Galileo's Paradox and
a New Way to Compare Sizes of Infinite Sets

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Abstract In this paper Galileo's paradox is going to be analyzed. It will be shown that there is no paradox, but rather an imprecise analysis in which very big numbers were not treated with the utmost care. A new method for comparison of the sizes of infinite sets will be introduced.

1 Introduction

In this paper, we are going to analyze the Galileo's paradox [1]. Galileo was among the first scientists that has noticed that it is difficult to compare sizes of infinite sets, and one of the first that came to idea to try to establish bijection as a criteria for testing the equivalence of the sizes of infinite sets. In the case he has analyzed, he had opinion that it is possible to establish bijection between the set of natural numbers and the set of squares of natural numbers. In this paper it is going to be explained that he made a mistake, that was later repeated by Cantor and other mathematicians.

In this paper, we are going to introduce a new method for comparison of the sizes of infinite sets. We are going to analyze an elementary experiment with enumerated balls and boxes and we are going to use simple fact that the number of natural numbers and their reciprocals is the same (a very similar approach is used in [3] and [4]). The result of the experiment will clearly show that the number of squares of natural numbers is negligible comparing to the number of natural numbers. Very similar methods can be used to show that the number of even numbers, prime numbers and so on, is smaller than the number of natural numbers. The mistake that Galileo made comes from the fact that very large numbers were not treated with proper care – they require specific approach, like it is the case with the infinitesimally small numbers. A mistake Galileo has made, and many others after him, is related to the fact that he has proved that there is infinitely many pairs of numbers and their squares and that mapping is one-to-one, but he failed to prove that this holds for all numbers, including all infinite numbers. In this paper a different approach has been used and it will be shown that it is not possible to create bijection between natural numbers and squares of natural numbers. We are going to work with infinitesimally small numbers with which we know how to deal. We are going to use the result from [2] where it is shown that there is no reason to believe that changing the order of summation of the conditionally convergent infinite series will lead to a different result – different result is the consequence of the omission of the elements of the series.

(One possible way to see infinite numbers more precisely is through an analogy with a black hole - a math-black hole would have the 1/0 in its center, while the infinity (∞) represent the event horizon. The 1/0 does not represent a number, but rather the end of time, like in the case of astronomic black hole. Analogy for the mistake made by Galileo is related to the fact that he established bijection only for the numbers that are out of math-black hole, or till the event horizon – all other numbers were ignored, and that should not be the case.)

2 Experiment

In this section, a simple experiment, we are going to call Galileo's experiment, is going to be presented. This represent a specific realization of a new method for comparison of the sizes of two infinite sets.

Galileo's experiment: Imagine that we have infinite number of balls with reciprocals of all natural numbers written on them exactly once, that are placed in the source box (SB) and that we have another, experimental box (EB).

In the moment 1, we are going to move ball with reciprocals of number 1 on it from SB to

EB, and remove from the EB the ball with reciprocal of number 1 on it. In the moment 2, balls with reciprocals of numbers 2 to 2^2 are transferred from SB to EB, and the ball with reciprocal of number 2 on it, is removed from EB. We continue the process at the moments $n \in N,\ n > 2$ – transfer the balls with reciprocals of numbers from $(n-1)^2 + 1$ to n^2 from SB to EB, and remove ball with reciprocal of number n on it from EB. So, in every step of the experiment we remove a natural number that generates a square.

Now, we can try to answer the following question: What is the number of the balls in EB at the end of the process? Here, we can conclude that the number of balls in the EB at the end of process can be zero since we cannot specify any number that is still inside the EB - for every number that we mention we can specify a moment when the ball has been removed from the box. In that approach it is quite difficult to address numbers that are infinitesimally small and represent reciprocals of infinite numbers. However, we have an opportunity to analyze the problem from a different point of view:

We are going to calculate the sum of the numbers that are in the box at the end of the process. Let's denote with S(n) a following sum (sum of numbers on the balls that are left after n steps of experiment):

$$S(n) = 1\mathbf{-1}+1/2+1/3+1/4\mathbf{-1/2}+ \ldots +1/((n-1)^2 + 1)+1/((n-1)^2 + 2)+\ldots+1/(n^2)\mathbf{-1/}\boldsymbol{n}.$$

(Negative inputs are marked with bold numbers and letters to be noticeable.) We can see that S(n) represents the sum off all numbers left in the EB after n steps of experiment (sum of all numbers in EB at the end of the process will be obtained when n approaches infinity). If the number of natural numbers and number of squares of natural numbers is equal, that sum must approach 0. However, it is going to be shown that $S(n)$ approaches infinity, which means that the number of squares is substantially smaller than the number of all natural numbers. (If the sum at the end of the process is infinite or irrational – there is infinitely many balls in the EB at the of the process; if the sum is rational, it is necessary to do additional analysis in order to find out if the number of balls in the box is finite or infinite).

Let us define *S* as

$$S = \lim_{n \to \infty} S(n).$$

It is not difficult to understand that the following holds (see e.g. [2])

$$S = \lim_{n \to \infty} S(n) = \lim_{n \to \infty} \ln(n^2) - \ln(n) = \lim_{n \to \infty} \ln(n) = \infty .$$

This concludes our analysis and we can safely conclude that number of squares of natural numbers is negligible comparing to the number of natural numbers, since the number of the balls in the EB at the end of the experiment must be infinite.

However, as it was mentioned, if you are asked to give an example of the ball with any specific number on it, that is still in the EB, you will not be able to do it (without using a limes). The reason is quite obvious – for any number you choose, you can specify a moment in time in which the ball with that number on it has been removed from the EB. However, that is not a unique situation in mathematics – it is well known that integral represents a sum on infinitely many summands from which we cannot specify a single one, and it does not prevents us from using integral calculus.

Conclusion

An analysis of Galileo's paradox was performed. A new method for the comparison of the sizes of infinite sets has been introduced. An experiment was performed and that enabled us to conclude that the number of squares of natural numbers is negligibly small comparing to the number of natural numbers. Using a very similar analysis it can be shown that the number of even, odd, or prime numbers, numbers that represents powers of 2 and so on, is smaller than the number of natural numbers.

References

[1] G. Galilei (1954)(1638). Dialogues concerning two new sciences. Trans. Crew an de Salvio. New York, Dover.

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