Quantum Mechanics as Real Geometry: Curvature, Torsion, and Holonomy

S. K. Kwon
Department of Physics, Pohang University of Science and Technology, Pohang 37673, Republic of Korea
(Correspondence: sekk@postech.ac.kr)

Abstract

While the complex formalism of quantum mechanics has been extraordinarily successful, the fundamental geometric origin of its core elements—the imaginary unit, operator non-commutativity, quantum phase, and the **intrinsic geometry with quantum spin**—has remained unresolved for more than a century.
Here we resolve these foundational questions by deriving an exact and physically equivalent formulation of the Schrödinger equation as a *Real Geometric Flow*—a moving-frame evolution on a $2N$-dimensional real state space.

In this restored framework, the global imaginary unit i fully decomposes into a set of local skew-symmetric generators that drive planar rotations in real space, revealing that the traditional complex formalism is a compressed representation of an intrinsic rotational geometry.
Within the real geometric flow, energy eigenvalues appear as the **Curvatures** of the state trajectory, energy gaps correspond to **Torsions** that couple internal planes—naturally generating the geometry of quantum spin—and quantum phase emerges naturally as the **Holonomy** of the real moving frame.

This formulation preserves the full empirical content of quantum mechanics while exposing the geometric mechanisms hidden beneath its notation, providing the first coherent explanation of curvature, torsion, non-commutativity, spin behavior, and quantum phase within a single framework.
By demonstrating that the algebraic postulates of quantum mechanics arise from a deeper geometric order—one that stands alongside curvature-based theories such as General Relativity—this work establishes a unified differential-geometric foundation for quantum dynamics.
It opens new directions for quantum control, quantum information, and fundamental physics.

1. Introduction

Quantum mechanics governs every level of modern science and technology, yet its

foundational structure—particularly the meaning of complex numbers, non-commutativity, and quantum phases—remains conceptually unresolved[1].
The necessity and physical meaning of these structures have remained unclear for nearly a century, leaving the theory conceptually asymmetric and difficult to interpret.
Here we provide a **complete real geometric formulation** of the Schrödinger equation[1], revealing that its dynamics is governed by curvature, torsion, and holonomy in a $2N$-dimensional real moving frame.

Quantum mechanics employs complex numbers at its foundation[2]: states are complex vectors, observables are Hermitian operators, and time evolution is generated by the Schrödinger equation

$$i\dot{\psi} = H\psi.$$

Although this formalism is successful and internally consistent, the geometric meaning of the imaginary unit, phase, and non-commutativity has never been clarified.
Existing geometric approaches—such as the Berry phase[5,6], the Fubini–Study metric[7], and geometric quantization—describe important structures on Hilbert space, but they do not provide a fully real dynamical representation of Schrödinger evolution itself.

A fundamental observation is that any complex vector $\psi \in \mathbb{C}^N$ can be represented as a real vector $\Phi \in \mathbb{R}^{2N}$ by separating its real and imaginary components[3].
In this realification, the imaginary unit i specifies the *type* of a 2D rotation generator, but it does not confine rotations to the (x_k, y_k)-planes.
Instead, the operator $-iH$ expands into the full algebra $\mathfrak{so}(2N)$, producing rotations on all possible (i,j)-planes of $\mathbb{R}^{2N}$—including (x_m, x_n), (y_m, y_n), and cross-planes such as (x_m, y_n) and (y_m, x_n).
Thus, the global complex structure does not correspond to a block-diagonal operator; it becomes a structured superposition of planar rotations whose geometry is fully determined by the matrix elements of H.
The Schrödinger equation therefore transforms into the real linear differential equation

$$\dot{\Phi} = A\Phi, \qquad A = \rho\left(-\frac{i}{\hbar}H\right),$$

which describes a curve on a $2N$-dimensional real manifold with instantaneous rotation rates determined by the structure of A.

This real moving-frame representation reveals a natural geometric interpretation of the basic quantities in quantum mechanics.
Energy eigenvalues become **curvatures** of the state-space trajectory; energy differences appear as **torsions** coupling distinct internal planes; and the geometric or quantum phase corresponds to the **holonomy** accumulated along closed paths. Non-commutativity of Hamiltonians acquires a direct geometric meaning as curvature of the associated connection on real state space[3].

Although the global notation i is formally consistent, it obscures the fact that its action on each component of the Hamiltonian produces distinct rotations in different two–dimensional planes.
In our formulation, the symbol i is revealed as a compressed representation of a collection of local geometric generators J_{ij}, each acting on a specific plane of the expanded real state space.

In this work, we develop this correspondence rigorously.
We derive the exact real Schrödinger equation for general N-state systems, decompose its generator into planar rotation components J_{ij}, and demonstrate the resulting curvature–torsion structure explicitly in two-level systems, spin-1/2.
We further show that Berry curvature and holonomy emerge naturally from the real moving-frame connection.
This real geometric viewpoint does not alter any physical prediction of quantum mechanics; instead, it clarifies the mathematical roles of complex structure, phases, and non-commutativity by embedding the theory in a transparent real geometric framework.

2. Real Schrödinger Representation

Consider a quantum state $\psi(t) \in \mathbb{C}^N$ satisfying the Schrödinger equation[4]

$$i\frac{d}{dt}\psi(t) = H\psi(t), \qquad H^\dagger = H.$$

Let the quantum state be written as

$$\psi_k = x_k + iy_k, \qquad \mathbf{\Phi} = (x_1, y_1, \dots, x_N, y_N)^T \in \mathbb{R}^{2N}.$$

Each Hermitian element of the Hamiltonian

$$H_{mn} = a_{mn} + ib_{mn}, \qquad a_{mn}, b_{mn} \in \mathbb{R},$$

contributes rotations in the realified state space.
An off-diagonal element, however, pairs with its conjugate counterpart and unfolds into four planar rotations coupling the coordinate pairs (x_m, x_n), (x_m, y_n), (y_m, x_n), and (y_m, y_n).
The term

$$a_{mn}I_2 + b_{mn}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the basic 2×2 building block from which these rotations assemble[9].
This block is exactly the real linear operator corresponding to multiplication by the complex scalar $a_{mn} + ib_{mn}$.

Collecting all such contributions produces a real skew-symmetric generator[9]

$$A = \sum_{i<j} \omega_{ij} J_{ij} \in \mathfrak{so}(2N),$$

where J_{ij} rotates the (i,j)-plane of $\mathbb{R}^{2N}$, and the coefficients ω_{ij} are determined directly from H_{mn}.

The real Schrödinger equation is therefore

$$\frac{d}{dt}\mathbf{\Phi}(t) = A\,\mathbf{\Phi}(t),$$

which is exactly equivalent to the complex Schrödinger equation and describes the evolution as a sequence of planar rotations in real state space.

Hence

$$A \in \mathfrak{so}(2N)$$

is the Lie algebra of real orthogonal transformations.
The corresponding time evolution in real space is

$$\mathbf{\Phi}(t) = e^{At}\mathbf{\Phi}(0), \qquad e^{At} \in \mathrm{SO}(2N).$$

2.1. Decomposition into planar rotation generators

The real formulation obtained here is not a mere coordinate realification.
Each matrix element H_{mn} contributes a real 2×2 rotation block under $-iH$, which expands the complex Hilbert space $\mathbb{C}^N$ into a fully real geometric space $\mathbb{R}^{2N}$ while preserving the dynamical and algebraic structure of quantum mechanics.

The algebra $\mathfrak{so}(2N)$ is generated by elementary rotation matrices

$$(J_{ij})_{kl} = -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \qquad 1 \le i < j \le 2N.$$

Any real Schrödinger generator admits the expansion

$$A = \sum_{i<j} \omega_{ij} J_{ij}.$$

Each pair (i,j) defines a two-dimensional internal plane of the real state space, and J_{ij} serves as a local imaginary direction — the generator of the intrinsic rotation in the (i,j)-plane.
Hence the global complex structure of quantum mechanics appears as a collection of local planar rotations.

In this expanded space, the operator $-iH$ acts as a composite generator of simultaneous rotations across all intrinsic and cross-planes.
The diagonal entries of H—equivalently the eigenvalues of the diagonal part H_0—

determine the curvatures, energy gaps set the torsion couplings, and closed-loop evolution yields geometric holonomy — realizing the $\mathrm{SO}(2N)$ rotations generated by the real Schrödinger operator $A \in \mathfrak{so}(2N)$.

2.2. Curvature–torsion structure of the state-space trajectory

To reveal the geometric structure of quantum evolution, the Hamiltonian is decomposed into its diagonal part

$$H_0 = \mathrm{diag}(H)$$

and its off-diagonal coupling

$$H' = H - H_0.$$

In the real representation, the diagonal entries of H_0 generate intrinsic planar rotations at rates

$$\kappa_n = \frac{(H_0)_{nn}}{\hbar},$$

which act as the curvatures of the state-space trajectory.
The off-diagonal elements

$$H'_{mn} = a_{mn} + ib_{mn}$$

introduce cross-plane couplings.
Their real and imaginary parts generate two independent skew-symmetric components, corresponding to orthogonal cross-plane rotations in $\mathbb{R}^{2N}$—the torsions that twist the motion between eigen-planes.

Consequently, **the real Schrödinger evolution naturally acquires a full Frenet–Serret–type structure:** the diagonal part of H determines the curvatures within intrinsic state planes, while the off-diagonal couplings a_{mn} and b_{mn} provide the complete torsion that governs how the instantaneous plane of motion bends and twists into neighboring planes.

2.3. Equivalence with complex time evolution

The real solution

$$\mathbf{\Phi}(t) = e^{At}\mathbf{\Phi}(0)$$

is exactly equivalent to the complex evolution

$$\psi(t) = e^{-iHt/\hbar}\psi(0),$$

since the realification map satisfies

$$R\,e^{-iHt/\hbar} = e^{At}R.$$

Thus, the real moving-frame formulation is not a change of coordinates but a mathematically exact reformulation of quantum dynamics itself.

3. Local Imaginary Directions and Decomposition of the Global Complex Structure

The complex structure of quantum mechanics is traditionally encoded by the global operator i, which acts uniformly on every component of a complex state vector.
In the real moving-frame formulation, however, this global action is revealed to be a compressed notation for a **collection of local planar rotations**, each acting on a specific two-dimensional subspace of $\mathbb{R}^{2N}$.

Let

$$\Phi = (x_1, y_1, \dots, x_N, y_N)^T \in \mathbb{R}^{2N}$$

be the realified state vector.
For every pair of coordinates (i, j), define the elementary rotation generator

$$(J_{ij})_{kl} = -\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}, \qquad 1 \le i < j \le 2N.$$

These matrices generate rotations in the (i, j)-plane and satisfy

$$J_{ij}^T = -J_{ij}, \qquad J_{ij}^2 = -P_{ij},$$

with P_{ij} the projector onto that plane.
Thus, the global imaginary unit corresponds to the direct sum of local planar rotation operators:

$$i \leftrightarrow J_{\text{glob}} \equiv \operatorname{diag}(J, J, \dots, J),$$

where each 2×2 block J is the real canonical generator of a 90° rotation.

For any Hermitian Hamiltonian H, the real generator induced by Schrödinger evolution is

$$A = \rho\left(-\frac{i}{\hbar}H\right) = \sum_{i<j} \omega_{ij} J_{ij} \in \mathfrak{so}(2N).$$

This manifests that the complex structure of quantum mechanics is **not a single global object**, but a structured family of local imaginary directions J_{ij}, each associated with its own two-dimensional plane in real state space.

4. Curvature–Torsion Structure of Quantum State Evolution

The real Schrödinger equation,

$$\dot{\Phi}(t) = A\Phi(t), \qquad A \in \mathfrak{so}(2N),$$

describes motion along a curve in the real state space $\mathbb{R}^{2N}$.
Because A is skew-symmetric, the evolution is composed entirely of infinitesimal rotations, generating a natural moving frame along the trajectory.

Diagonalization of the Hamiltonian,

$$H = \sum_n E_n |n\rangle\langle n|,$$

reveals its intrinsic geometric content.
Under realification, each eigenstate corresponds to a distinct two-dimensional plane

$$\Pi_n \subset \mathbb{R}^{2N},$$

on which the dynamics reduces to a planar rotation with angular velocity

$$\kappa_n = \frac{E_n}{\hbar}.$$

These quantities represent the **curvatures** of the motion, setting the instantaneous rotation rate within each eigen-plane in the same way that curvature guides a Frenet–Serret frame.

Energy differences

$$\Delta E_{mn} = E_m - E_n$$

generate rotations in cross-planes Π_{mn} that couple different eigen-components.
These cross-plane rotations define the **torsions**, measuring how the instantaneous plane of motion twists into neighboring planes.
Even though diagonalization removes explicit off-diagonal torsion terms, the geometric torsion encoded in ΔE_{mn} remains conceptually valid, extending the torsion that originally arises from off-diagonal components in the real representation.

For two-level systems, the real generator $A \in \mathfrak{so}(4)$ naturally decomposes into two curvature components and two torsion components, consistent with a Frenet–Serret-type structure:

$$A = \kappa_1 J_{12} + \kappa_2 J_{34} + \tau_c J_c + \tau_d J_d.$$

Thus, the energy spectrum specifies the curvature of the state trajectory, while energy gaps manifest as torsion couplings between internal planes.

This yields a unified geometric interpretation of the fundamental quantities of quantum mechanics.

From this framework, we further see that nature intrinsically contains a **torsion-based spin geometry**.
Magnetic interactions activate these torsional degrees of freedom rather than introducing spin as an external structure.
The characteristic spin-½ behavior follows directly from the π-rotation holonomy of a torsion-coupled internal plane, requiring no additional assumptions.

5. Non-commutativity as Curvature in Real State Space

Non-commutativity in quantum mechanics, usually expressed as $[H_1, H_2] \neq 0$, acquires a direct geometric meaning in the real moving-frame formulation.
Let two Hamiltonians H_1, H_2 induce real generators $A_1, A_2 \in \mathfrak{so}(2N)$.
Since each A_i is a genuine rotation generator on the real state space, the commutator

$$[A_1, A_2] = \rho\left(-\frac{i}{\hbar}[H_1, H_2]\right)$$

measures the failure of their associated rotations to commute.
This failure is not algebraic in nature—it is geometric.

Interpreting $A(t)$ as a **connection** on the real state bundle, the curvature 2-form

$$F = dA + A \wedge A$$

encodes this non-commutativity[10].
When the Hamiltonian depends on parameters $\lambda \in M$,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$$

quantifies how the real moving frame twists under transport in parameter space.

Hence, non-commutativity—traditionally viewed as a postulate of operator algebra—is revealed to be the curvature of the state-space connection.
The canonical structure of quantum mechanics is thus naturally embedded in the differential geometry of real rotations.
Operator algebra becomes curvature and torsion, and quantum kinematics becomes parallel transport on the real state bundle.

Remarkably, the origin of operator non-commutativity can be understood without invoking any gauge-theoretic machinery.
It arises directly from the curved motion of the real state-space frame.
Non-commutativity is therefore not an abstract algebraic rule but a **geometric law**—a

direct consequence of curved motion in real state space.

6. Holonomy and Quantum Phase

A central feature of quantum mechanics is the emergence of geometric phase upon cyclic evolution[8].
In the real moving-frame formulation, the quantum phase is revealed as the holonomy of the real connection generated by

$$A(t) \in \mathfrak{so}(2N).$$

For any closed path γ in time or parameter space, the accumulated evolution of the real frame is

$$U(\gamma) = \mathcal{P}\exp\left(\oint_{\gamma} A\right) \in \mathrm{SO}(2N),$$

a genuine rotation in the $2N$-dimensional real state space.
When this rotation is projected back into the complex representation, it yields the total phase of the quantum state—both dynamical and geometric.

Under adiabatic evolution, the eigen-plane associated with a nondegenerate eigenstate traces a closed curve in state space. Its holonomy reduces precisely to the Berry phase, completing the geometric interpretation:

- **Dynamical phase** arises from curvature $\kappa_n = E_n/\hbar$,
- **Geometric (Berry) phase** arises from the parameter-space curvature and torsion of the induced connection[5,6,8] $F = dA + A \wedge A$,
- **Total phase** is the **holonomy of the real moving frame**, unifying dynamical and geometric contributions under a single geometric mechanism.

Thus, quantum phase—long treated as an abstract complex number—emerges naturally as the rotational holonomy of a real moving frame, directly reflecting the curvature and torsion encoded in the generator $A(t)$.
This places all aspects of quantum phase on the same geometric footing as curvature-driven motion and torsion-driven mixing, completing the **curvature–torsion–holonomy** structure of quantum dynamics.

7. Holonomy as the Unifying Principle

The real moving-frame formulation of quantum mechanics provides a unified geometric

interpretation of Schrödinger evolution, complex amplitudes, and quantum phase.
The central observation—that the global imaginary unit decomposes into local planar rotations—leads to a natural picture in which quantum dynamics becomes a sequence of rotations in a $2N$-dimensional real state space.
In this representation, the energy spectrum determines the curvatures of the trajectory, energy gaps determine torsions coupling internal planes, and the geometric phase arises as holonomy.
This unification is not accidental: the algebraic rules of quantum mechanics arise precisely as the shadows of the underlying real rotational symmetries encoded in the moving-frame connection.

A key advantage of this viewpoint is conceptual transparency.
Traditional treatments regard complex amplitudes, phases, and non-commutativity as algebraic postulates.
Here they emerge from explicit geometric operations: curvature, torsion, and connection on a real vector bundle.
This allows quantum dynamics to be interpreted using the same tools that describe moving frames and parallel transport in differential geometry.
Earlier geometric approaches captured isolated aspects of this structure—such as Berry curvature or the Fubini–Study metric[7]—but none reconstructed the full real moving-frame dynamics hidden beneath the complex notation.

Importantly, this framework preserves full equivalence with the complex Schrödinger equation.
No approximation or new physical hypothesis is introduced.
Instead, the real moving-frame representation reorganizes the existing structure of quantum mechanics into a transparent geometric language.
This also resolves the apparent mystery of why quantum theory requires complex numbers: they encode local rotations in real state space and can be reconstructed from the real generator $A \in \mathfrak{so}(2N)$.

The approach generalizes without obstruction to high-dimensional systems, interacting systems, and continuous degrees of freedom.
Berry connection and curvature appear as natural extensions of the state-space connection restricted to eigen-planes.
Time-dependent Hamiltonians and non-adiabatic processes are governed by the same real differential equation $\dot{\Phi}(t) = A\Phi(t)$, and non-commuting Hamiltonians correspond directly to curvature of the induced connection.
Viewed this way, the mathematical structure of quantum mechanics becomes a direct expression of the geometry of its state space, rather than a set of formal axioms imposed from the outside.

This geometric formulation does not modify any empirical predictions of quantum

mechanics.
Rather, it provides a unifying perspective linking energy spectra, phases, non-commutativity, and dynamics to a single real geometric framework.
It may thus serve as a foundation for visualizing high-dimensional quantum evolution, for reformulating quantum control and quantum information tasks, and potentially for relating quantum mechanics more closely to geometric methods in classical mechanics and gauge theory.
In this light, quantum mechanics appears not as an algebraic convention but as a manifestation of a deeper geometric order in which curvature, torsion, and holonomy govern every aspect of quantum evolution.

8. Conclusions

We have shown that the Schrödinger equation is exactly equivalent to a real moving-frame evolution in a $2N$-dimensional state space.
Under realification, the global imaginary unit becomes a set of local planar rotations J_{ij}; energy eigenvalues become curvatures of the state-space trajectory; energy differences become torsions; and quantum phases arise as holonomy.
These geometric structures reveal that the basic features of quantum mechanics—complex amplitudes, phases, non-commutativity, and unitary evolution—are expressions of a single real differential framework.

This formulation preserves the full empirical content of quantum mechanics.
It clarifies the mathematical structure of the theory by exposing the curvature–torsion–holonomy geometry hidden in the complex notation.
The approach is completely general, extending seamlessly to arbitrary N-state systems and continuous degrees of freedom, and it places quantum mechanics on the same geometric footing as classical differential geometry.

By interpreting quantum evolution as real rotational motion, this framework provides a unified and intuitive lens for understanding high-dimensional quantum dynamics.
It also opens new directions for quantum control, quantum information, and geometric representations of many-body systems, with potential extensions to quantum fields, open-system dynamics, and geometric tensor-network methods.

The real Schrödinger equation therefore stands not as a coordinate rewrite, but as a **geometric completion** of quantum mechanics—a restoration of the intrinsic rotational geometry that the complex formalism had obscured for a century.

Figure 1 | Real geometric structure of a two-level quantum system

a, A complex state $\psi = (\psi_1, \psi_2)^T \in \mathbb{C}^2$ is realified as $\Phi = (x_1, y_1, x_2, y_2)^T \in \mathbb{R}^4$, revealing two orthogonal internal state planes corresponding to the real and imaginary components of each basis amplitude.

b, The real Schrödinger generator $A \in \mathfrak{so}(4)$,

$$A = \kappa_1 J_{12} + \kappa_2 J_{34} + \tau_1 J_{13} + \tau_2 J_{24},$$

decomposes into curvature components $\kappa_{1,2} = E_{1,2}/\hbar$ acting within the two eigen-planes, and torsion components $\tau_{1,2} \propto (E_1 - E_2)/\hbar$, reflecting the torsion coupling between eigen-planes which encodes how energy gaps twist the real moving frame.

c, A closed-path evolution generates the real holonomy

$$U(\gamma) = \mathcal{P}\exp\left(\oint_\gamma A\right),$$

a genuine rotation in $\mathrm{SO}(4)$.
When projected back to the complex representation, this holonomy yields the total quantum phase—combining both the dynamical phase from curvature and the geometric phase from torsion-induced twisting.

Methods

M.1. Real Schrödinger equation via local imaginary generators

Let $\psi(t) \in \mathbb{C}^N$

$$i\hbar\dot{\psi}(t) = H\psi(t), \qquad H^\dagger = H.$$

Write each component as $\psi_k = x_k + iy_k$ and define the real vector

$$\Phi = (x_1, y_1, \dots, x_N, y_N)^T \in \mathbb{R}^{2N}.$$

We introduce the elementary real rotation generator on the (u, v)-plane

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad J^2 = -I_2.$$

Each matrix element $H_{mn} = a_{mn} + ib_{mn}$ induces a real 2×2 rotation block on the corresponding real subspace as

$$H_{mn} = a_{mn} I_2 + b_{mn} J.$$

As a result, the full Hermitian matrix H induces a real generator $A \in \mathfrak{so}(2N)$, which can be written as a linear combination of planar rotation generators J_{ij} acting on pairs of coordinates (x_k, y_k):

$$A = -\sum_m \frac{H_{mm}}{\hbar} J_{2m-1,2m} + \sum_{m<n} \frac{\Re H_{mn}}{\hbar} \left(J_{2m,2n-1} - J_{2m-1,2n}\right) + \sum_{m<n} \frac{\Im H_{mn}}{\hbar} \left(J_{2m-1,2n-1} + J_{2m,2n}\right).$$

Here J_{ij} is the skew-symmetric matrix that generates rotations in the (i, j)-plane of $\mathbb{R}^{2N}$.
The real Schrödinger equation is then

$$\dot{\Phi}(t) = A\,\Phi(t),$$

which is exactly equivalent to the complex Schrödinger equation, but now expressed as a sum of local imaginary rotations acting on each plane of the real state space.

M.2. Curvatures and torsions in real state space

For a time-independent Hamiltonian with eigenvalues E_n and eigenvectors $|n\rangle$, the real generator A block-diagonalizes into rotations in the corresponding two-dimensional planes:

$$A = \bigoplus_{n=1}^{N} \begin{pmatrix} 0 & E_n/\hbar \\ -E_n/\hbar & 0 \end{pmatrix} + \text{ cross-plane couplings.}$$

Each block defines a curvature $\kappa_n = E_n/\hbar$, and the cross-plane components encode torsions associated with energy differences $\tau_{mn} \propto (E_m - E_n)/\hbar$.

Thus, the full generator can be expressed as

$$A = \sum_{i<j} \omega_{ij} J_{ij}\,, \qquad \omega_{ij} \in \mathbb{R},$$

where J_{ij} are the basis elements of $\mathfrak{so}(2N)$.
The decomposition explicitly identifies curvature and torsion components of the state-space trajectory.

M.3. Holonomy and non-commutativity

If the Hamiltonian depends on parameters $\lambda_\mu \in M$, the real generator becomes a parameter-dependent connection

$$A_\mu(\lambda) = \rho(-\frac{i}{\hbar} H(\lambda)).$$

The curvature 2-form of this connection is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

which measures non-commutativity:

$$[A_\mu, A_\nu] = \rho(-\frac{i}{\hbar}[H_\mu, H_\nu]).$$

For a closed loop $\gamma \subset M$, the real frame undergoes the holonomy

$$U(\gamma) = \mathcal{P}\exp\left(\oint_\gamma A_\mu d\lambda^\mu\right) \in \mathrm{SO}(2N).$$

When projected back into the complex representation, this holonomy reproduces the dynamical and geometric (Berry) phases. Thus, quantum phases correspond precisely to the holonomy of the real moving-frame connection.

M.4. Continuous systems and field-theoretic extension

The geometric formulation extends seamlessly to continuous quantum systems.
For a free particle with Hamiltonian

$$H = \frac{p^2}{2m},$$

a Fourier-space decomposition

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \tilde{\psi}(k,t)e^{ikx}dk$$

shows that each momentum mode satisfies

$$i\frac{d}{dt}\tilde{\psi}(k,t) = \frac{\hbar^2 k^2}{2m}\tilde{\psi}(k,t).$$

Writing $\tilde{\psi}(k,t) = u(k,t) + iv(k,t)$ yields the real moving-frame evolution

$$\frac{d}{dt}\begin{bmatrix} u(k,t) \\ v(k,t) \end{bmatrix} = \begin{bmatrix} 0 & \hbar\omega_k \\ -\hbar\omega_k & 0 \end{bmatrix}\begin{bmatrix} u(k,t) \\ v(k,t) \end{bmatrix}, \qquad \hbar\omega_k = \hbar^2 k^2/2m.$$

Thus, each momentum component forms an independent 2D curvature plane in the infinite-dimensional real space $\mathbb{R}^{2\infty}$, and the full time-evolution becomes a simultaneous geometric rotation across all k-planes.
Potentials and interactions preserve this structure.

In field theories, mode expansions

$$\psi(x,t) = \sum_k a_k(t) e^{ikx}$$

lead to the same intrinsic planar rotations for each mode, while interaction terms generate cross-mode torsion.

Therefore, the curvature–torsion–holonomy structure developed for finite-dimensional quantum systems generalizes without modification to continuous systems and quantum fields.

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