# Mass Generation and Fermion Mixing from Hopf Bifurcations

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## Abstract

We put forward a theoretical framework in which fermion masses and mixing angles arise from the *unstable oscillations* of the Higgs condensate near a Hopf bifurcation. These fluctuating modes generate phase differences corresponding to the observable mixing angles and the strong CP phase. Our analysis provides a minimal route from universal Hopf bifurcations to the CKM/PMNS phenomenology.

**Key words:** Fermion masses, mixing angles, Stuart-Landau oscillators, Hopf bifurcation, Higgs condensate, CKM and PMNS matrices, Beyond the physics of Standard Model.

## 1. Introduction

The Standard Model of particle physics (SM) partitions fermion masses and mixings in two categories of free parameters. While Yukawa eigenvalues supply fermion masses via electroweak symmetry breaking, unitary mixing matrices (CKM for quarks, PMNS for leptons) describe rotations between the flavor and mass eigenstates.

To date, the origin of these free parameters remains a mystery. Building off [1], here we find that the mathematical structure of mass hierarchies and mixing matrices resembles the universal behavior of *weakly nonlinear oscillators near a Hopf bifurcation*. Specifically, our paper points out that:

- The Higgs condensate slightly above the electroweak transition exhibits *three unstable oscillatory modes*.
- Each mode obeys a *Stuart–Landau (SL) equation*, which is the universal normal form of dynamics near a Hopf bifurcation.

- Masses emerge from the multifractal geometry of the Feigenbaum attractor, a universal marker of transition to chaos.
- The mass basis and flavor basis differ by *relative oscillator phases*, giving rise to the observed mixing matrices.

The paper is organized in the following way: next section is a condensed review of electroweak physics; section 3, along with Appendix A, show how the oscillatory modes of the Higgs condensate produce fermion masses and mixing/CP violation. Appendix B discusses hierarchical mass generation from multifractal structure of the Feigenbaum attractor; Appendix C builds the case that the Cabibbo angle emerges as reciprocal of the Feigenbaum constant.

In the interest of transparency and accessibility, the paper is written in an introductory/pedagogical format. Whenever possible, attempts are made to rely on intuitive concepts rather than formal jargon and heavy mathematics. Many nuanced technical details are left out for follow-up studies and independent reviews or rebuttals.

## 2. Electroweak physics and the construction of mixing matrices

The goal of this section is to go over four major features of the electroweak sector of SM, namely,

1. How the Higgs field converts Yukawa couplings into fermion masses,
2. Why one must rotate from flavor to mass bases,
3. How the mismatch of rotations produces quark and lepton mixing matrices and,
4. The parametrization of CKM and PMNS matrices.

To start with, consider the Yukawa Lagrangian describing the interaction of quarks with the Higgs field [see e.g. 2],

$$L_Y = -\overline{Q}_L Y_d \Phi\, d_R - \overline{Q}_L Y_u \Phi\, u_R + h.c. \qquad (1)$$

Here,

$$Q_L = \begin{pmatrix} u_L & c_L & t_L \\ d_L & s_L & b_L \end{pmatrix} \qquad (2)$$

is the left-handed quark doublet (3-component vector in family space),

$$u_R = \begin{pmatrix} u_R \\ c_R \\ t_R \end{pmatrix}, \qquad d_R = \begin{pmatrix} d_R \\ s_R \\ b_R \end{pmatrix} \tag{3}$$

are the 3-component vectors of right-handed singlets; $Y_u, Y_d$ are the 3 x 3 complex Yukawa matrices in family space,

$$\Phi = \begin{pmatrix} 0 \\ \dfrac{v+h}{\sqrt{2}} \end{pmatrix} \tag{4}$$

is the Higgs condensate after electroweak symmetry breaking and

$$\Phi = i\sigma^2 \Phi^* \tag{5}$$

After inserting the Higgs vacuum expectation value $v$, the Yukawa terms turn into mass terms,

$$L_{mass} = -\bar{d}_L M_d d_R - \bar{u}_L M_u u_R + h.c. \tag{6}$$

with mass matrices

$$M_d = \frac{v}{\sqrt{2}} Y_d \ , \qquad M_u = \frac{v}{\sqrt{2}} Y_u$$

These mass matrices *are not diagonal* in general and therefore do not define fermion masses directly and in a unique way. Since physical particles are true *mass eigenstates*, one wants a basis where all mass terms are diagonal, called a *mass basis*. Obtaining the mass basis require $M_d$ and $M_u$ to be diagonalized using a set of unitary matrices according to

$$U_u^\dagger M_u V_u = diag(m_u, m_c, m_t) \tag{7}$$

$$U_d^\dagger M_d V_d = diag(m_d, m_s, m_b) \tag{8}$$

These define the following field rotations

$$u_L = U_u u_L^{mass}; \qquad u_R = V_u u_R^{mass} \tag{9}$$

$$d_L = U_d d_L^{mass}; \qquad d_R = V_d d_R^{mass} \tag{10}$$

After performing these rotations, all mass terms are diagonal

$$L_{mass} = -\overline{u}_L^{mass} m_u^{diag} u_R^{mass} - \overline{d}_L^{mass} m_d^{diag} d_R^{mass} + h.c. \tag{11}$$

However, gauge interactions are written using the flavor basis, i.e.,

$$L_W = \frac{g}{\sqrt{2}} \overline{u}_L \gamma^\mu d_L W_\mu^+ + h.c. \tag{12}$$

When one inserts the mass-basis rotations, a mismatch appears,

$$\overline{u_L} \gamma^\mu d_L = \overline{u}_L^{mass} (U_u^\dagger U_d) \gamma^\mu d_L^{mass} \tag{13}$$

Because there is no reason for the mass and flavor matrices to be the same, their mismatch is the source of *observable flavor mixing* and (for quarks) is characterized by the CKM matrix,

$$\boxed{V_{CKM} = U_u^\dagger U_d} \tag{14}$$

Similarly, for charged leptons, diagonalization yields

$$M_l \rightarrow U_l^\dagger M_l V_l = diag \tag{15}$$

whereas diagonalization of the Majorana or Dirac neutrino mass matrix gives a rotation $U_\nu$ and flavor mixing in the lepton sector is described by the PMNS matrix

$$\boxed{U_{PMNS} = U_l^\dagger U_\nu} \tag{16}$$

Historically, Cabibbo introduced a single angle $\theta_C$ that mixes two quark families/generations,

$$V_{CKM}^{(2)} = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \tag{17}$$

This mixing arises in the following way: when there are only two generations present, the Yukawa mass matrices $M_u, M_d$ are 2 x 2. Each is diagonalized by a single angle:

$$U_u = \begin{pmatrix} \cos\alpha_u & \sin\alpha_u \\ -\sin\alpha_u & \cos\alpha_u \end{pmatrix}; \quad U_d = \begin{pmatrix} \cos\alpha_d & \sin\alpha_d \\ -\sin\alpha_d & \cos\alpha_d \end{pmatrix}$$

Then

$$V_{CKM}^{(2)} = U_u^\dagger U_d = \begin{pmatrix} \cos\theta_C & \sin\theta_C \\ -\sin\theta_C & \cos\theta_C \end{pmatrix} \tag{18}$$

where,

$$\boxed{\theta_C = \alpha_d - \alpha_u} \tag{19}$$

The Cabibbo angle (19) measures the difference between the diagonalization angles of the up-type and down-type quark mass matrices. It encodes the relative orientation of the flavor bases induced by the two sectors.

This interpretation can be extrapolated directly to the three generations of SM: the CKM has 3 angles and 1 CP phase and these correspond to the relative geometry of the two diagonalization matrices $U_u$ and $U_d$ in $SU(3)$ flavor space.

Because the CKM matrix is close to unity, one introduces the expansion,

$$\boxed{\lambda = \sin\theta_C \approx 0.225} \tag{20}$$

and defines four independent parameters $\lambda, A, \rho, \eta$. The so-called Wolfenstein parametrization of the CKM matrix reads

$$V_{CKM} \approx \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{21}$$

This parameterization makes the hierarchical structure transparent, as nearly all entries are either linear or quadratic functions of (20).

It can be shown that the PMNS matrix may be also formulated in a hierarchical way [3], via

$$m_\nu = U_\nu m_\nu^{diag} U_\nu^T \tag{22}$$

$$m_l = U_L m_l^{diag} U_R^\dagger \tag{23}$$

$$V_{PMNS} = U_L^\dagger U_\nu \tag{24}$$

which leads to the following representation,

$$m_l m_l^{\dagger} \propto m_\tau^2 \begin{vmatrix} \lambda^6 & \lambda^5 & \lambda^3 \\ \lambda^5 & \lambda^4 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \quad (25)$$

$$U_L m_l^{diag} U_L^T \propto m_\tau \begin{vmatrix} \lambda^4 & \lambda^3 & \lambda^3 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{vmatrix} \quad (26)$$

We close this section with a remark on the CP phase. The so-called *Jarlskog invariant* $J$ is defined in relation to the CKM parametrization in terms of cosines and sine functions of the mixing angles ($c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$). The invariant contains the *CP phase* $\delta_s$ as in [4]

$$J = c_{12} s_{12} c_{23} s_{23} c_{13}^2 s_{13} \sin\delta_s \quad (27)$$

The invariant (27) ties CP violation to $s_{12}, s_{23}, s_{13} \neq 0$ and $\sin\delta_s \neq 0$. As explained in Appendix A, SM *requires three generations* to comply with Jarlskog invariant (27) and its consequences.

## 3. Mixing matrices from Hopf bifurcations

We now switch gears and bring up the recently proposed mechanism of Higgs condensation based on the Stuart-Landau (SL) equation and the formation of a *stable limit cycle* via Hopf bifurcations [1].

A *Hopf bifurcation* is a mathematical event where a system that used to sit quietly at a *fixed point* (steady state) suddenly starts oscillating when a control parameter crosses a threshold. A prime example of this behavior is a laser source, where gradually increasing power leads to the formation of coherent oscillations from an initial steady state. The limit cycle generated by the Hopf bifurcation is a *stable, self-sustained oscillatory state* whose phase-space trajectory is a circle surrounding the fixed point (figs. 1 and 2). There is a natural connection of limit cycles to the SL equation, because the latter is the *universal normal form* (that is, the most general leading-order equation) governing oscillations produced by Hopf bifurcations.

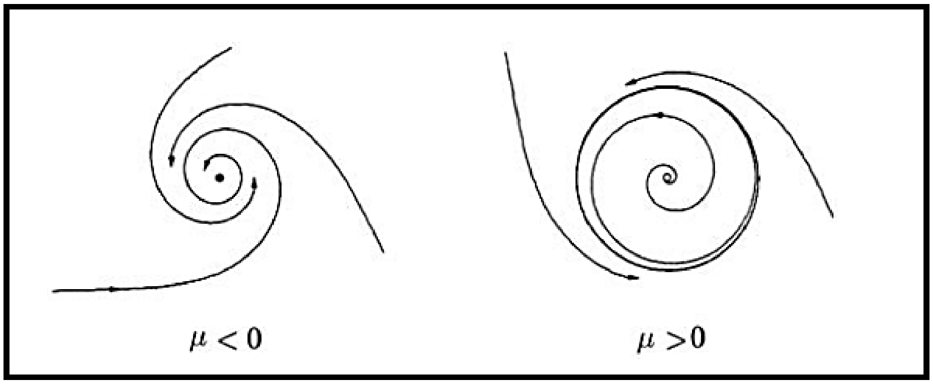


**Fig. 1:** Planar Hopf bifurcation driven by parameter $\mu$.

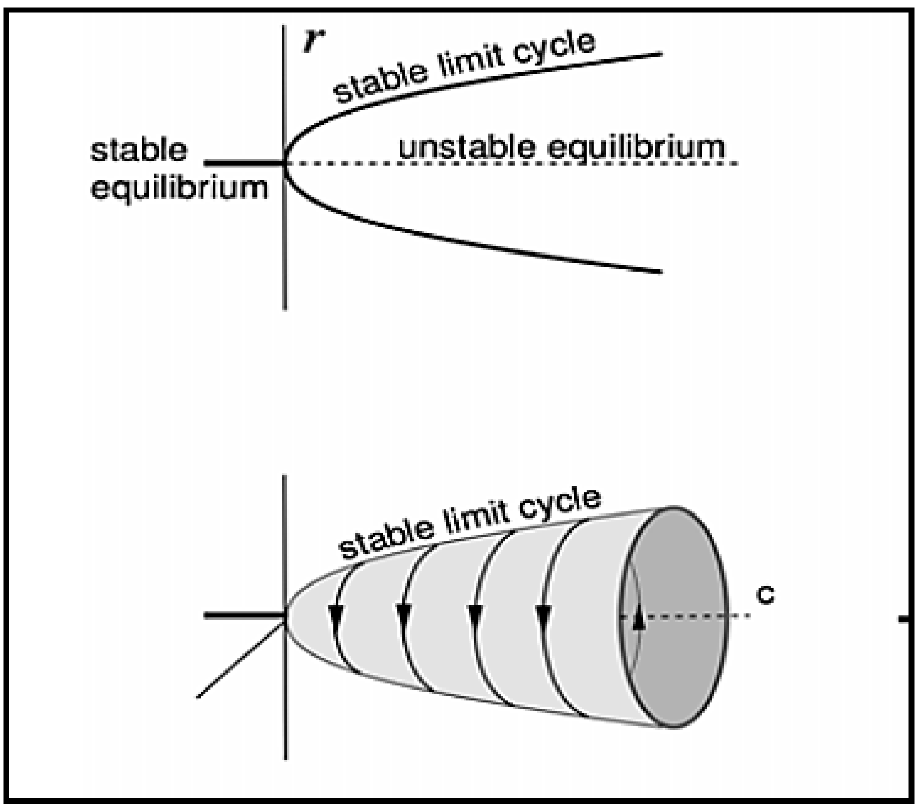


**Fig. 2:** Hopf bifurcation in one and three dimensions.

In field theory language:

- Before the bifurcation → the field sits at a constant location fixed by the vacuum expectation value ($v$).
- After the bifurcation → the field oscillates on a limit cycle with a well-defined frequency.

That is to say that, if the Higgs effective potential is modified by quantum or thermal corrections, or by interactions with heavy fields, then $v$ undergoes **a** Hopf bifurcation, replacing the static vacuum with a self-sustained oscillatory limit cycle [1].

All fermion masses come from couplings to the Higgs field:

$$m_f = \frac{y_f}{\sqrt{2}} v$$

It follows that, if the Higgs has a time-dependent oscillatory component, then every fermion mass becomes a slight oscillatory function, as in, for example,

$$m_f(t) = m_f^0 + A\sin(\omega t + \varphi) \tag{28}$$

It also follows from (28) that each fermion mass gets its own “phase shift” from the Higgs oscillation. While two fermions with the same mass get the same phase, fermions with different Yukawa couplings acquire different phases from the same Higgs oscillation. “Mixing” amounts to the *difference in phases*. The best analogy is provided by *two coupled oscillators* — if they have different natural frequencies and share a driving signal, they lock with a constant relative phase difference.

When the Hopf bifurcation takes place, the Higgs field may not oscillate in just one mode. The bifurcation can support *three independent and unstable limit-cycle modes*, loosely corresponding to

- amplitude,
- radial fluctuation,
- angular (Goldstone-like) fluctuation.

Each mode gives a phase and amplitude contribution to each fermion mass. Thus, for three fermion generations:

- up, charm, top quarks $(u,c,t)$ receive a triplet of oscillatory corrections,
- down, strange, bottom quarks $(d,s,b)$ receive another triplet,
- the *relative* phase structure between the two triplets forms a 3×3 unitary matrix which reflects the structure of the CKM or PMNS matrix:

A crucial observation here is that, out of the three unstable Higgs modes, the dominant contribution comes from amplitude fluctuations and generates the structure of SM and its mass content [3, 5]; the radial fluctuation and the Goldstone mode, while leaving their imprint on the mixing matrices, are likely to be short-lived and vanish in the long run.

APPENDIX A elaborates on the mathematical underpinning of the three-mode oscillation model. It is apparent from the discussion there that 3 is the *minimum number* that supplies all the following conditions:

- nontrivial mixing,

- irreducible CP violation,
- $SU(3)$ - compatible flavor structure,
- basis for 3×3 mass matrices,
- correct number of degrees of freedom to match CKM/PMNS phenomenology.

It can be reasonably inferred that the three Higgs oscillation modes *are not arbitrary*—they are the most natural, symmetry-compatible choice that reproduces the observed structure of the SM flavor sector.

## 4. Conclusions

The main takeaway points of this paper can be summarized as follows:

- A three-component Stuart–Landau (SL) normal form with an antisymmetric cubic coupling naturally arises from a Hopf bifurcation of an unstable 3-modes Higgs sector.
- After the Hopf, the system generically flows to a *phase-locked 3-cycle* with nontrivial cyclic phase — precisely the structure needed to seed

fermion mass matrices with phases that produce mixing angles and CP violation.

- $SU(3)$ (or cyclic $\mathbb{Z}_3$) group structure gives a sound group-theoretic origin for the antisymmetric cubic coupling $\varepsilon_{ijk}$ that enforces the cyclic phase.
- Three generations couple to the three unstable Higgs modes in alignment with the CP violating condition mandated by the Jarlskog invariant.
- Hierarchical structure of masses is driven by the surviving Higgs mode via the Feigenbaum route to chaos.
- Cabibbo angle emerges as the reciprocal of the Feigenbaum constant and explains the core structure/texture of both CKM and PMNS matrices.
- Breaking of parity and time-reversal symmetry follows directly from the asymmetry of Hopf limit cycles [1].

## APPENDIX A: Mode instability of the Higgs condensate and fermion mixing

We model the unstable Higgs modes by a complex triplet field $\Phi = (\Phi_1, \Phi_2, \Phi_3)$, whose components live in an internal flavor space and are allowed to weakly mix with each other. Because Hopf bifurcations are inherently dynamical, involving gain/losses and dissipative features, one can write a *complex Ginzburg-Landau equation (CGLE)* for the spatially uniform modes (dropping spatial dependence for clarity),

$$\partial_t \Phi_i = (\mu + i\omega)\Phi_i - g\left|\Phi_i^2\right|\Phi_i + \xi \sum_{j \neq i} \Phi_j + \gamma \sum_{jk} \varepsilon_{ijk} \Phi_j^* \Phi_k^* \qquad \text{(A1)}$$

where,

- $\mu$ is a real growth/damping control parameter,
- $\omega$ is a base frequency,
- $g > 0$ is the coefficient of nonlinear damping,
- $\xi$ is the symmetric linear coupling between modes,

- $\gamma$ is a (complex, in general) coefficient multiplying the totally antisymmetric tensor $\varepsilon_{ijk}$. This cubic term complies with $SU(3)$ group structure and couples the three modes in a cyclic way. Since it enables cyclic interactions among the three components, it provides a natural source for phase circulation and the formation of the CP phase on the CKM matrix.

Increasing $\mu$ through 0 makes all three modes unstable and oscillatory; nonlinearities of (A1) then determine which limit cycles form and their relative phases.

Near the Hopf threshold, the long-time dynamics are governed by the center manifold spanned by the critical eigenmodes. According to the *center manifold reduction*, (A1) can be reduced to amplitude equations for *three complex slow variables* $Z_i(t)$ and the SL 3-mode normal form is

$$\boxed{\dot{Z}_i = (\mu + i\omega)Z_i - g\left|Z_i^2\right|Z_i + \xi\sum_{j\neq i} Z_j + \Gamma\sum_{jk}\varepsilon_{ijk}Z_j^* Z_k^*} \tag{A2}$$

with constants $g, \xi, \Gamma$ (in general complex). By the center manifold theory, this amplitude equation version of (A1) guarantees that, to leading order, (A2) captures all possible behaviors close to the Hopf threshold. Note that (A2) is exactly a set of *three coupled SL oscillators* with an antisymmetric cubic coupling.

Now, write $Z_i = r_i \exp(i\phi_i)$ with real $r_i \geq 0$ and phases $\phi_i$. Inserting into (A2) and separating real/imaginary parts gives:

1) **Amplitude equations**

$$\dot{r}_i = \mu r_i - g r_i^3 + \xi \sum_{j \neq i} r_j \cos(\phi_j - \phi_i) + |\Gamma| \sum_{j,k} \varepsilon_{ijk} r_j r_k \cos(\phi_j + \phi_k + \arg\Gamma - \phi_i) \quad \text{(A3)}$$

2) **Phase equations**

$$r_i \dot{\phi}_i = \omega r_i + \xi \sum_{j \neq i} r_j \sin(\phi_j - \phi_i) + |\Gamma| \sum_{j.k} \varepsilon_{ijk} r_j r_k \sin(\phi_j + \phi_k + \arg\Gamma - \phi_i) \quad \text{(A4)}$$

These are real dynamical equations that determine steady amplitudes $r_i$ and relative phases $\psi_{ij} = \phi_i - \phi_j$.

Next, consider the case of equal amplitudes $r_i = r$ and look for stationary phase differences $\psi_{12}, \psi_{23}$, constant in time. In this situation, amplitude equations are reduced to

$$0 = \mu r - g r^3 + \xi r \sum_{j \neq i} \cos \psi_{ij} + |\Gamma| r^2 \sum_{j,k} \varepsilon_{ijk} \cos(\Phi_{ijk}) \qquad \text{(A5)}$$

(where $\Phi_{ijk}$ denote linear combinations of phases and $\arg \Gamma$), while phase equations yield algebraic constraints on $\psi_{ij}$. A particularly important solution is the *cyclic 3-state*:

$$\psi_{12} = \psi_{23} = \psi_{31} = \frac{2\pi}{3} \ (\mathrm{mod}\, 2\pi) \qquad \text{(A6)}$$

i.e., the three phases are evenly spaced and describe a “rotating 3-cycle”, where the three modes form a cyclic phase-locked pattern. In this case, the antisymmetric $\varepsilon_{ijk}$ yields an irreducible phase circulation that cannot be removed by a single overall rephasing – it is a genuine geometric phase echoing the CP phase of the CKM matrix.

Let's look now at the stability of (A2) in the Hopf bifurcation picture. There are three situations of interest:

- For $\mu < 0$, one has a stable trivial fixed point $Z_i = 0$.
- At $\mu = 0$, the Hopf threshold is reached and small oscillations appear.
- For small $\mu > 0$, nonlinear terms select one of several attractors: a) single -mode limit cycles (only one $r_i$ is nonzero), b) two-mode patterns, c) symmetric *three-mode periodic limit cycle* (the one of interest in our context).

Linear stability of the cyclic solution is determined by the Jacobian of (A3) and (A4) evaluated at the stationary $r, \psi$. One typically finds parameter regions where the symmetric 3-cycle is *stable* for $\Gamma$ nonzero, which means that the three-mode cycle is not just allowed but dynamically preferred.

A natural question is now*: How does this formalism connect to the electroweak sector of SM?* To answer it, recall that, after electroweak symmetry breaking, charged fermions gain mass according to

$$M_f = \frac{v}{\sqrt{2}} Y_f \tag{A7}$$

where $M_f$ is the mass matrix and $Y_f$ are the Yukawa matrices describing how strongly each fermion family couples to the Higgs vacuum. Each type of fermion (e.g. up-quarks vs. down quarks) has its own Yukawa/mass matrix, namely, the up-sector $M_u$ and the down-sector $M_d$. As stated in section 2, they are not diagonalized by the same rotation and the mismatch between the left-handed rotations produces the mixing matrix,

$$V = U_{uL}^{\dagger} U_{dL} \tag{A8}$$

In the Higgs model of three SL oscillatory modes $Z_i = r_i \exp(i\phi_i)$, projecting the Yukawa operator onto these three modes gives three flavor matrices $Y^{(i)}$ and one can define an effective Yukawa as in

$$\boxed{Y_f = Z_1 Y^{(1)} + Z_2 Y^{(2)} + Z_3 Y^{(3)}} \tag{A9}$$

In this picture, the amplitudes $r_i$ generate ***mass hierarchies***, while phases $\phi_i$ generate ***mixing and CP violation***. The latter observation can be understood as follows: since the CKM matrix (A8) and the rotations $U_{uL}$ and $U_{dL}$ come from the structure of $M_u$ and $M_d$, the mixing angles are controlled by how the SL phases differ between the up and down sectors. Symbolically,

$$\theta_{ij} = function\, of\, (\phi_{1u} - \phi_{1d}, \phi_{2u} - \phi_{2d}, \phi_{3u} - \phi_{3d}) \tag{A10}$$

and the CP phase $\delta_s$ is the cyclic phase mismatch between the up and the down sectors.

## APPENDIX B: Mass generation from multifractal geometry of the Feigenbaum attractor

As the previous section alludes to, the surviving amplitude $r_1$ generates the hierarchical composition of masses via successive period-doubling bifurcations of the primary limit cycle [1]. At the endpoint of these bifurcations (called the *Feigenbaum point*), dynamics is neither periodic nor chaotic, and trajectories evolve on a critical invariant set with strong scale invariance. The invariant measure supported on this attractor is non-uniform and exhibits power-law scaling that changes from point to point. Hence the Feigenbaum point describes an attractor whose geometry may be effectively approximated using the following multifractal relationship [6]

$$\sum_j p_j^q s_j^{\tau(q)} = 1 \tag{B1}$$

Here, $s_j$ are the fractal scales of the multifractal set acting with probabilities/weights $p_j$ and $q, \tau(q)$ are scaling exponents. A limiting case of (B1) for $q = 0$ and $\tau(0) = 2$ reduces to the so-called *sum-of-squares* constraint of SM which links the square of elementary particle masses to the square of the Fermi scale viz. [7]

$$m_W^2 + m_Z^2 + m_H^2 + \sum_f m_f^2 = v^2 \tag{B2a}$$

$$s_j = \frac{m_j}{v} \Rightarrow \sum_j s_j^2 = 1 \tag{B2b}$$

where $W, Z$ and $H$ stand for the electroweak and the Higgs bosons, respectively, and the sum in the left-hand side is taken over the whole spectrum of SM fermions. Moreover, according to [8], all particle masses and couplings (Yukawa and Higgs self-interaction couplings) are functions of the dimensional deviation $\varepsilon = 4 - D << 1$ as in,

$$s_H(\varepsilon) = \frac{m_H(\varepsilon)}{v} = \sqrt{2\lambda(\varepsilon)}$$

$$s_f(\varepsilon) = \frac{m_f(\varepsilon)}{v} = \frac{1}{\sqrt{2}} y_f(\varepsilon) \tag{B3}$$

$$s_W(\varepsilon) = \frac{m_W(\varepsilon)}{v} = \frac{g(\varepsilon)}{2}; \quad s_Z(\varepsilon) = \frac{m_Z(\varepsilon)}{v} = \frac{\sqrt{g(\varepsilon)^2 + g'(\varepsilon)^2}}{2}$$

Comparing (B3) with the entries of (A7) shows that each mass component of (A7) behaves as if *it have been generated* by the vacuum $v$ via appropriate coupling coefficients. The conceptual difference between (B3) and (A7) is that, in the geometric interpretation of SM as multifractal set, couplings *are not stemming* from the standard Higgs mechanism of electroweak symmetry breaking, but from the notion of *fractal scale* entering the sum-of-squares relationship (B2).

Another instructive point is worth mentioning here. If $M_{Pl}$ denotes the Planck scale, by recasting (B3) as

$$(\frac{m_i}{v})^2 = y_i^2 = \frac{M_{Pl}^2}{v^2}\frac{m_i^2}{M_{pl}^2} = const \times \frac{m_i^2}{M_{pl}^2} = const \times \varepsilon_i^2 \qquad \text{(B4)}$$

shows that, up to a normalization constant, couplings represent continuous *dimensional deviations* from the four-dimensionality of ordinary spacetime, that is $\varepsilon = 4 - D = O(m/M_{Pl}) << 1$.

## APPENDIX C: Cabibbo angle as reciprocal of the Feigenbaum constant

As the SM structure approaches the Feigenbaum point via bifurcations, fermion masses are arranged in a geometric progression having the form [9]

$$m_n \propto \delta^{-n} \qquad \text{(C1)}$$

where $\delta$ is the Feigenbaum constant for the quadratic map ($\delta = 4.669...$).

Two mass values in the series (C1) separated by $p = 2^k;\ k = 0,1,2,...$

$$m_{n+p} \propto \delta^{-(n+p)} \tag{C2}$$

satisfy

$$\frac{m_{n+p}}{m_n} \propto \delta^{-p} \tag{C3}$$

or

$$\sqrt{\frac{m_{n+p}}{m_n}} = \delta^{-p/2} = \delta^{-2^{k-1}} \tag{C4}$$

Experimental data shows that mass ratios scale as powers of the Cabibbo angle $\lambda$ as in [3],

$$\frac{m_e}{m_\mu} = O(\lambda^4) \qquad \frac{m_\mu}{m_\tau} = O(\lambda^2)$$

$$\frac{m_c}{m_t} = O(\lambda^4) \qquad \frac{m_s}{m_b} = O(\lambda^2)$$

$$\frac{m_u}{m_t} = O(\lambda^8) \qquad \frac{m_d}{m_b} = O(\lambda^4)$$

Unlike several proposals linking the Cabibbo angle to hypothetical conditions or extra relationships among particle masses [10-17], our papers imply, based on the above considerations,

$$\sin\theta_C \propto \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} = \lambda(1-\lambda) \tag{C5}$$

or

$$\boxed{\sin\theta_C \propto \lambda = \delta^{-1} = (4.669..)^{-1}} \tag{C6}$$

compatible with the measured value of the Cabibbo angle $\lambda = 0.223 - 0.225$ [18].

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