown operator $\hat{x}(\mathbf{q}, t)$:

$$S(t, t_0) = S(\hat{x}(\mathbf{q}, t)). \tag{27}$$

Then adopting the formal operation $\frac{\partial}{\partial \hat{x}(\mathbf{q},t)}$, the scattering operator can be expanded into the Maclaurin series:

$$S(\hat{x}(\mathbf{q}, t)) = S(0) + S'(0)\hat{x}(\mathbf{q}, t) + \frac{1}{2!}S''(0)[\hat{x}(\mathbf{q}, t)]^2 + \cdots + \frac{1}{n!}S^{(n)}(0)[\hat{x}(\mathbf{q}, t)]^n + \cdots . \tag{28}$$

To determine Eq. (28), let us consider the solution of the Schrödinger equation in the first-order approximation:

$$|\Phi(t)\rangle \approx |\Phi(0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'|\Phi(0)\rangle. \tag{29}$$

Comparing Eq. (29) with Eq. (28), we immediately identify

$$S(0) = 1, \;\; S'(0) = 1, \;\; \hat{x}(\mathbf{q}, t) = -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'.$$

Combining $S(0) = 1$ and $S'(0) = 1$, we suppose

$$\left.\frac{\partial S(\hat{x}(\mathbf{q}, t))}{\partial \hat{x}(\mathbf{q}, t)}\right|_{\hat{x}(\mathbf{q},t)=0} = S(\hat{x}(\mathbf{q}, t))|_{\hat{x}(\mathbf{q},t)=0}\,. \tag{30}$$

Obviously, the sufficient condition for Eq. (30) to hold is

$$\frac{\partial S(\hat{x}(\mathbf{q}, t))}{\partial \hat{x}(\mathbf{q}, t)} = S(\hat{x}(\mathbf{q}, t)).$$

Hence, we obtain

$$S(t, t_0) = S(\hat{x}(\mathbf{q}, t)) = \exp[\hat{x}(\mathbf{q}, t)] = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]. \tag{31}$$

From Eq. (31), it follows that $S(0) = 1,\ S'(0) = 1,\ S''(0) = 1, \cdots,\ S^{(n)}(0) = 1, \cdots$. Thus, we rewrite Eq. (28) as

$$S(\hat{x}(\mathbf{q}, t)) = 1 + \hat{x}(\mathbf{q}, t) + \frac{1}{2!}[\hat{x}(\mathbf{q}, t)]^2 + \cdots + \frac{1}{n!}[\hat{x}(\mathbf{q}, t)]^n + \cdots = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right].$$

This shows that the supposition Eq. (30) is valid. As a consequence, the scattering operator Eq. (16) has been derived in another way.

The third approach is to obtain the scattering operator, based on the causality condition of the scattering operator. From the physical point of view, the scattering operator should satisfy the causal product relation

$$S(t_N, t_0) = S(t_N, t_{N-1}) \cdots S(t_2, t_1)S(t_1, t_0). \tag{32}$$

Evidently, Eq. (16) is valid, since it satisfies the key causality axiom of scattering. Eq. (32) can be viewed as a functional equation for finding $S(t, t_0)$. Eq. (32) tells us that for the time interval, the rule of sum must be satisfied, while for the scattering operator, the rule of product must be satisfied.

Hence, it is obvious that the scattering operator has to take on the form of $S(t, t_0) = \exp(\hat{x}(\mathbf{q}, t))$, where $\hat{x}(\mathbf{q}, t)$ is an unknown operator dependent on t and t_0. Here, the condition

$$\hat{x}(t_N, t_0) = \hat{x}(t_N, t_{N-1}) + \cdots + \hat{x}(t_2, t_1) + \hat{x}(t_1, t_0) \tag{33}$$

should be satisfied. Substituting the formal solution $\Phi(t) = S(t, t_0)\Phi(t_0)$ into the Schrödinger equation yields

$$\mathrm{i}\hbar\frac{\partial}{\partial t}[\exp(\hat{x}(\mathbf{q}, t))\Phi_S(\mathbf{q}, t_0)] = \hat{H}[\exp(\hat{x}(\mathbf{q}, t)\Phi_S(\mathbf{q}, t_0)],$$

and by further arranging,

$$\mathrm{i}\hbar\frac{\partial \exp(\hat{x}(\mathbf{q}, t))}{\partial t}\Phi_S(\mathbf{q}, t_0) = \hat{H}\exp(\hat{x}(\mathbf{q}, t))\Phi_S(\mathbf{q}, t_0).$$

From the requirement that the Schrödinger equation should hold for arbitrary initial condition, $\Phi_S(\mathbf{q}, t_0)$, we get the equation for the operator $\hat{x}(\mathbf{q}, t)$:

$$\mathrm{i}\hbar\frac{\partial \hat{x}(\mathbf{q}, t)}{\partial t}\exp(\hat{x}(\mathbf{q}, t)) = \hat{H}\exp(\hat{x}(\mathbf{q}, t)), \tag{34}$$

where the chain rule $\frac{\partial}{\partial t} = \frac{\partial \hat{x}(\mathbf{q},t)}{\partial t}\frac{\partial}{\partial \hat{x}(\mathbf{q},t)}$ was used. Thus, we immediately determine

$$\hat{x}(\mathbf{q}, t) = -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'. \tag{35}$$

Obviously, this result satisfies Eq. (33). Finally, we arrive at

$$S(t, t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right].$$

Thus, we again confirm that Eq. (16) is valid from the point of view of causality.

The fourth approach is to obtain the scattering operator, based on the unitarity condition. Since the scattering operator should be unitary, the following relations

$$S(t, t_0)S^{-1}(t, t_0) = S(t, t_0)S^*(t, t_0) = 1 \tag{36}$$

and

$$S^{-1}(t, t_0)S(t, t_0) = S^*(t, t_0)S(t, t_0) = 1 \tag{37}$$

must hold. Eqs. (36) and (37) can be regarded as the functional equation for determining the form of the scattering operator. Evidently, the function satisfying Eqs. (36) and (37) is uniquely the exponential function. Thus, the form of function for the scattering operator can be set as

$$S(t,t_0) = \exp(\hat{x}(\mathbf{q},t)). \quad (38)$$

Then the Schrödinger equation becomes

$$\mathrm{i}\hbar\frac{\partial}{\partial t}[\exp(\hat{x}(\mathbf{q},t)\Phi_S(\mathbf{q},t_0)] = \hat{H}[\exp(\hat{x}(\mathbf{q},t)\Phi_S(\mathbf{q},t_0)].$$

Hence, we get the equation for the operator $\hat{x}(\mathbf{q},t)$:

$$\mathrm{i}\hbar\frac{\partial\hat{x}(\mathbf{q},t)}{\partial t}\exp(\hat{x}(\mathbf{q},t)) = \hat{H}\exp(\hat{x}(\mathbf{q},t)).$$

By Eqs. (34) and (35), we obtain the same result:

$$S(t,t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right].$$

Thus, we again confirm that Eq. (16) is valid from the requirement for the unitarity of the scattering operator.

The fifth approach is to obtain the scattering operator, based on the expansion of the wave function by a system of eigenfunctions. Without loss of generality, it is possible to expand an arbitrary time-dependent wave function into

$$\Phi_S(\mathbf{q},t) = \sum_n a_n(t)\,\phi_n(\mathbf{q}), \quad (39)$$

where $\{\phi_n : (n \in \mathbb{N})\}$ is a set of orthonormal eigenfunctions offered by an eigenvalue equation

$$\hat{L}\phi = L\phi. \quad (40)$$

The expansion coefficients $\{a_n(t) : (n \in \mathbb{N})\}$ are determined from the orthonormality of the eigenfunctions by

$$\langle\phi_n(\boldsymbol{q})|\Phi_S(\boldsymbol{q},t)\rangle = \left\langle\phi_n(\boldsymbol{q})\middle|\sum_m a_m(t)\,\phi_m(\boldsymbol{q})\right\rangle$$
$$= \sum_m a_m(t)\langle\phi_n(\boldsymbol{q})|\phi_m(\boldsymbol{q})\rangle = \sum_m a_m(t)\,\delta_{nm} = a_n. \quad (41)$$

By inserting Eq. (39) into (1), we go through

$$\mathrm{i}\hbar\frac{\partial\sum_n a_n(t)\,\phi_n(\boldsymbol{q})}{\partial t} = \hat{H}\sum_n a_n(t)\,\phi_n(\boldsymbol{q}),$$

$$\mathrm{i}\hbar\sum_n\frac{da_n(t)}{dt}\phi_n(\boldsymbol{q}) = \sum_n\hat{H}\left[a_n(t)\,\phi_n(\boldsymbol{q})\right],$$

$$\sum_n a_n(t)\left[\mathrm{i}\hbar\frac{1}{a_n(t)}\frac{da_n(t)}{dt}\right]\phi_n(\boldsymbol{q}) = \sum_n a_n(t)\,\hat{H}\phi_n(\boldsymbol{q}). \quad (42)$$

Eq. (42) is simplified as

$$\sum_n a_n(t)\left[\mathrm{i}\hbar\frac{d\ln a_n(t)}{dt} - \hat{H}\right]\phi_n(\boldsymbol{q}) = 0.$$

From the arbitrariness of $a_n(t)$, it follows that

$$\left[\mathrm{i}\hbar\frac{d\ln a_n(t)}{dt} - \hat{H}\right]\phi_n(\mathbf{q}) = 0,$$

and furthermore, from the arbitrariness of eigenfunctions $\phi_n(\mathbf{q})$,

$$\mathrm{i}\hbar\frac{d\ln a_n(t)}{dt} - \hat{H} = 0. \quad (43)$$

From Eq. (43), we obtain

$$a_n(t) = a_n(t_0)\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]. \quad (44)$$

Then Eq. (39) is finalized as

$$\begin{aligned}\Phi_S(\mathbf{q},t) &= \sum_n a_n(t)\,\phi_n(\mathbf{q})\\ &= \sum_n a_n(t_0)\exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]\phi_n(\mathbf{q})\\ &= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]\sum_n a_n(t_0)\phi_n(\mathbf{q})\\ &= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]\Phi_S(\mathbf{q},t_0). \quad (45)\end{aligned}$$

Thus, we identify

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]\Phi_S(\mathbf{q},t_0).$$

Thus, the time-evolution operator Eq. (16) has been derived repeatedly in five ways. The fact that the five approaches with different starting point produce one and the same result, Eq. (16), demonstrates that the obtained scattering operator is exact and reliable.

3 Hermiticity, unitarity and causality of scattering operator

The Hermiticity, unitarity and causality together with convergence are indispensable for a consistent scattering operator. Let us consider the generalized time-evolution operator and the Dyson series from the aspects of the Hermiticity, unitarity and causality. Indeed, these conditions for the scattering operator are the main criteria for ascertaining whether an obtained scattering operator is correct or not.

It is easily verified that the generalized time-evolution operator is a Hermitian operator. In fact, since $\hat{H}(\mathbf{q},t)$ is Hermitian, the integral of the Hamilton operator on a given time interval, $\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt'$, is also Hermitian. On the other hand, since $\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(\boldsymbol{q}, t')dt'\right]$ is expanded into a power series with respect to $-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(\boldsymbol{q}, t')dt'$, it is also Hermitian.

The generalized time-evolution operator is unitary. In fact, since

$$
\begin{aligned}
S(t, t_0)S^{-1}(t, t_0) &= S(t, t_0)S^{*}(t, t_0) \\
&= \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]\exp\left[\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right] = 1,
\end{aligned} \tag{46}
$$

the generalized time-evolution operator is unitary. Since the generalized time-evolution operator is derived based on the causality condition, it evidently satisfies the causality condition. Eventually, the generalized time-evolution operator entirely satisfies the three requirements for the scattering operator: Hermiticity, unitarity and causality.

Let us consider whether the Dyson series satisfies the Hermiticity, unitarity and causality conditions. We begin with the Dyson series

$$
\begin{aligned}
S(t, t_0) = \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \cdot \\
\hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n).
\end{aligned} \tag{47}
$$

For convenience, let us determine its second-order term. We calculate $2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\hat{H}(t_1)\hat{H}(t_2)$ to get

$$
\begin{aligned}
&2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\hat{H}(t_1)\hat{H}(t_2) \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\hat{H}(t_2)\hat{H}(t_1) \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\hat{H}(t_1)\hat{H}(t_2) \\
&\quad - \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1\hat{H}(t_2)\hat{H}(t_1) \\
&= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\hat{H}(t_1)\hat{H}(t_2) \\
&\quad + \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\left[\hat{H}(t_1)\hat{H}(t_2) - \hat{H}(t_2)\hat{H}(t_1)\right].
\end{aligned} \tag{48}
$$

With this result, we finalize the second-order term of the Dyson series as

$$
\begin{aligned}
&\left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\hat{H}(t_1)\hat{H}(t_2) \\
&\qquad = \frac{1}{2}\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]^2 + R^{(2)}(t),
\end{aligned} \tag{49}
$$

where $R^{(2)}(t)$ is the term relative to the noncommutativity of the Hamilton operators distinguished by time. Generalizing this result, we write the Dyson series as

$$
\begin{aligned}
S(t, t_0) &= \sum_{n=0}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \cdot \\
&\quad \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \\
&\quad \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n) + \sum_{n=2}^{\infty} R^{(n)}(t) \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]^n + \sum_{n=2}^{\infty} R^{(n)}(t) \\
&= \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right] + \sum_{n=2}^{\infty} R^{(n)}(t).
\end{aligned} \tag{50}
$$

Eq. (50) is rewritten simply as

$$
S(t, t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right] + R(t). \tag{51}
$$

Obviously, the scattering operator in this form cannot satisfy the causal relation of scattering, Eq. (32). From Eq. (51), it follows that the Dyson series satisfies the causality axiom of scattering only if $R(t)$ that is related to the characteristic of approximate calculation vanishes. From this, it turns out that the Dyson series in general is not unitary. Importantly, Eq. (51) explains the relationship between the generalized time-evolution operator and the Dyson series. In fact, if $R(t)$ vanishes, the Dyson series coincides with the generalized time-evolution operator. This fact clearly shows that the Dyson series is an approximation of the generalized time-evolution operator.

Let us consider whether the Dyson series satisfies the unitarity condition. Assume that the Dyson series is Hermitian. For the Dyson series to be unitary,

$S(t, t_0)S^{-1}(t, t_0) = 1$ and $S^{-1}(t, t_0)S(t, t_0) = 1$

should hold. In view of Eq. (51), we calculate to get

$$S(t,t_0)S^*(t,t_0) = \left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right) + R(t)\right]\times$$
$$\left[\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right) + R^*(t)\right]$$
$$= 1 + \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)R^*(t) + R(t)\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)$$
$$+ R(t)R^*(t).$$

Then, according to the unitarity condition, it is required that

$$\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)R^*(t) + R(t)\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)$$
$$+R(t)R^*(t) = 0. \quad (52)$$

From the requirement that the scattering operator should be Hermitian, the commutation relation

$$R(t)\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)R(t).$$

should hold. Thus, we rewrite Eq. (52) as

$$2\mathrm{Re}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)R^*(t)\right] + R(t)R^*(t)$$
$$= 2\mathrm{Re}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right)R^*(t) + R(t)R^*(t)\right] = 0. \quad (53)$$

With Eq. (53), we obtain

$$R(t) = 0 \quad (54)$$

and

$$R(t) = -2\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right), \quad (55)$$

respectively. Eq. (55) results in

$$S(t,t_0) = -\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right).$$

This scattering operator with a negative sign does not satisfy the causality condition Eq. (32). In fact, the sign of this scattering operator becomes undetermined when applying the causality theorem. Therefore, only $R(t) = 0$ is valid. This means that to guarantee the unitarity of the scattering operator, the Dyson series should go over to the generalized time-evolution operator. Moreover, this indicates that one must make a crucial approximation of ignoring the time-ordering operator to ensure the unitarity of the scattering operator when applying methods of successive approximation.

As a serious respect, the Dyson series is not a Hermitian operator. As seen from Eq. (48), the Dyson series contains non-Hermitian terms such as $\hat{H}(t_1)\hat{H}(t_2)$ which is a product of noncommutative operators. In fact, since $\hat{H}(t_1)$ does not commute with $\hat{H}(t_2)$, the product of two Hermitian operators, $\hat{H}(t_1)\hat{H}(t_2)$, is not Hermitian. This fact shows that in order to ensure the Hermitian property of the scattering operator, we must avoid formulations of the scattering operator that involve the time-ordering operator. To make the Dyson series a Hermitian opertor, it is necessary to make a crucial approximation of cutting off the non-Hermitian term $R(t)$ in the Dyson series.

We may consider this matter in another way. The product of the Hermitian operators, $\hat{H}(t_1)$ and $\hat{H}(t_2)$, is not Hermitian but $\hat{H}(t_1)\hat{H}(t_2) + \hat{H}(t_2)\hat{H}(t_1)$ is Hermitian. Therefore, for the non-Hermitian operator $\hat{H}(t_1)\hat{H}(t_2)$, a key approximation that preserves the Hermitian property can be chosen as

$$\hat{H}(t_1)\hat{H}(t_2) \approx \frac{1}{2}\left(\hat{H}(t_1)\hat{H}(t_2) + \hat{H}(t_2)\hat{H}(t_1)\right).$$

The introduction of this approximation leads to the removal of $R^{(2)}(t)$ in Eq. (49). This means that in order for the Dyson series to be Hermitian, it is inevitable that it goes over to the generalized time-evolution operator.

Altogether, the Dyson series violates the Hermiticity, unitarity and causality conditions as the main requirements for scattering operator. This result concludes that the Dyson series is not consistent even if it is convergent.

4 Convergence of scattering matrix

Let us consider whether the scattering matrix in terms of the generalized time-evolution operator always ensures convergence. Obviously, a finite order approximation of Eq. (18) is convergent. Now, it is necessary to examine whether the infinite series of the generalized time-evolution operator ensures convergence. According to the mean value theorem of integral calculus, we set

$$\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt' = \hat{H}(\mathbf{q},\bar{t})(t-t_0), \quad (56)$$

where $\bar{t} \in [t_0, t]$ becomes a parameter. With the help of Eq. (56), we eliminate the integral symbol from the time-evolution operator to get

$$S(t,t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right] = \exp\left[-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}(\mathbf{q},\bar{t})\right]. \quad (57)$$

Next, Eq. (57) is expanded into the Maclaurin series:

$$S(t,t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}(\mathbf{q},\bar{t})\right] = \sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}(\mathbf{q},\bar{t})\right)^n. \quad (58)$$

The solution of the time-dependent Schrödinger equation is represented with the help of the time-evolution operator as

$$\Phi_S(\mathbf{q},t) = \left[\sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t-t_0)\hat{H}(\mathbf{q},\bar{t})\right)^n\right]\Phi_0(\mathbf{q}). \quad (59)$$

From the finiteness condition of the state function, $\Phi_S(\mathbf{q},t)$, it is possible to suppose that there exists a definite number, E, which satisfies

$$|\Phi_S(\mathbf{q},t)| = \left|\sum_n \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t-t_0)\right]^n\left[\hat{H}(\mathbf{q},\bar{t})\right]^n\Phi_0(\mathbf{q})\right| \le \left|\sum_n \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t-t_0)\right]^n E^n\Phi_0(\mathbf{q})\right|. \quad (60)$$

Then the transition probability from an initial state i to a final state f becomes

$$W_{i\to f} = |\langle f|S(t,t_0)|i\rangle|^2 \le \left|\sum_n \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t-t_0)E\right]^n\right|^2 = \left|\exp\left(-\frac{\mathrm{i}}{\hbar}(t-t_0)E\right)\right|^2 = 1. \quad (61)$$

This is in keeping with our common knowledge that the transition probability always should be less than one. Eventually, the scattering matrix in terms of the generalized time-evolution operator ensures convergence. Thus, it is proved that using the generalized time-evolution operator leads to the scattering matrix without infinity.

Let us consider the case where the Hamiltonian is expressed by the use of the Hamiltonian density $\mathcal{H}_I$ as $\hat{H}_I = \int\mathcal{H}_I(x)d^3x$. The wave equation in the Dirac interaction picture is

$$\mathrm{i}\hbar\frac{\partial\Phi_I}{\partial t} = \hat{H}_I\Phi_I. \quad (62)$$

Since Eq. (62) takes the same form as Eq. (1), the solution of Eq. (62) becomes

$$\Phi_I(t) = \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}_I(t')dt'\right]\Phi_I(t_0). \quad (63)$$

As an example, in the case of the interaction between electron-positron field and electromagnetic field, the Hamiltonian density of interaction is written by the use of field operators as

$$\mathcal{H}_I = -j_\mu(x)A^\mu(x) = -\frac{e}{2}\left[\bar{\psi}\gamma_\mu\psi\right]A^\mu = eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu. \quad (64)$$

By Eqs. (18) and (64), the scattering operator is given as

$$S = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n\left[\int eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu d^4x\right]^n. \quad (65)$$

For the first approximation of the scattering operator

$$S^{(1)} = \left(-\frac{\mathrm{i}}{\hbar}\right)\int eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu d^4x,$$

the element of the scattering matrix, $M^{(1)} = S^{(1)}_{i\to f} = \langle\Phi_f|S^{(1)}|\Phi_i\rangle$, in general, is convergent. This means that there exists a positive number M' satisfying

$$\left|M^{(1)}\right| = \left|\left\langle f|S^{(1)}|i\right\rangle\right| \le \left|\left\langle f\left|-\frac{\mathrm{i}}{\hbar}M'\right|i\right\rangle\right| = \frac{M'}{\hbar}|\langle f|i\rangle|.$$

Therefore, for the scattering operator Eq. (65) and an appropriate M satisfying $M \ge M'$, the modulus of the scattering amplitude becomes

$$\left|S_{i\to f}\right| = \left|\left\langle f\left|\sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n\left[\int eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu d^4x\right]^n\right|i\right\rangle\right| = \left|\left\langle f\left|\sum_n \frac{1}{n!}\left(S^{(1)}\right)^n\right|i\right\rangle\right| \le \left|\sum_n \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}M\right)^n\right| = \left|\exp\left(-\frac{\mathrm{i}}{\hbar}M\right)\right| = 1,$$

where $\langle f|i\rangle < 1$ has been taken into consideration. This demonstrates that the scattering matrix always converges. Thus, the transition probability from an initial state i to a final state f:

$$W_{i\to f} = |S_{i\to f}|^2 \quad (66)$$

is convergent too. Eventually, the generalized time-evolution operator gives the scattering matrix without infinity.

Since the generalized time-evolution operator is consistent, it is necessary to adopt the generalized representation in place of the Heisenberg picture:

$$\hat{F}_H(t) = \exp\left(\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\hat{F}_S\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)$$

as

$$\hat{F}_H(t) = \exp\left(\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right)\hat{F}_S\exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right),$$

where $\hat{F}_H(t)$ is an operator in the Heisenberg picture and $\hat{F}_S$ an operator in the Schrödinger picture.

5 Discussion

The focus of our work has been on building an alternative mathematical formulation of the scattering theory which is consistent and perfect in every respect. One and the same time-evolution operator distinguished from the Dyson series has been derived in five ways of reflecting the necessary requirements for the scattering operator. The generalized time-evolution operator obtained is a nonperturbative expression derived without taking any approximation. The Hermiticity, unitarity, causality and convergence of the generalized time-evolution operator have been proved rigorously. Therefore, the generalized time-evolution operator is consistent and perfect and does not have exception. The generalized time-evolution operator is not related to the Dyson series, and thus does not require the Feynman diagram. This indicates that a scattering theory which does not need renormalization in principle is possible.

Our work showed that the Dyson series is not a consistent scattering operator which satisfies the general requirements for the scattering operator. Of course, it is commonly acknowledged that the Dyson series is the standard model of the time-evolution operator. However, this fact is not sufficient to draw the conclusion that the Dyson series is a unique model. Above all, the Dyson series is faced with the infinity problem which is unreasonable from the physical and mathematical aspects. In addition, the Dyson series does not guarantee the Hermiticity, unitarity and causality. If these conditions are violated, then elements of the scattering matrix cannot be interpreted as transition probabilities. This means that even if we might obtain a convergent scattering matrix by using a refined renormalization method, it would not be perfect because it cannot satisfy the other basic requirements for the scattering operator. To guarantee the unitarity condition in general gauge field theories, one usually introduces additional ghost and antighost fields which become factitious. On the contrary, since the generalized time-evolution operator is unitary, within the framework of our formulation, it is not required to use these additional fields. The fact that the Dyson series does not satisfy indispensable requirements needs a radical departure from the realm of the conventional researches.

By a simple consideration, let us discuss whether it is inevitable that the infinity problem in the scattering matrix arises. We presuppose that the Schrödinger equation

$$i\hbar\frac{\partial\Phi_S}{\partial t} = \hat{H}\Phi_S \tag{67}$$

has a correct solution. It means that its solution

$$\Phi_S(\mathbf{q}, t) = S(t, t_0)\Phi_S(\mathbf{q}, t_0) \tag{68}$$

represented in terms of the time-evolution operator should be finite. To understand the truth of the infinity problem, we begin with the formal solution of the Schrödinger equation:

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')\Phi(t')dt'. \tag{69}$$

By the mean value theorem of integral calculus:

$$\int_a^b f(x)g(x)dx = f(c)g(c)(b-a) \quad (c \in [a, b]),$$

Eq. (69) is written as

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{i}{\hbar}(t - t_0)\hat{H}(\bar{t})|\Phi(\bar{t})\rangle, \tag{70}$$

where $\bar{t} \in [0, t]$. For $\hat{H}(\bar{t})|\Phi(\bar{t})\rangle$ in Eq. (70), we can imagine an eigenvalue equation with parameter $\bar{t}$:

$$\hat{H}(\bar{t})|\Phi(\bar{t})\rangle = E(\bar{t})|\Phi(\bar{t})\rangle. \tag{71}$$

Then, Eq. (70) becomes

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{i}{\hbar}(t - t_0)E(\bar{t})|\Phi(\bar{t})\rangle. \tag{72}$$

Since the Schrödinger equation presupposes the finiteness of solution, Eq. (71) is finite and thus Eq. (72) too is finite. In this case, in a formal manner, the scattering operator can be taken as

$$S(t, t_0) = 1 - \frac{i}{\hbar}(t - t_0)E(\bar{t})\frac{\Phi(\bar{t})}{\Phi(t_0)}. \tag{73}$$

Of course, since $|\Phi(\bar{t})\rangle$ is unknown, the solution is formal but Eq. (73) is enough to verify that the scattering operator should be finite. In fact, according to the definition of the Schrödinger equation, $|\Phi(\bar{t})\rangle, |\Phi(t_0)\rangle$ and $E(\bar{t})$ should be finite and nonzero valued. Consequently, the scattering matrix is finite. Considering in this simple way, we can draw the conclusion that there is no infinity problem as far as $\hat{H}$ is defined correctly.

Also, we can understand the truth of the infinity problem, purely based on mathematical logic. Purely from the point of view of mathematics, such a mathematical theory that one must separate a finite quantity from a given infinity cannot be assessed as being consistent. Obviously, Eqs. (67) and (68) are mathematically identical. If the scattering matrix based on the Dyson series and the Feynman diagram diverges, it means that the scattering operator is not exact. Therefore, in this case, we should ascribe the infinity problem to the scattering operator wrongly written. This fact shows that the Dyson series as the commonly adopted scattering operator is not exact.

In this work, we have shown that there exists an exact scattering operator able to avoid the time-ordering operator responsible for the infinity problem. In fact, our formulation is free from the infinity problem in principle. Furthermore, our formalism satisfies all the requirements for the scattering operator encompassing the Hermiticity, unitarity, causality and convergence conditions.

6 Conclusions

Our aim has been to derive a consistent time-evolution operator that satisfies all the requirements for the scattering operator such as the Hermiticity, unitarity, causality and convergence. We have presented a new mathematical representation of scattering matrix using of the generalized time-evolution operator that provides the generalized Heisenberg picture. Using five mathematically rigorous methods, we have obtained an identical result, i.e., the generalized time-evolution operator, Eq. (16), which shows that the obtained time-evolution operator is exact. The time-evolution operator that has been derived in the nonperturbative ways satisfies all the requirements for the scattering operator, so it is consistent and exact. Within the framework of our formulation using this time-evolution operator, there is no necessity of using the time-ordering operator and the Feynman diagram, and thus the infinity problem of the scattering matrix does not arise. Thus, it is demonstrated that it is possible to formulate a consistent scattering theory irrelevant to the infinity problem.

Our study has clarified the imperfect aspects of the Dyson series by demonstrating the fact that the Dyson series does not satisfy the Hermiticity, unitarity and causality conditions. The obtained time-evolution operator is the nonperturbative expression, whereas the Dyson series is based on a perturbative expansion. The Dyson series which is given by a perturbative expansion is impossible to satisfy the Hermiticity, unitarity and causality conditions for the scattering operator, since it necessarily contains products of noncommutative operators. Eq. (51) obviously explains why the Dyson series violates the main requirements for the scattering operator.

What is best is to obtain a perfectly exact time-evolution operator other than a perturbative approximation. With this intention, we have derived an exact time-evolution operator distinct from the Dyson series. It should be emphasized that the obtained time-evolution operator is the exact mathematical representation for the scattering operator that exists uniquely. The fact that the present time-evolution operator with a starting point different from the Dyson series satisfies all the requirements for the scattering operator unlike the Dyson series demonstrates that the former is superior to the latter and the ultimate solution to the problem of the scattering matrix has been given.

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