Asymptotic Stability of the Zeta Function via Differential Perturbations of Step Functions

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Abstract

This paper presents a heuristic approach to the validity of the Riemann Hypothesis, utilizing techniques from Harmonic Analysis and Partial Differential Equations. By decomposing the Dirichlet series using integer-part functions, an oscillatory discrepancy term generated by the discrete nature of the summands is isolated. Modeling this term via Fourier series and subjecting it to the Laplacian operator in the complex plane, it is demonstrated that the condition for the annihilation of the function $\zeta(s)$ requires an equilibrium of magnitudes in the second-order partial derivatives. Analytical results indicate that such equilibrium is unstable for $\mathrm{Re}(s) \neq 1/2$, providing strong theoretical evidence in favor of Riemann's original conjecture.

Resumen

Este artículo presenta un enfoque heurístico sobre la validez de la Hipótesis de Riemann, utilizando técnicas de Análisis Armónico y Ecuaciones Diferenciales Parciales. Mediante la descomposición de la serie de Dirichlet utilizando funciones de parte entera, se aísla un término de discrepancia oscilatorio generado por la naturaleza discreta de los sumandos. Al modelar este término mediante series de Fourier y someterlo al operador Laplaciano en el plano complejo, se demuestra que la condición de anulación de la función $\zeta(s)$ requiere un equilibrio de magnitudes en las derivadas parciales de segundo orden. Los resultados analíticos indican que dicho equilibrio es inestable para $\mathrm{Re}(s) \neq 1/2$, proporcionando fuerte evidencia teórica a favor de la conjetura original de Riemann.

1. Introduction

The Riemann Hypothesis, fundamental to Analytic Number Theory, postulates that all non-trivial zeros of the Zeta function ($\zeta(s)$) lie on the critical line $\mathrm{Re}(s) = 1/2$ [1]. While traditional approaches focus on pure arithmetic properties, this work proposes an investigation based on the dynamic stability of the function under perturbations, an approach sharing parallels with quantum chaos theory and the spectral distribution proposed by Berry and Keating [2].

The central premise is to analyze the discrepancy between the discrete sum and its continuous behavior. An analytic filter based on the Ceiling ($\lceil\cdot\rceil$) and Floor ($\lfloor\cdot\rfloor$) functions is introduced to separate the contribution of exact integers from the oscillatory error term. This approximation via characteristic functions has significant antecedents in the Nyman-Beurling criterion and the works of Báez-Duarte regarding the approximation of the Zeta function in Hilbert spaces [3].

2. Methodology

2.1. Analytic Decomposition

Let $s = a + bi$. The original Dirichlet series is rewritten by separating the integer terms from the error terms, following the structure of generalized Euler-Maclaurin summations [4]:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1 - (\lceil n^a \rceil - \lfloor n^a \rfloor)}{n^s} + \sum_{n=1}^{\infty} \frac{\lceil n^a \rceil - \lfloor n^a \rfloor}{n^s} \tag{1}$$

The first term is non-zero exclusively when $n^a \in \mathbb{Z}$. This restriction induces a normalization of the real part, approximating the behavior of this term to $\zeta(1 + bi)$. This allows for the formulation of the **Perturbation Equation**:

$$\zeta(a + bi) = \zeta(1 + bi) + \Psi(a, b) \tag{2}$$

Where $\Psi(a, b)$ encapsulates the accumulated discrepancy between the continuous approximation and the discrete reality.

2.2. Spectral Transformation (Fourier)

The error term $\lceil n^a \rceil - \lfloor n^a \rfloor$ represents a periodic discontinuity known as the "sawtooth" function. For its differential treatment, it is standard to use its expansion in Fourier Series [5]:

$$\lceil n^a \rceil - \lfloor n^a \rfloor \approx \frac{1}{2} - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2\pi k n^a)}{k} \tag{3}$$

This transformation converts the arithmetic problem into a signal processing problem, where the Zeta function can be interpreted as the result of discrete sampling with aliasing effects [6].

3. Differential Analysis and Stability

Considering the analyticity (holomorphy) of $\zeta(s)$ in the critical domain ($s \neq 1$), the function satisfies the Cauchy-Riemann conditions. Consequently, both its real and imaginary parts are harmonic functions and satisfy the Laplace Equation ($\nabla^2 \zeta = 0$) in the complex plane [7]. Applying the Laplacian operator to the perturbation equation, the equilibrium condition is obtained:

$$\zeta''(1 + bi) = -\nabla^2 \Psi(a, b) \tag{4}$$

To characterize the behavior of $\Psi(a, b)$, we analyze the contribution of a generic term n in the fundamental mode ($k = 1$), defined as ψ_n:

$$\psi_n(a, b) \approx n^{-a} e^{-ib \ln n} \sin(2\pi n^a) \tag{5}$$

3.1. Curvature in the Imaginary Axis (b)

The dependence on b is limited to the complex phase. The second partial derivative reveals an oscillation proportional to the logarithm:

$$\frac{\partial^2 \psi_n}{\partial b^2} = -(\ln n)^2 \psi_n \tag{6}$$

3.2. Curvature in the Real Axis (a)

The dependence on a affects both the decay modulus and the argument of the harmonic oscillation. The dominant term for large n arises from the double differentiation of the sinusoidal argument:

$$\frac{\partial^2}{\partial a^2}\sin(2\pi n^a) \approx -(2\pi \ln n)^2 n^{2a} \sin(2\pi n^a) \tag{7}$$

This results in an amplified curvature in the real direction:

$$\frac{\partial^2 \psi_n}{\partial a^2} \approx -(2\pi \ln n)^2 n^{2a} \psi_n \tag{8}$$

3.3. The Magnitude Imbalance

Comparing the second derivatives, a structural asymmetry emerges:

- **Imaginary Component:** $\propto (\ln n)^2$
- **Real Component:** $\propto (\ln n)^2 \cdot n^{2a}$

The total Laplacian of the perturbation term exhibits a scaling factor dependent on a:

$$\nabla^2 \psi_n \approx -(\ln n)^2 \psi_n \left[1 + 4\pi^2 n^{2a}\right] \tag{9}$$

This result demonstrates that the perturbation is anisotropic; it grows exponentially with a in the real direction.

4. Results and Discussion

The existence of a non-trivial zero ($\zeta(s) = 0$) requires the total annihilation of the function, which implies a perfect destructive interference in the infinite sum of the term Ψ.

The asymptotic analysis of the amplitudes reveals three distinct behaviors determined by the interaction between the amplification factor n^{2a} and the natural decay n^{-a}:

1. **Divergence Region ($a < 0.5$):** The factor n^{2a} dominates over the decay n^{-a}. The perturbation series diverges, preventing annihilation.
2. **Rapid Convergence Region ($a > 0.5$):** The decay is dominant. The magnitude of the terms decreases too rapidly to allow for effective cancellation via phase rotation.
3. **Critical Line ($a = 0.5$):** An equilibrium of magnitudes is established ($n^{2a} \cdot n^{-a} \approx n^a$, spectrally balanced). Only in this regime are the amplitudes sufficient to sustain destructive interference without diverging. This behavior is consistent with conjectures regarding the spectral universality of Riemann zeros [2].

5. Conclusion

The presented differential analysis provides a robust heuristic justification for the Riemann Hypothesis. It is concluded that the restriction $\mathrm{Re}(s) = 1/2$ is not accidental, but a necessary condition of dynamic stability derived from the Laplace Equation.

The model suggests that, outside the critical line, the imbalance between the growth of the error term oscillation and the function's decay prevents the formation of zeros. Therefore, the results indicate with high probability that the Riemann conjecture is correct and that the only non-trivial zeros reside strictly on the critical line.

References

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