Short note: Trigonometric interpolation for $2N$ points

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Abstract

The construction of a trigonometric interpolant for an even number of interpolated points is not unique. To resolve this, we propose to minimize the amplitude of the highest-frequency term, which leads to the Gauss interpolation formula.

1 Introduction

A trigonometric polynomial

$$T_N(x) = a_0 + \sum_{n=1}^{N} [a_n \cos(nx) + b_n \sin(nx)] \tag{1}$$

can naturally interpolate $2N + 1$ points $\{(x_i, y_i)\}_{i=0}^{2N}$ because it has $2N + 1$ parameters $\{a_n\}_{n=0}^{N}$ and $\{b_n\}_{n=1}^{N}$. Attempting to interpolate an even number of points $2N$, one cannot "close" the highest-frequency term; the situation is under-constrained and an infinite number of solutions exists. We have not conducted a thorough literature review; this text is only a short commentary on [1].

2 Criterion

Let us consider in the rest of this text $2N = K$ points $\{(x_i, y_i)\}_{i=0}^{2N-1}$ to be interpolated with (1). To choose one representation from all possibilities, one often applies a condition to the highest frequency term

$$T_N^{\mathrm{HF}}(x) = a_N \cos(Nx) + b_N \sin(Nx) = A_N \cos(Nx + \varphi_N). \tag{2}$$

Common choices are $b_N = 0$ (balanced), $a_N = 0$ (skew-balanced) or $a_N = b_N$. The author of [1] argues in favor of a different choice, namely using the Gauss interpolation formula (GIF, see later) and provides several arguments for it:

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- The GIF guarantees the existence of the interpolation (not always true for the balanced and skew-balanced constraints).
- The GIF provides the minimal Lebesgue constant if $\{x_i\}$ are equidistant in both, L^∞ and L^2 norms.
- The GIF provides the minimal L^2-norm solution T_N for equidistant points.

The whole idea we have is to add to the previous list the requirement for A_N to be minimal. The question of "naturalness" is not mathematical, yet our criterion seems to us as the most natural.

Here is one possible point of view: Imagine we want to interpolate M_1 points using some trigonometric interpolation where we loosen our requirements: this interpolation may have arbitrarily many terms (with the total number of parameters $M_2{=}\#a_n+\#b_n$, $M_2 \geq M_1$), providing us with various Lebesgue constants and having various norms. But what is actually done in the "odd" scenario? We disregard all possibilities except the one with $M_2 = M_1$; we automatically choose to "fill" only the lowest-frequency modes, strictly cutting the higher ones by putting all corresponding coefficients to zero. The idea of using some higher number of terms just to lower the Lebesgue constant or the norm of the interpolant does not even come to our mind. We should simply do the same for an even number of points: the cut cannot be sharp since the Nx- frequency term cannot be zero, but we should do our best and suppress it as much as possible — that is, minimize A_N.

If one is convinced about the naturalness of this choice, simple calculations lead to the answer. The author of this text was hoping for a new interpolation rule, but was unfortunate. As shown by calculations which follow, the optimal choice to minimize A_N is also given by GIF. Thus the author joins the author of [1] and recommends GIF as the most natural way for interpolating $2N$ points in any scenario.

2.1 Computations

The text [1] provides all the necessary tools, which simplifies our task. Let us recall the GIF [2]:

$$T^{\mathrm{GIF}}(x) = \sum_{k=0}^{K-1} y_k l_k(x), \qquad l_k(x) = \left[\prod_{j=0,\, j\neq k}^{K-1} \frac{\sin\left(\frac{x-x_j}{2}\right)}{\sin\left(\frac{x_k-x_j}{2}\right)}\right] \times \cos\left(\frac{x-x_k}{2}\right).$$

All possible interpolations can be expressed in terms of one real parameter λ (formula (2.4) of [1])

$$T_\lambda(x) = T^{\mathrm{GIF}}(x) + \lambda l(x), \tag{3}$$

where $l(x)$ is a trigonometric polynomial of the order N which for all x_n satisfies $l(x_n) = 0$. The highest-frequency term of $T_\lambda(x)$ looks like ([1], section 3)

$$a_N = \frac{(-1)^N}{2^{2N-1}}\left[\lambda\cos(\sigma) + \eta\sin(\sigma)\right], \quad b_N = \frac{(-1)^N}{2^{2N-1}}\left[\lambda\sin(\sigma) - \eta\cos(\sigma)\right]$$

with

$$\sigma = \frac{1}{2}\sum_{k=0}^{K-1} x_k, \qquad \eta = \sum_{k=0}^{K-1} \gamma_k y_k, \qquad \gamma_k = \prod_{j=0,\, j\neq k}^{K-1} 1/\sin\left[\left(x_k - x_j\right)/2\right].$$

The amplitude in (2) is expressed

$$A_N(\lambda) = \sqrt{a_N^2 + b_N^2} = \frac{1}{2^{2N-1}}\sqrt{\lambda^2 + \eta^2},$$

the minimum is clearly attained for $\lambda = 0$. The choice $\lambda = 0$ gives GIF, see (3), and the value of the amplitude is

$$A_N^{\min} = \frac{|\eta|}{2^{2N-1}}.$$

References

[1] Anthony Austin. On trigonometric interpolation in an even number of points. *ETNA - Electronic Transactions on Numerical Analysis*, 58:271–288, 01 2023.

[2] Carl Friedrich Gauss. Theoria interpolationis methodo nova tractata. In *Carl Friedrich Gauss Werke*, volume 3, pages 265–327. Königliche Gesellschaft der Wissenschaften zu Göttingen, 1866.