Einstein Equations from a Timeless Euclidean Model: Operational Reconstruction and the Compensation Principle

A. N. Smirnov

Abstract

This paper considers a timeless Euclidean model on $\mathbb{E}^4$ with a single fundamental field Φ satisfying $\Delta_{\mathbb{E}^4}\Phi = 0$, and shows that, within this model, the equations of General Relativity arise as an effective geometric realization of causal reconstruction. The underlying operational structure is provided by causal reconstruction and observable transformations M, defined relative to a local foliation; slow variations of the foliation direction are allowed, which gives rise to a reconstructed effective metric g. The requirement of causal reconstruction is formulated in the form of the compensation principle, whose variational form is written as $\delta S_{\text{eff}} + \delta S_g = 0$.

It is shown that, under standard structural restrictions on the geometric sector (locality, diffeomorphism invariance, second order in derivatives, and the correct SR limit), the compensation formulation leads to the minimal Einstein–Hilbert gravitational action (with a boundary term) and, consequently, to the equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T^{\text{eff}}_{\mu\nu}, \qquad \nabla^\mu T^{\text{eff}}_{\mu\nu} = 0,$$

where $T^{\text{eff}}_{\mu\nu}$ is defined as the metric variation of the effective sector, and the normalization of G is fixed by the Newtonian limit.

In addition, an operational local formulation of the equivalence principle is obtained: in the adiabatic regime $L_{\text{field}} \ll L_{\text{fol}}$, the gravitational component is locally eliminable, so that local inertiality arises on scales $L_{\text{obs}} \ll L_{\text{fol}}$. It is also shown that the hierarchy between the characteristic scales of nongravitational effective fields and the gravitational sector follows from the requirements of causal reconstruction; equivalently, in observable regions one has the local estimate $\varepsilon = L_{\text{field}}/L_{\text{fol}} \le \varepsilon_{\max} \ll 1$, which ensures the weak-field regime.

Thus, the universality of gravity arises as a structural consequence: all operationally accessible effective fields use the same reconstructed

effective metric g, whereas any observable field-dependent splitting of light cones or metrics would falsify the model.

1 Introduction

In the previous work [1], a timeless Euclidean model on $\mathbb{E}^4$ was studied, with a real scalar field $\Phi(x)$ satisfying the Laplace equation

$$\Delta_{\mathbb{E}^4}\Phi = 0, \qquad \Delta_{\mathbb{E}^4} := \delta^{AB}\partial_A\partial_B,$$

on the class of admissible configurations defined by the operational restrictions of a localized observer. No explicit solutions were constructed; rather, the properties of the class were analyzed. In particular, it was shown that foliation and mode decomposition give rise to an operational inertial frame with its own event structure, causality, and inertia, while observable transformations between inertial frames take the Lorentz form with a finite universal limiting speed $v_{\max}$, whereas the Galilean limit is excluded.

The present work develops this framework further and derives the gravitational dynamics of the reconstructed geometry. We pass from foliations by hyperplanes to the more general case in which the foliation direction $n_A(x)$ may vary slowly, and the slices Σ_s need not be hyperplanes. The key idea is that, in this regime, the requirements of causal reconstruction lead to the necessity of a compensating geometric sector. In the framework considered here, energy–momentum arises as an emergent quantity of the effective description on the slices, whereas the curvature of the reconstructed metric is associated with the condition of consistent reconstruction rather than with the introduction of fundamental matter.

1.1 Background and Motivation

Background. The following are adopted as the initial minimal framework:

- the Euclidean space $\mathbb{E}^4$ without fundamental time and without an a priori causal structure;
- a real scalar field $\Phi(x)$ satisfying the Laplace equation $\Delta_{\mathbb{E}^4}\Phi = 0$ (see (1));
- operational causal reconstruction: events and causal relations are introduced as reconstructed structures determined by a localized observer on the foliation slices $\Sigma_s^{(\mathbf{n})}$.

In the previous work [1], it was shown that observable transformations M take the Lorentz form with a finite universal limiting speed $v_{\max}$, and that the reconstructed causal structure within each inertial frame is consistent with the null cone of the effective metric. This provides the recovery of local SR kinematics for a flat foliation.

Motivation. The next natural question is whether causal reconstruction can be preserved after introducing effective degrees of freedom ψ_I, arising as mode excitations on the slices. In the general case, their contribution leads to a mismatch between the local transfer law and the requirements of inter-slice consistency. In other words, the flat SR structure alone no longer guarantees the fulfillment of the conditions of causal reconstruction in the presence of the full effective sector.

This circumstance motivates the introduction of an additional compensating sector. In the present work, this sector is the reconstructed geometry: curvature is not postulated as a fundamental interaction, but arises as a way to compensate the mismatch induced by the effective fields and thereby restore the admissibility conditions for reconstruction.

It is precisely in this sense that gravity, in the framework considered here, differs from the other effective fields: it is responsible not for yet another type of local dynamics on a fixed background, but for the consistency of the reconstructed geometric structure itself. This then leads to the compensation principle, to a variational derivation of the Einstein equations with an effective source, and to the operational origin of the hierarchy of scales that ensures the weak-field regime in observable regions.

1.2 Aim and Structure of the Work

The aim of the present work is to derive rigorously the dynamics of the reconstructed effective metric g from the requirements of causal reconstruction in the timeless Euclidean model and to show that, at the working order, this dynamics is equivalent to the Einstein equations with an effective source.

The tasks are as follows:

- to fix the minimal reconstructive framework (the model on $\mathbb{E}^4$ with $\Delta_{\mathbb{E}^4}\Phi = 0$, local foliation, observable transformations M, effective metric g) and the necessary structural assumptions of the effective description (locality, diffeomorphism invariance of the reconstructed description, and second order in derivatives);
- to formulate the compensation principle as the condition of vanishing of the total reconstruction mismatch $\mathfrak{E}[g, \psi] = 0$ and its variational form,

and to obtain from it the minimal gravitational sector (the Einstein–Hilbert action with a boundary term) and the Einstein equations with the source $T^{\mathrm{eff}}_{\mu\nu}$ (see §3–§4);

- to establish the operational form of the equivalence principle as the local eliminability of the gravitational component associated with foliation curvature in the adiabatic regime (see §3);
- to show the operational origin of the hierarchy of scales $L_{\text{field}} \ll L_{\text{fol}}$ and the recovery of the Newtonian limit, and also to obtain the local upper bound $\varepsilon = L_{\text{field}}/L_{\text{fol}} \leq \varepsilon_{\max} \ll 1$, which ensures the weak-field regime in observable regions and yields falsifiable constraints on deviations from the SR/GR regime (see §5).

The structure of the paper is as follows. Section 2 fixes the fundamental framework and the operational structures of causal reconstruction (foliation, observer localization, mode description, and transfer). Section 3 formulates the compensation principle and the operational form of the equivalence principle. Section 4 provides a brief variational bridge from the compensation principle to the Einstein–Hilbert action and the Einstein equations. Section 5 analyzes the hierarchy of scales, the weak-field regime, and the Newtonian limit, as well as the exclusion of regimes with $\varepsilon = \mathcal{O}(1)$ in observable regions in the sense of the requirements of causal reconstruction. Section 6 discusses the limitations of applicability and the prospects.
Notation remark: the main notations related to foliation and induced geometry (Σ_s, h_{ab}, K_{ab}, $R^{(3)}[h]$) are introduced in §2; the hierarchy parameters L_{field}, L_{fol}, ε are introduced in §5.

2 The Fundamental Euclidean Model and the Foliation Structure

This section fixes the basic objects of the model and the operational structures required for causal reconstruction: foliations, observer localization, mode description, and the transfer law on the operationally accessible subspace. These elements introduce no new physical assumptions beyond those of [1], but serve as a self-contained reminder and a specification of the notation required for the subsequent exposition.

The question of how deviations from the SR regime are described through the curvature of the reconstructed geometric data, as well as their relation to the effective gravitational dynamics, will be considered in the next section.

2.1 The Laplace Equation and Admissible Configurations

Let $\mathbb{E}^4$ be equipped with the Euclidean metric δ_{AB} and global coordinates x^A, $A = 0, 1, 2, 3$. The Laplacian is defined by $\Delta_{\mathbb{E}^4} := \delta^{AB}\partial_A\partial_B$. The fundamental field is a real function $\Phi : \mathbb{E}^4 \to \mathbb{R}$, $\Phi \in C^2$, satisfying

$$\Delta_{\mathbb{E}^4}\Phi(x) = 0 \quad \text{for all } x \in \mathbb{E}^4. \tag{1}$$

This is the *only fundamental equation of the model.* No fundamental variational functionals, sources, or nonlinearities for Φ are introduced.

In the present work, we simply fix this framework as the fundamental model, following [1], where the reasons for choosing the Laplace equation as the minimal local $O(4)$-invariant elliptic equation without a distinguished time are discussed in detail.

This equation contains no distinguished time and does not define any fundamental dynamics; at this level, no internal (gauge) symmetries or distinguished directions in $\mathbb{E}^4$ are postulated, beyond the geometric $O(4)$ symmetry of the Euclidean metric itself, and no interactions are introduced, whether linear or nonlinear.

The solutions of (1) form the class of harmonic configurations of the field Φ on $\mathbb{E}^4$. At the interpretive level, each such configuration describes a complete "world" of the model: in the timeless framework, there are no independent initial conditions and no external evolution parameter, so the entire content of the description is contained in the choice of Φ, rather than in its "evolution in time."

In what follows, we shall not need to fix any particular solution. All constructions are formulated for an arbitrary configuration from a certain admissible class. By *admissible configurations* we shall mean, from this point on, solutions of (1) satisfying additional operational restrictions formulated below (see, in particular, §2.2).

Our interest is not in explicit solutions of equation (1) as such, but in the physical consequences of the imposed operational restrictions—in particular, the structure of the emergent effective fields and their properties in interaction with a localized observer.

2.2 Requirements of Causal Reconstruction

We work on the Euclidean space $\mathbb{E}^4$ with a field Φ satisfying the Laplace equation on the class of admissible configurations (see §2.1). Fix a foliation by level hyperplanes

$$\Sigma_s^{(\mathbf{n})} = \{\, x \in \mathbb{E}^4 \mid n_A x^A = s \,\}, \qquad n_A n^A = 1,\ s \in \mathbb{R}, \tag{2}$$

where the choice of n_A fixes an inertial frame, and the parameter s is used as the *operational foliation parameter* ("operational time") relative to that inertial frame.

Definition (causal reconstruction). Let O be a localized observer whose body, at fixed s, occupies a region $\Omega_0(s) \subset \Sigma_s^{(\mathbf{n})}$. By the *accessible data* of Φ we mean the information actually obtained by the observer through interactions within $\Omega_0(s)$ (for example, through acts of measurement/registration and the internal state of the observer's memory). Let also $\Omega(s) \subset \Sigma_s^{(\mathbf{n})}$ be the reconstruction domain, where $\Omega_0(s) \subset \Omega(s)$. Hereafter, unless stated otherwise, the dependence of $\Omega_0(s)$ and $\Omega(s)$ on s will be omitted from the notation.

Causal reconstruction is a procedure which, from the observer's accessible data (localized in Ω_0), constructs a consistent description of the causal network $\mathcal{C}_\mathbf{n} = (V_\mathbf{n}, \prec_\mathbf{n})$ in the reconstruction domain Ω, where the vertex set $V_\mathbf{n}$ is represented as

$$V_\mathbf{n} = V_{\text{obs}} \cup V_{\text{rec}}, \qquad V_{\text{obs}} \subset \Omega_0, \quad V_{\text{rec}} \subset \Omega,$$

and the order relation $\prec_\mathbf{n}$ is interpreted as "can influence" and is defined operationally in accordance with the transfer direction $\mathbf{n}$.

Requirement of causal reconstruction. Reconstruction in each inertial frame must satisfy:

(i) Localizability and transfer of modes. On each slice $\Sigma_s^{(\mathbf{n})}$, there exists a local mode representation of the field in the reconstruction domain Ω, given by coefficients $a_\alpha^{(\mathbf{n})}(s)$ (where α runs over a fixed set of modes), for which a *local linear transfer* in s is defined:

$$a_\alpha^{(\mathbf{n})}(s+ds) = \sum_\beta A_{\alpha\beta}^{(\mathbf{n})}[\Phi]\, a_\beta^{(\mathbf{n})}(s), \tag{3}$$

where $A^{(\mathbf{n})}[\Phi]$ depends only on the local configuration Φ in a neighborhood of Ω_0 on the slice $\Sigma_s^{(\mathbf{n})}$. In the SR regime, there exists a finite universal limiting speed $v_{\max}$ restricting operationally admissible causal relations.

(ii) Compatibility of the equation with transfer and change of inertial frame. For any direction $\mathbf{n}$ in the regime considered (for $\Phi \in \mathcal{S}$), the equation $\Delta_{\mathbb{E}^4}\Phi = 0$ admits a representation and a transfer of the form (3) in the domain Ω. Under a change $\mathbf{n} \mapsto \mathbf{n}'$, the corresponding transfer matrices are related by a transformation of the mode representation (see [1]).

(iii) Consistency under small rotations. Under $\mathbf{n} \mapsto \mathbf{n}'$ with $\theta = \arccos(\mathbf{n}\cdot\mathbf{n}') \to 0$, the reconstructions are consistent in the sense

$$V_{\mathbf{n}} \triangle V_{\mathbf{n}'} \to \varnothing, \tag{4}$$

and the orderings coincide on the common vertex set, i.e. the relations $\prec_{\mathbf{n}}$ and $\prec_{\mathbf{n}'}$ coincide on $V_{\mathbf{n}} \cap V_{\mathbf{n}'}$. Here (4) is understood relative to a fixed reconstruction protocol and its thresholds.

These conditions are operational: causality is not postulated, but arises as an admissibility criterion for reconstruction in the presence of a bound on the speed of interactions. They generalize the conditions of causal reconstruction used in [1] to derive the SR structure; here they fix the class of configurations compatible with the existence of an observer.

Admissibility of configurations. Denote by $\mathcal{S} \subset \{\Phi \in C^2(\mathbb{E}^4) \mid \Delta_{\mathbb{E}^4}\Phi = 0\}$ the class of admissible configurations for which causal reconstruction exists in the sense of the definition above and conditions (i)–(iii) are satisfied. Henceforth, we assume $\Phi \in \mathcal{S}$.

2.3 Observer Localization and a Common Reference Basis

The observer is not an external agent: it is described as a localized structure in $\mathbb{E}^4$ within the field configuration $\Phi(x)$. Fix the foliation (2). For causal reconstruction, we choose a reconstruction domain $\Omega(s) \subset \Sigma_s^{(\mathbf{n})}$, compact in three directions, and a region occupied by the observer's body $\Omega_0(s) \subset \Omega(s)$. Hereafter, unless stated otherwise, the dependence of $\Omega(s)$ and $\Omega_0(s)$ on s will be omitted from the notation.

Let $d\mu_s$ denote the volume element induced on $\Sigma_s^{(\mathbf{n})}$. Fix an orthonormal family $\{u_\alpha\}_{\alpha\in\Lambda} \subset L^2(\Omega, d\mu_s)$, such that $\operatorname{supp} u_\alpha \subset \Omega$ and

$$\int_\Omega u_\alpha(x)\, u_\beta(x)\, d\mu_s = \delta_{\alpha\beta}. \tag{5}$$

Represent the restriction of the field to the slice $\Sigma_s^{(\mathbf{n})}$ in the domain Ω as

$$\Phi(x)\big|_\Omega = \sum_{\alpha\in\Lambda} a_\alpha(s)\, u_\alpha(x), \qquad a_\alpha(s) = \int_\Omega u_\alpha(x)\, \Phi(x)\, d\mu_s. \tag{6}$$

Operationally accessible modes. We further assume the nesting of regions

$$\Omega_0 \Subset K \Subset \Omega \subset \Sigma_s^{(\mathbf{n})},$$

where K is an interior region separated from $\partial\Omega$ and used to suppress boundary effects of the elliptic setting. It is assumed that the region K is chosen sufficiently far from $\partial\Omega$ and sufficiently large compared with the microstructure of the observer's body Ω_0, so that, within the given reconstruction protocol, the influence of the choice of boundary conditions on $\partial\Omega$ and of the microdetails of Ω_0 on the operationally defined transfer in K is negligible.

We shall assume that the operationally used modes form a subset $\Lambda_K \subset \Lambda$ such that $\operatorname{supp} u_\alpha \subset K$ or, more generally, the modes u_α are effectively localized in K and only weakly sensitive to the choice of boundary conditions on $\partial\Omega$. It is precisely on the subset Λ_K that the transfer law (3) and the invariance relation (8) hold with the accuracy required by the given reconstruction protocol; hereafter, all statements about transfer and invariance are understood after restriction to Λ_K.

In each fixed inertial frame $\mathbf{n}$, we choose a *common reference basis* $\{u_\alpha\}$ as part of the reconstruction protocol and inter-observer compatibility.

The internal modes of the observer O are expressed as local linear combinations of the reference modes:

$$\chi_\beta(x) = \sum_{\alpha\in\Lambda} C^{(O)}_{\beta\alpha}\, u_\alpha(x), \qquad b_\beta(s) = \sum_{\alpha\in\Lambda} C^{(O)}_{\beta\alpha}\, a_\alpha(s), \tag{7}$$

where $\operatorname{supp} \chi_\beta \subset \Omega_0 \subset \Omega$ for all β.

Different admissible choices of the reference basis correspond to different possible protocols of causal reconstruction. Hereafter, we fix one arbitrary basis satisfying the conditions of localizability and compatibility formulated in §2.2.

2.4 Transfer Invariance and the Emergence of Causality

Conditions (i)–(iii) of §2.2 mean that, on the operationally accessible subspace of modes, there exists a *local linear transfer* of the coefficients $a^{(\mathbf{n})}(s)$ along the foliation parameter s, defined by the operator $A^{(\mathbf{n})}[\Phi]$ in the form (3). Here $A^{(\mathbf{n})}[\Phi]$ depends only on the local configuration Φ accessible to the observer through the data region $\Omega_0 \subset \Omega$, and is used in the local sense (for small ds).

Interior region. Hereafter, all statements about transfer and its invariance are understood as referring to the subspace of modes localized in $K \Subset \Omega$, introduced in §2.3, that is, to Λ_K. It is precisely this restriction that ensures stability with respect to variations of the boundary conditions on $\partial\Omega$ within the protocol thresholds.

Invariance of the transfer law under change of inertial frame. Consistency of reconstruction under small rotations $\mathbf{n} \mapsto \mathbf{n}'$ requires that the transfer law have the same form in different inertial frames on the intersecting class of operationally accessible modes (localized in K). Since the basis on each slice is constructed anew according to the same rule $\mathcal{U}[\Phi; \mathbf{n}, s]$ (see §2.3), a change of inertial frame induces a mapping between mode representations, and the invariance of the transfer law is expressed by a relation of the form

$$A^{(\mathbf{n}')}[\Phi] \;=\; U(\mathbf{n}, \mathbf{n}')\, A^{(\mathbf{n})}[\Phi]\, U(\mathbf{n}, \mathbf{n}')^{-1} \quad \text{on the subspace of modes localized in } K, \tag{8}$$

where $U(\mathbf{n}, \mathbf{n}')$ is the transition matrix between basis representations induced by the rule $\mathcal{U}$. (An explicit construction of $U(\mathbf{n}, \mathbf{n}')$ will be given below.)

Remark 2.1 (Interpretation of (8))**.** *The rotation* $\mathbf{n} \to \mathbf{n}'$ *means the replacement of the family of slices* $\Sigma_s^{(\mathbf{n})} \to \Sigma_s^{(\mathbf{n}')}$*. Since the basis* $\{u_\alpha\}$ *on each slice is defined by the same rule* $\mathcal{U}$ *and is not transferred as a fixed set of functions, relation* (8) *expresses the invariance of the* transfer law *(its form) under a change of inertial frame, rather than the covariance of a fixed set* $\{u_\alpha\}$*.*

Emergence of causality. In the adopted framework, causality in each inertial frame arises as a condition of operational consistency: reconstruction of the causal structure $\mathcal{C}_\mathbf{n}$ is possible only if there exists a local mode description and a consistent transfer law (3)–(8) on the operationally accessible modes. In this sense, the direction of "operational time" and the causal structure are emergent and depend on the choice of foliation; no global set of events is introduced at the fundamental level.

Remark on events. In the present work, we do not use a detailed mechanism for identifying events. It is sufficient that causal reconstruction in each inertial frame defines a causal network $\mathcal{C}_\mathbf{n} = (V_\mathbf{n}, \prec_\mathbf{n})$ with the decomposition $V_\mathbf{n} = V_{\text{obs}} \cup V_{\text{rec}}$, where V_{obs} corresponds to acts of local registration in the region Ω_0, and V_{rec} to reconstructed vertices in the reconstruction domain Ω on the basis of the transfer law. For details, see [1].

2.5 Direct and Observable Transformations

In the absence of a global set of events, the transition between inertial frames defined by the normals $\mathbf{n}$ and $\mathbf{n}'$ admits two conceptually distinct descriptions [1].

Direct transformations. Direct transformations correspond to the action of Euclidean symmetry on the fundamental data (the field configuration and the foliation) and, in the general case, lead to different reconstructed causal networks:

$$D_{\mathbf{n}\to\mathbf{n}'} : \mathcal{C}_{\mathbf{n}}^{(O)} \longmapsto \mathcal{C}_{\mathbf{n}'}^{(O')}. \tag{9}$$

In general, there is no canonical bijection between $\mathcal{C}_{\mathbf{n}}^{(O)}$ and $\mathcal{C}_{\mathbf{n}'}^{(O')}$. In the present work, direct transformations will not be used further.

Observable transformations (SR regime). Observable transformations are constructed at the level of reconstructed descriptions, preserving operational event structure. In the SR regime of inter-observer event consistency, they are given by the universal mapping

$$M_{\mathbf{n}\to\mathbf{n}'} : (t, \mathbf{r}) \mapsto (t', \mathbf{r}'). \tag{10}$$

where $(t, \mathbf{r})$ are defined in §2.6. As shown in [1], the transformations (10) take the Lorentz form with invariant $v_{\max}$; equivalently, the null cone of the effective metric $\eta_{\mu\nu}$ is preserved:

$$\eta_{\mu\nu}\,\Delta x^\mu \Delta x^\nu = 0 \implies \eta_{\mu\nu}\,\Delta x'^\mu \Delta x'^\nu = 0.$$

In all subsequent sections, Lorentz symmetry and the associated invariants refer exclusively to the observable transformations $M_{\mathbf{n}\to\mathbf{n}'}$ in the SR regime.

2.6 Operational Coordinates and the SR Invariant

In a fixed inertial frame $\mathbf{n}$, the foliation parameter s is used to define operational time. Following [1], we introduce the time normalization

$$t := \frac{s}{v_t}, \tag{11}$$

where v_t is the normalization scale of operational time. In the SR regime, as shown in [1], one has $v_t = v_{\max}$, where $v_{\max}$ is the universal limiting speed.

The spatial coordinates $\mathbf{r} \in \mathbb{R}^3$ are defined as the coordinates of a point $x \in \Sigma_s^{(\mathbf{n})}$ in an orthonormal frame $\{e_i\}_{i=1}^3 \subset T_x\Sigma_s^{(\mathbf{n})}$ satisfying $e_i \cdot e_j = \delta_{ij}$ and

$e_i \cdot n = 0$ (the scalar product is taken with respect to δ_{AB}). The corresponding increments $\Delta x^\mu = (\Delta t, \Delta \mathbf{r})$ ($\mu = 0, 1, 2, 3$) are used as coordinate differences in the reconstructed description.

In the SR regime, the observable transformations $M_{\mathbf{n}\to\mathbf{n}'}$ (see §2.5) take the Lorentz form with invariant $v_{\max}$ [1]. Equivalently, the quadratic form

$$Q(\Delta t, \Delta \mathbf{r}) := v_{\max}^2 (\Delta t)^2 - \|\Delta \mathbf{r}\|^2. \tag{12}$$

is preserved. In particular, the null cone $Q = 0$ defines the boundary of operationally admissible causal relations in the SR regime.

3 Operational Consistency and the Two Principles of General Relativity

The aim of this section is to obtain, in an operational formulation, two basic principles of General Relativity, without introducing them as independent postulates, but deriving them from the requirements of causal reconstruction in the timeless Euclidean model.

- First, we formalize the *compensation principle*: the requirement that the joint dynamics of the effective degrees of freedom and the compensating geometric sector ensure the fulfillment of the conditions of causal reconstruction, which leads to a variational formulation (and subsequently to the choice of the gravitational action).
- Second, we derive the *operational form of the equivalence principle*: the local eliminability of the "gravitational" component (associated with the rotation/curvature of the foliation) in a sufficiently small region under the adiabaticity condition.

As technical preparation, we introduce the minimally necessary definitions of the effective fields ψ_I on the slices Σ_s, their local evolution with respect to the foliation parameter s, and we also formulate the local consistency of observable transformations M as the preservation of the null cone of the metric g under a change of observer chart at a fixed point $p \in \Sigma_s$, thereby distinguishing this operation from transfer between neighboring slices.

3.1 Foliation Curvature and Effective Force

In the SR regime, the foliation is given by hyperplanes $\Sigma_s^{(\mathbf{n})}$ with constant normal $n_A = \text{const}$, and local causal reconstruction is based on the transfer

(3) and its invariance (8). To go beyond the SR regime, let us allow the normal to become a smooth field $n_A(x)$, satisfying the unit condition $n_A n^A = 1$ and the integrability condition (that is, locally there exists a function $s(x)$ such that $n_A \propto \partial_A s$), so that the slices are given by the level sets $s(x) = \text{const}$.

The curvature of the foliation is characterized by the extrinsic curvature of the slices. Define

$$K_{ab} := \tfrac{1}{2}\,\mathcal{L}_n h_{ab}, \qquad K := h^{ab} K_{ab}, \tag{13}$$

where h_{ab} is the induced metric on Σ_s, and $\mathcal{L}_n$ is the Lie derivative along n^A. In the SR regime, $n_A = \text{const}$ (in the Euclidean sense, $\bar{\nabla} n = 0$), and therefore $K_{ab} = 0$.

Reconstructed effective metric. In order to formulate the local cone consistency of observable transformations and to describe subsequently the compensating geometric sector, we introduce the reconstructed metric form associated with the normal field $n_A(x)$:

$$g_{AB}(x) = -h_{AB}(x) + v_{\max}^{-2}\, n_A(x) n_B(x), \qquad h_{AB}(x) = \delta_{AB} - n_A(x) n_B(x), \tag{14}$$

where h_{AB} is the projector onto the tangent subspace to the slice, so that $h_{AB} n^B = 0$. In the SR regime, when $K_{ab} = 0$, the metric g locally reduces to η, and the observable transformations M take the Lorentz form with invariant $v_{\max}$ [1].

Transition from foliation curvature to effective metric geometry. In the SR regime, the Lorentz structure of the reconstructed description is locally determined by the choice of foliation and by the unit normal field $n_A(x)$ to the hypersurfaces Σ_s. The corresponding metric expression should be understood as the SR limit of the observable causal-metric structure induced by the foliation-based organization of reconstruction.

In the GR regime, the residual mismatch of reconstruction requires compensation already at the geometric level. In the framework considered here, this compensation is achieved through a deformation of the foliation structure itself. In the reconstructed description, such a deformation corresponds to an effective metric $g_{\mu\nu}$, which encodes the resulting local causal structure and curvature. At the same time, $g_{\mu\nu}$ should not be identified with the ordinary metric induced by the embedding of hypersurfaces in $\mathbb{E}^4$: it belongs to the level of observable description and is determined by the properties of operational reconstruction, in particular by the structure of observable transformations.

Thus, the metric formulation used below serves as an effective geometric representation of the compensating mechanism arising from foliation curvature. By construction, the metric $g_{\mu\nu}$ must locally reduce to the SR form in the limit of vanishing residual mismatch and weak deformation of the foliation.

Effective force. In a fixed inertial frame, inertiality means the absence of acceleration relative to the chosen foliation direction: deviation from the integral curves of n^A requires an external action. When $n_A = n_A(x)$, the normal field defines a locally varying direction of "operational time," and an observer, while remaining within its reconstructed chart, interprets this as the appearance of an additional effective action required to ensure causal reconstruction when passing between neighboring slices.

This new effective force is associated with the geometric data of the foliation, in particular with K_{ab} and derivatives of K_{ab}, and at the level of effective description will be preliminarily identified with *gravity.* The next task is to formalize the condition of operational consistency of reconstruction in the presence of this force and to show that it leads to the standard gravitational dynamics.

Adiabaticity and hierarchy of scales. For a curved foliation, causal reconstruction remains possible only in a regime of scale separation. Introduce the *foliation curvature scale* L_{fol} through the estimates

$$\|K_{ab}\| \sim \mathcal{O}(L_{\mathrm{fol}}^{-1}), \qquad \|D_c K_{ab}\| \sim \mathcal{O}(L_{\mathrm{fol}}^{-2}), \tag{15}$$

where D is the covariant derivative on Σ_s, and the norms are induced by h_{ab} on the interior region $K \Subset \Omega$.

Let L_{field} be the characteristic variation scale of the operationally accessible effective degrees of freedom, that is, the typical spatial scale of modes from Λ_K or of effective fields ψ_I on Σ_s. Define the small parameter

$$\varepsilon := \frac{L_{\mathrm{field}}}{L_{\mathrm{fol}}}. \tag{16}$$

The *requirement of causal reconstruction in the GR regime* is that, on the subspace Λ_K, the transfer (3) and its invariance (8) must hold within the protocol accuracy, which is achieved only in the adiabatic regime $\varepsilon \ll 1$. If, however, $L_{\mathrm{field}} \sim L_{\mathrm{fol}}$, that is, $\varepsilon \sim 1$, then corrections of order unity arise to the local transfer and to the consistency of reconstructions, and therefore the causal network $\mathcal{C}_{\mathbf{n}}$ cannot be constructed stably in the operational sense. Such a regime therefore lies outside the admissible class $\mathcal{S}$.

3.2 Effective Fields as Operational Degrees of Freedom

Let $\{\varphi_\alpha^{(\mathbf{n})}(y;s)\}$ be an admissible local basis of modes on Σ_s (see [1, Sec. 3.2]). We define the effective fields as local functionals of the fundamental field, specified on the slice Σ_s in the reconstruction domain (and, when necessary, after restriction to Λ_K):

$$\psi_I(s,y) = \Psi_I[\Phi;\Sigma_s](y), \qquad \psi_I(s,y) = W_I{}^{\alpha}(s,y)\, a_\alpha^{(\mathbf{n})}(s) + \mathcal{O}\big(Da, DW\big), \tag{17}$$

where $a_\alpha^{(\mathbf{n})}(s)$ are the mode coefficients in the chosen representation (in a fixed inertial frame $\mathbf{n}$, the upper index $(\mathbf{n})$ may hereafter be omitted), D is the covariant derivative on Σ_s, and the matrix $W_I{}^{\alpha}(s,y)$ is locally invertible on the working subspace. The terms $\mathcal{O}(Da, DW)$ denote contributions containing at least one D-derivative and suppressed in the working approximation (in particular, in the adiabatic regime).

Local evolution along the foliation. The transfer of modes along the foliation parameter is defined by the operator $A^{(\mathbf{n})}[\Phi;\, s]$ in the sense of (3). At the working order, we write locally

$$a_\alpha^{(\mathbf{n})}(s{+}ds) = A_\alpha{}^{\beta}[\Phi;\, s]\; a_\beta^{(\mathbf{n})}(s) + \mathcal{O}(ds^2). \tag{18}$$

The corresponding evolution of the effective fields takes the form

$$\psi_I(s{+}ds, y) = \mathcal{U}_I{}^{J}(s,y;ds)\, \psi_J(s,y) + \mathcal{O}\big(ds^2, D\psi, DW\big), \tag{19}$$

where the transfer operator over the step ds is defined pointwise in (s,y) by

$$\mathcal{U}(s,y;ds) := W(s{+}ds,y)\, A(s)\, W^{-1}(s,y) = WAW^{-1}{+}(\partial_s W)W^{-1}ds{+}\mathcal{O}(ds^2). \tag{20}$$

Here all quantities on the right-hand side of the last equality are taken at the point (s,y). For a locally invertible transformation $W_I{}^{\alpha}(s,y)$, the description in terms of the mode coefficients a_α and the description in terms of the effective fields ψ_I are equivalent at the working order; the evolution of ψ_I is induced from the evolution of a_α by a local change of basis.

Compatibility with observable transformations. We require that the action of the observable transformations M on ψ define a finite-dimensional representation $R(M)$, and that the local evolution be compatible with it at the working order:

$$R(M)\,\mathcal{U} = \mathcal{U}\, R(M) + \mathcal{O}(\varepsilon), \qquad \varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}}, \tag{21}$$

where ε is the adiabaticity parameter (see §5).

Minimal conditions (as consequences of the reconstructive framework). From locality on Σ_s, transfer invariance, adiabaticity of the foliation, and the requirement of the existence of a stable effective description (in particular, the absence of Ostrogradsky instabilities for local evolution), there follow:

1. *Locality and second order in* y^a*:* at the working order (within the class of local finite-dimensional descriptions ψ_I), no terms of order higher than second in D arise.
2. M*-covariance and* 3*-diffeomorphism invariance* on Σ_s; compatibility is expressed by (21).
3. *Adiabaticity:* dependence on K_{ab} and DK_{ab} enters as $\mathcal{O}(L_{\text{fol}}^{-1})$ and higher.
4. *Reality of observables:* for complex ψ, observables (operators) are Hermitian; upon introducing a variational formulation below, the corresponding action is chosen to be real (Hermitian). In the present discussion, it is immaterial whether the effective fields are real or complex.

Optional complexification. When necessary, we introduce complex combinations of real fields:

$$\psi_c = \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \qquad (\psi_1, \psi_2) \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{22}$$

that is, the effective $U(1)$ phase is equivalent to an internal $SO(2)$ on a pair of real fields. The evolution on the slice is defined pointwise in y by an orthogonal operator $\mathcal{U} \in O(N)$; the unitary form appears only after complexification and is not required for the derivations in §4–§5.

Remark on the variational formulation. In the present subsection, only the operational degrees of freedom ψ_I and their local evolution are fixed. The variational formulation (the effective action S_{eff} and the definition of the source $T^{\text{eff}}_{\mu\nu}$) will be introduced below as a way to quantify the reconstruction mismatch and to formulate the compensation principle, which leads to the equations of gravitational dynamics.

3.3 Reconstruction Mismatch and the Compensation Principle

In the SR regime (flat foliation, $\nabla n = 0$), local causal reconstruction is ensured by the existence of the transfer (3) and its invariance (8) on the operationally accessible subspace Λ_K, while observable transformations M are consistent with the null cone (see §3.4). In the presence of operational degrees of freedom ψ_I, a *mismatch* generally arises between the predicted transfer, defined by the local law on Λ_K, and that required for inter-slice consistency of reconstruction. The need to eliminate this mismatch is precisely what leads to the introduction of a compensating geometric sector, realized through curvature of the reconstructed geometric data.

Reconstruction mismatch as a consequence of the requirements of causal reconstruction. The requirements of §2.2 fix that, on the subspace Λ_K, the transfer (3), the invariance (8), and the local cone consistency of observable transformations M (see §3.4) must hold. Consequently, the *admissibility* of a configuration in the sense $\Phi \in \mathcal{S}$ is equivalent to the requirement that the resulting reconstruction mismatch can be eliminated completely. In other words, only those configurations are physically relevant for which causal reconstruction is compatible with the complete vanishing of the corresponding mismatch on Λ_K.

Operational origin of the mismatch. More explicitly, the reconstruction mismatch arises because, in the presence of effective degrees of freedom ψ_I, the requirements of causal reconstruction impose on the operationally accessible subspace Λ_K two interrelated groups of conditions: first, consistency of the local transfer of modes between neighboring slices in the sense of (3); second, compatibility of this transfer and of the local change of observer chart with the observable causal structure, as expressed by the cone-consistency conditions (28), (29). In general, these conditions need not be automatically satisfied by the same reconstructed data (g, ψ).

Schematically, one may distinguish two types of local residual mismatches. First, the *transfer mismatch*, measuring the difference between the coefficients reconstructed on the neighboring slice and the coefficients predicted by the local transfer law:

$$\mathcal{R}^{\mathrm{tr}}_{\alpha} := a^{\mathrm{rec}}_{\alpha}(s{+}ds) - A_{\alpha}{}^{\beta}(s)\, a_{\beta}(s).$$

Second, the *local cone-consistency mismatches*, measuring deviations from the conditions (28) and (29), which are necessary for the local change of observer chart and the transfer between neighboring slices to remain compatible with the observable causal structure.

Alongside this, for admissible configurations one still requires consistency of the representations of the transfer law under change of inertial frame in the sense of (8). This condition, however, pertains not to a separate local mismatch for a fixed observer, but to the structural compatibility of the class of reconstructions.

The functional $\mathfrak{E}[g,\psi]$ is then understood as a positive local integral functional built from such residual mismatches (and their norms weighted by the protocol thresholds) on Λ_K. Accordingly, the condition

$$\mathfrak{E}[g,\psi]=0$$

means that the transfer mismatch and the local cone-consistency mismatches vanish within the accuracy thresholds of the reconstruction protocol, while the structural admissibility conditions for the class of reconstructions, including (8), are satisfied. In this sense, the compensation principle represents a compact variational form of the requirement that causal reconstruction remain jointly consistent.

Definition (mismatch functional). Introduce a functional $\mathfrak{E}[g,\psi]\geq 0$, constructed from local reconstructed data, such that

$$\mathfrak{E}[g,\psi]=0 \quad\Longleftrightarrow\quad \text{the requirements of causal reconstruction are satisfied on } \Lambda_K. \tag{23}$$

Since adiabaticity and scale separation $\varepsilon = L_{\text{field}}/L_{\text{fol}} \ll 1$ have been fixed earlier (see §5 and the discussion in §3.1), $\mathfrak{E}$ may be chosen in the form of a local integral functional admitting an expansion in ε and in the number of D-derivatives on Σ_s. Invariance of the requirements of reconstruction under admissible changes of observer chart means that $\mathfrak{E}$ may be chosen to be invariant (at the working order) under these transformations.

The compensation principle as an admissibility condition. The requirements of causal reconstruction (see §2.2) mean that physically admissible are only those configurations for which the total reconstruction mismatch can be compensated. This condition constitutes the *compensation principle*:

$$\mathfrak{E}[g,\psi]=0. \tag{24}$$

Here $\mathfrak{E}[g,\psi]$ denotes the total reconstruction mismatch, including the contribution of the effective sector and the compensating geometric response.

Variational form of the compensation principle. To obtain local equations of compensation, consider variations of $\mathfrak{E}[g,\psi]$ with respect to the reconstructed data (g,ψ). From condition (24) there follows the necessary variational condition

$$\delta\,\mathfrak{E}[g,\psi]=0. \tag{25}$$

Here δ denotes variations with respect to the reconstructed data (g, ψ) at fixed elements of the reconstruction protocol and fixed choice of the operationally accessible subspace Λ_K. It is precisely relation (25) that is used below as the local variational form of the admissibility condition.

Sectoral realization of the compensation principle. Since the compensating geometric sector is introduced exclusively to eliminate the reconstruction mismatch, at the working order the functional $\mathfrak{E}[g, \psi]$ splits into two contributions:

$$\mathfrak{E}[g, \psi] = S_{\text{eff}}[g, \psi] + S_g[g]. \tag{26}$$

Here S_{eff} is a local integral functional encoding the contribution of the effective degrees of freedom ψ_I, whereas S_g is the functional of the compensating geometric sector, depending only on g. No independent mixed contribution $S_{\text{mix}}[g, \psi]$ arises at the working order: the geometric sector is introduced not as a new independent interaction channel, but precisely as a compensator of the reconstruction mismatch.

At this stage, S_{eff} and S_g are understood as variational functionals realizing the decomposition of the total reconstruction mismatch. Their interpretation as the effective action of the field sector and the gravitational action will be specified below after the minimal structural requirements on the geometric sector have been fixed.

Substituting (26) into (25) yields

$$\delta S_{\text{eff}}[g, \psi] + \delta S_g[g] = 0. \tag{27}$$

Relation (27) is the variational realization of the compensation principle and expresses the exact balancing of the contribution of the effective sector by the geometric compensator within the working order under consideration.

3.4 Local Consistency of Observable Transformations

Definition. Let $p \in \Sigma_s$. Following the distinction introduced earlier between direct and observable transformations (see §2.5), by an *observable transformation* M we mean a linear change of observer chart at the point p, that is, a linear operator $M : T_p\mathcal{U} \to T_p\mathcal{U}$ acting on the coordinate increments $\Delta x^\mu = (\Delta t, \Delta \mathbf{r})$, consistent with $g(p)$, and on the observable fields $\psi_I \mapsto R_I{}^J(M)\psi_J$. The requirement of local consistency consists in preserving the null cone of the metric g in $T_p\mathcal{U}$:

$$g_{\mu\nu}(p)\,\Delta x^\mu \Delta x^\nu = 0 \implies g_{\mu\nu}(p)\,\Delta x'^\mu \Delta x'^\nu = 0, \qquad \Delta x'^\mu := M^\mu{}_\nu\,\Delta x^\nu, \;\; \Delta x \neq 0. \tag{28}$$

In addition, preservation of spatial and temporal orientation is assumed.

Separation from transfer along the foliation. The transformation M acts at a fixed point p and refers to a change of observer chart. Evolution with respect to the foliation parameter s is described separately: the transfer of effective fields is defined by the operator $\mathcal{U}$ from (19), whereas the transfer of tangent vectors between neighboring slices $\Sigma_s \to \Sigma_{s+ds}$ is defined by the operator Π, introduced below. These operations are not identified with M.

Cone consistency under transfer. Let $\Pi^\mu{}_\nu(s+ds, s) : T_p\mathcal{U} \to T_{p_+}\mathcal{U}$ be a linear operator transferring tangent vectors, induced by the procedure of joint reconstruction between neighboring slices. We require preservation of the null cone under transfer:

$$g_{\mu\nu}(\Sigma_s)\,\Delta x^\mu \Delta x^\nu = 0 \;\Rightarrow\; g_{\mu\nu}(\Sigma_{s+ds})\,\Delta x_+^\mu \Delta x_+^\nu = 0, \quad \Delta x_+^\mu := \Pi^\mu{}_\nu(s+ds, s)\,\Delta x^\nu + \mathcal{O}(ds^2). \tag{29}$$

Here Π acts on tangent vectors and *does not coincide* with the operator $\mathcal{U}$ from (19), which acts on effective fields; M acts in $T_p\mathcal{U}$ at fixed p and defines a change of chart, see (28).

Class of admissible observable transformations. Henceforth, by admissible transformations we mean observable transformations M that: (i) preserve the null cone of g in the sense of (28); (ii) preserve spatial orientation ($\det M > 0$); (iii) preserve the chosen temporal orientation; (iv) preserve the time-scale gauge fixed in the SR regime (see §2.6), or equivalently, the normalization of the chosen direction n in the local chart.

In the SR regime (flat foliation, $g = \eta$ locally), such M have the Lorentz form; thus, the set of admissible M is isomorphic to the proper orthochronous component of the Lorentz group $SO^+(1,3)$ *in the sense of local structure.* Here the notation $SO^+(1,3)$ is used as a shorthand, not as a previously established global group classification.

Theorem 3.1 (SR regime for M)**.** *If $g(p) = \eta$ and conditions (i)–(iv) hold, then the set of all admissible M in a neighborhood of the point p coincides with the proper orthochronous Lorentz group $SO^+(1,3)$ with invariant $v_{\max}$ [3, 4, 5].*

Proof. We work at the point p with $g(p) = \eta$. An admissible M is linear on $T_p\mathcal{U}$ and preserves the null cone: $\eta_{\mu\nu}x^\mu x^\nu = 0 \Rightarrow \eta_{\mu\nu}(Mx)^\mu(Mx)^\nu = 0$. Hence $M^\mathsf{T}\eta M = \lambda\,\eta$ for some $\lambda \neq 0$. Condition (iv) (preservation of the gauge) fixes $\lambda = 1$, and therefore $M \in O(1,3)$. Preservation of orientations then implies $M \in SO^+(1,3)$. □

3.5 Acceleration as Foliation Rotation and the Equivalence Principle

In the SR regime, the difference between inertial frames corresponds to a rotation of the foliation direction: the angle between the normals $\mathbf{n}$ and $\mathbf{n}'$ determines the relative velocity (see [1]). In the GR regime, we allow a smooth dependence $n_A = n_A(x)$; this means that the chosen direction of "operational time" changes from point to point and thereby introduces an effective acceleration.

Operational acceleration. Define the projector $P^A{}_B = \delta^A{}_B - n^A n_B$ and the operational acceleration along the integral curve of n^A:

$$a_A := P_A{}^B\,(n^C \bar{\nabla}_C n_B), \tag{30}$$

where $\bar{\nabla}$ is the Levi–Civita connection of the flat metric δ_{AB}. In the SR regime, $n_A = \text{const}$ (in the Euclidean sense, $\bar{\nabla} n = 0$), and therefore $a_A = 0$. When $n_A = n_A(x)$, the quantity a_A is nonzero in general and is interpreted as a local "gravitational" accelerating component in the reconstructed description.

Operational equivalence principle (local formulation). Let p be a point of the reconstructed domain. In the adiabatic regime $\varepsilon = L_{\text{field}}/L_{\text{fol}} \ll 1$, the results of local experiments depend on the reconstructed data only through $g(p)$ and its first derivative (up to $\mathcal{O}(\varepsilon)$). Then there exists a local gauge (a change of observer chart) in which

$$g_{\mu\nu}(p) = \eta_{\mu\nu}, \qquad \partial_\rho g_{\mu\nu}(p) = 0, \tag{31}$$

and, consequently, the local gravitational component can be eliminated to first order. Thus, in a sufficiently small neighborhood of the point p, the results of local experiments are indistinguishable (up to $\mathcal{O}(\varepsilon)$) from the results in a locally accelerated frame in flat space.

Remark 3.2 (Relation to foliation rotation)**.** *Locally, foliation curvature corresponds to a smooth variation of $n^A(x)$; uniform acceleration corresponds to the same mechanism, viewed as a local change of foliation (a rotation of the direction of operational time) in the flat SR limit.*

4 Variational Derivation of the Einstein Equations

In the previous section, the compensation principle was obtained as the variational form of the admissibility condition following from the requirements of causal reconstruction:

$$\delta S_{\text{eff}}[g, \psi] + \delta S_g[g] = 0, \tag{32}$$

where S_{eff} encodes the contribution of the effective degrees of freedom ψ_I, while S_g describes the compensating geometric sector. In the present section, we fix the standard minimal bridge from (32) to the Einstein equations, without reproducing the full textbook derivation.

Status of metric variables in the variational formulation. Here and below, variation with respect to $g_{\mu\nu}$ is understood as variation with respect to the effective metric variables of the reconstructed description. This does not mean that the metric $g_{\mu\nu}$ is introduced as a fundamental independent structure alongside the foliation: in the sense of the model, it arises as a geometric representation of the compensating mechanism induced by deformation of the foliation structure. Accordingly, the metric variational formulation below is used as a closed effective description of the gravitational sector at the level of reconstructed geometry, locally consistent with the SR limit.

4.1 Source as the Metric Variation of the Effective Sector

The present work does not aim to derive the full form of the nongravitational effective action. What is used below is only the fact that, for the class of admissible configurations $\Phi \in \mathcal{S}$ considered here, which allow stable causal reconstruction and a local effective description at the working order, the effective sector admits a local variational representation. In other words, within the decomposition of the mismatch functional $\mathfrak{E}[g, \psi]$ (see §3.3), one introduces a local functional

$$S_{\text{eff}}[g, \psi] = \int d^4x \sqrt{|g|}\, \mathcal{L}_{\text{eff}}(\psi, \nabla\psi; g), \tag{33}$$

whose Euler–Lagrange equations are equivalent to the local evolution (19) in the adiabatic regime and at the working order under consideration.

In the variational formulation of the compensation principle, it is natural to define

$$T^{\text{eff}}_{\mu\nu} := -\frac{2}{\sqrt{|g|}}\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}}, \tag{34}$$

and to interpret $T^{\text{eff}}_{\mu\nu}$ as the effective energy–momentum tensor, that is, as the source of the geometric sector. Such an identification is standard in metric variational schemes and is consistent with the SR limit: in the locally flat regime, $T^{\text{eff}}_{\mu\nu}$ coincides with the symmetrized energy–momentum tensor obtained from the Noether construction (more precisely, with its standard symmetrized form) under the usual assumptions of locality and translational invariance of the effective description.

In the local variational representation of the effective sector, consistent with the reconstructed geometry, S_{eff} inherits the local diffeomorphism invariance of the description. It then follows, on solutions of the equations of motion for ψ, that

$$\nabla^{\mu}T^{\text{eff}}_{\mu\nu} = 0. \tag{35}$$

This is the standard consequence of diffeomorphism invariance of a local metric functional (see, for example, [3, 4]).

4.2 Minimal Choice of the Gravitational Action

Since the reconstructed geometric description does not depend on the choice of its local coordinate parametrization, the geometric sector $S_g[g]$, which represents the same reconstructed structure in metric variables, must be invariant under local diffeomorphisms of the reconstructed description. Thus, diffeomorphism invariance is not postulated here independently, but is inherited from the coordinate nonessentiality of the reconstruction itself. At the same time, in order for $S_g[g]$ to provide the minimal local geometric realization of the compensation principle, we shall use the following structural restrictions:

1. *Locality:* S_g is a local functional of the metric $g_{\mu\nu}$ and a finite number of its derivatives.
2. *Diffeomorphism invariance:* S_g is invariant under local diffeomorphisms of the reconstructed description.
3. *Second-order equations:* variation with respect to $g_{\mu\nu}$ does not lead to equations of order higher than second.
4. *Correct SR limit:* for $K_{ab} = 0$ and $\varepsilon \to 0$, the geometric sector reduces to the flat SR regime.

In four dimensions, these restrictions fix the gravitational sector at the working order. According to Lovelock's theorem, the only local scalar (up to a total divergence) leading to second-order equations is $R[g]$ [2] (see also the standard discussions in [3, 4]).

In addition, let us allow the local diffeomorphism-invariant term without derivatives, $\sqrt{|g|}$, corresponding to a cosmological term. Its presence is compatible with the structural requirements listed above; however, the derivation (or estimate) of Λ from more detailed considerations of the model lies beyond the scope of the present paper. In what follows, Λ is treated as an effective parameter.

Accordingly, we adopt the gravitational action in the minimal form

$$S_g[g] = \frac{1}{16\pi G} \int d^4x \, \sqrt{|g|} \left(R[g] - 2\Lambda \right) \; + \; S_{\partial\Omega}, \tag{36}$$

where $S_{\partial\Omega}$[1] is the Gibbons–Hawking–York boundary term, required for a well-posed variational formulation with the metric fixed on the boundary $\partial\Omega$ [3].

4.3 The Einstein Equations as a Consequence of Compensation

Here and below, variations with respect to $g^{\mu\nu}$ are considered in the effective metric variables of the reconstructed description. In other words, the metric variational formulation is used here as an effective geometric representation of the compensating mechanism arising from curvature of the foliation structure.

Consider variations with respect to $g^{\mu\nu}$ at fixed boundary data. From (32) we obtain

$$\frac{\delta S_g}{\delta g^{\mu\nu}} = -\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}}. \tag{37}$$

Variation of (36) gives the standard result

$$\frac{2}{\sqrt{|g|}} \frac{\delta S_g}{\delta g^{\mu\nu}} = \frac{1}{8\pi G} \left(G_{\mu\nu}[g] + \Lambda g_{\mu\nu} \right), \tag{38}$$

---

[1] The boundary term $S_{\partial\Omega}$ is required for a well-posed variational formulation of the geometric sector with the metric fixed on $\partial\Omega$: without it, the variation of the Einstein–Hilbert action contains uncompensated boundary contributions depending on the normal derivatives of $\delta g_{\mu\nu}$. In the context of the present work, this means that the variational realization of the compensation principle in the form $\delta S_{\text{eff}} + \delta S_g = 0$ requires inclusion of the standard Gibbons–Hawking–York term.

after which, using definition (34), it follows from (37) that

$$G_{\mu\nu}[g] + \Lambda g_{\mu\nu} = 8\pi G \, T^{\mathrm{eff}}_{\mu\nu}. \tag{39}$$

Thus, the Einstein equations arise here as the local variational realization of the compensation principle: the contribution of the effective sector to the reconstruction mismatch is balanced by the geometric compensator, written in the metric variables of the reconstructed description. At the same time, (39) refers to the effective spacetime geometry, rather than to the fundamental Euclidean space $\mathbb{E}^4$.

5 Hierarchy of Scales, Weak Field, and the Operational Resolution of the Hierarchy Problem

This section formalizes the *operational origin of the hierarchy of scales* $L_{\mathrm{field}} \ll L_{\mathrm{fol}}$ (and, accordingly, of the weak field in observable regions). The key claim is that the small quantity

$$\varepsilon := \frac{L_{\mathrm{field}}}{L_{\mathrm{fol}}}$$

is not introduced as an additional assumption, but *follows* from the requirements of causal reconstruction: the joint cone consistency of observable transformations at a point $p \in \Sigma_s$ and the consistency of the cone under transfer between neighboring slices (see §3.4). In the adiabatic regime $\varepsilon \ll 1$, the geometry varies slowly, whereas the effective fields on the slices have shorter characteristic scales; the leading order reproduces the Newtonian limit, while the corrections are suppressed by powers of ε.

5.1 Two-Scale Separation

Definitions of scales. Let $\Omega \subset \Sigma_s$ be a compact region with smooth boundary. We define the foliation curvature scale L_{fol} through local geometric invariants on Ω:

$$L_{\mathrm{fol}}^{-2} \;\sim\; \max\Big\{ \, \|K_{ab}K^{ab}\| \; , \; |R^{(3)}[h]| \, \Big\}, \tag{40}$$

where the norms are induced by h_{ab} on Ω, $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$, and $R^{(3)}[h]$ is the scalar curvature of h.

We define the scale of variation of the effective fields L_{field} on the operationally accessible subspace of excitations admissible under the reconstruction protocol (in particular, after restriction to Λ_K). Set

$$L_{\text{field}}^{-2} \sim \max_{\psi \in \mathcal{H}_{\text{acc}}} \frac{\int_\Omega \sqrt{h}\, |D\psi|^2}{\int_\Omega \sqrt{h}\, \psi^2}, \tag{41}$$

where D is the covariant derivative on Σ_s, and $\mathcal{H}_{\text{acc}}$ is the class (subspace) of excitable modes of the effective fields that are operationally accessible on Ω for a fixed reconstruction protocol. Then, at the working order,

$$D\psi = \mathcal{O}(L_{\text{field}}^{-1}), \qquad D^2\psi = \mathcal{O}(L_{\text{field}}^{-2}), \qquad D^2 := h^{ab} D_a D_b. \tag{42}$$

Order counting. Introduce the small parameter

$$\varepsilon := \frac{L_{\text{field}}}{L_{\text{fol}}} \ll 1. \tag{43}$$

In the adiabatic regime (43), we have

$$K_{ab} = \mathcal{O}(L_{\text{fol}}^{-1}), \quad D_c K_{ab} = \mathcal{O}(L_{\text{fol}}^{-2}), \quad R^{(3)}[h] = \mathcal{O}(L_{\text{fol}}^{-2}), \tag{44}$$

while for the effective fields ψ the estimates (42) hold. Accordingly, the densities of local contributions scale as

$$\sqrt{|g|}\, R[g] \;\sim\; \mathcal{O}(L_{\text{fol}}^{-2}), \qquad \sqrt{|g|}\, \mathcal{L}_{\text{eff}} \;\sim\; \mathcal{O}(L_{\text{field}}^{-2}), \tag{45}$$

and corrections containing additional derivatives or additional geometric invariants of the foliation are suppressed by powers of ε.

Adiabatic expansion. At a fixed observational scale $L_{\text{obs}} \ll L_{\text{fol}}$, the fields and functionals admit an expansion in ε:

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \varepsilon\, g_{\mu\nu}^{(1)} + \varepsilon^2 g_{\mu\nu}^{(2)} + \cdots, \qquad \psi = \psi^{(0)} + \varepsilon\, \psi^{(1)} + \cdots,$$

where $g^{(0)}$ is a locally flat metric on scales of order L_{field}, consistent with §3.4, while the corrections $g^{(k)}$ vary on the scale L_{fol}.

Remark 5.1 (On the absence of an independent mixed sector)**.** *In the present model, no independent mixed functional $S_{\text{mix}}[g, \psi]$ is introduced (see §3.3). Therefore, in the adiabatic expansion, only the contribution of the geometric sector $S_g[g]$ and the contribution of the effective sector $S_{\text{eff}}[g, \psi]$ are taken into account; any additional terms excluded by the structure of the construction are not considered.*

5.2 Structural Origin of the Hierarchy

Operational consistency. Let L_{obs} be the characteristic spatial scale of a local measurement procedure in a neighborhood of a point $p \in \Sigma_s$ (detector aperture, interferometer baseline, etc.). The requirements of preservation of the null cone under a change of observer chart at the point p and under transfer between neighboring slices (see (28), (29), and §3.4) mean that, on the scale L_{obs}, the foliation structure must not significantly alter the local cone geometry. This leads to the conditions

$$|K|\, L_{\mathrm{obs}} \ll 1, \qquad |DK|\, L_{\mathrm{obs}}^2 \ll 1, \tag{46}$$

where $|K|$ and $|DK|$ denote characteristic order-of-magnitude values of the tensor of extrinsic curvature $K_{ab} = \frac{1}{2}\mathcal{L}_n h_{ab}$ and of its covariant derivative $D_c K_{ab}$ on the reconstruction domain under consideration. Thus, on the observational scale, the curvature of the foliation and its spatial variation remain small, so that local reconstruction preserves an SR-compatible causal structure. Regularity $n_A, h_{ab} \in C^2$ on the domain under consideration is assumed.

For the class of local reconstruction protocols considered here, the observational scale L_{obs} is of the same order of magnitude as the characteristic scale of the resolvable effective excitations, L_{field}. In other words, reconstruction is based precisely on such local procedures as are sensitive to the operationally accessible modes at their own characteristic scale. Since L_{fol} sets the characteristic scale of variation of the foliation structure, conditions (46) imply

$$\varepsilon := \frac{L_{\mathrm{field}}}{L_{\mathrm{fol}}} \ll 1, \tag{47}$$

that is, the adiabatic regime arises here as a consequence of operational consistency for the class of observational protocols considered, rather than as an external tuning of parameters (see also the upper bound (51)).

Physical content. In the regime (47):

1. the geometric sector defines a slowly varying (quasistatic on scales L_{field}) reconstructed metric g;
2. the effective fields ψ_I evolve on the background g in the adiabatic approximation, and their contribution to compensation is encoded by the source $T^{\mathrm{eff}}_{\mu\nu}$ after the variational formulation is introduced;
3. the observable transformations M realize local SR kinematics with corrections of order $\mathcal{O}(L_{\mathrm{obs}}^2/L_{\mathrm{fol}}^2)$.

Summary on the origin of the adiabatic regime. Let us emphasize that, in the present work, the condition $\varepsilon \ll 1$ is not introduced as an external small-parameter assumption. Rather, it follows from the very requirement of admissibility of causal reconstruction. Indeed, if local cone consistency is preserved on the observational scale L_{obs}, both under a change of observer chart and under transfer between neighboring slices, then the foliation structure must vary slowly on that scale, which yields the conditions (46). For the class of local reconstruction protocols considered here, L_{obs} is of the same order as L_{field}, since the observer is sensitive precisely to the operationally accessible modes on their characteristic spatial scale. It follows directly that the hierarchy $L_{\mathrm{field}} \ll L_{\mathrm{fol}}$ holds, that is,

$$\varepsilon = \frac{L_{\mathrm{field}}}{L_{\mathrm{fol}}} \ll 1.$$

It is in this sense that the adiabatic regime is not an additional postulate, but a structural consequence of the requirements of causal reconstruction and, accordingly, of the admissibility condition for configurations $\Phi \in \mathcal{S}$.

5.3 Weak Field and the Newtonian Limit

In this subsection, we merely fix the standard Newtonian limit of the Einstein equations (39) in the notation of the present work; for detailed calculations, see, for example, [3, 6, 7].

Weak field and quasistaticity. Let

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \qquad |\gamma_{\mu\nu}| \ll 1,$$

and choose the harmonic gauge

$$\partial^\mu \bar{\gamma}_{\mu\nu} = 0, \qquad \bar{\gamma}_{\mu\nu} := \gamma_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\gamma, \quad \gamma := \eta^{\alpha\beta}\gamma_{\alpha\beta}.$$

In the Newtonian regime, we assume slow sources $|\mathbf{v}| \ll v_{\mathrm{max}}$, quasistaticity $\partial_0 \gamma_{\mu\nu} \approx 0$, weak field $|\phi|/v_{\mathrm{max}}^2 \ll 1$, and low pressure $p \ll \rho_{\mathrm{eff}} v_{\mathrm{max}}^2$. We use the standard parametrization

$$g_{00} = 1 + \frac{2\phi}{v_{\mathrm{max}}^2}, \qquad g_{0i} = 0, \qquad g_{ij} = -\Big(1 - \frac{2\phi}{v_{\mathrm{max}}^2}\Big)\delta_{ij}, \tag{48}$$

where $i, j = 1, 2, 3$, and δ_{ij} is the Euclidean metric on a locally chosen spatial chart.

Newtonian limit. It then follows from (39) that the Poisson equation holds:

$$\nabla^2\phi = 4\pi G\,\rho_{\text{eff}}, \qquad \rho_{\text{eff}} := T^{\text{eff}}_{00}, \qquad \nabla^2 := \delta^{ij}\partial_i\partial_j, \tag{49}$$

which reproduces the Newtonian limit in the adiabatic regime.

Estimate of corrections. Corrections to (49), due to the finite curvature of the foliation and slow nonstationarity, are suppressed by the hierarchy of scales. At the level of orders of magnitude,

$$\frac{|\delta(\nabla^2\phi)|}{|\nabla^2\phi|} = \mathcal{O}\!\left(\frac{L^2_{\text{obs}}}{L^2_{\text{fol}}}\right) + \mathcal{O}\!\left(\frac{T^2_{\text{obs}}}{L^2_{T}}\right), \tag{50}$$

where L_{obs} and T_{obs} are the spatial scale and duration of the experiment, and L_T is the characteristic timescale of variation of the metric.

Status of the leading order. Let us emphasize that, in the admissible adiabatic regime, the leading order of the effective geometric dynamics is governed by the Einstein equations (39). Accordingly, at this order the reconstructed geometric description reproduces the GR regime, whereas deviations in observable regions are suppressed by powers of the hierarchy parameter ε and, in particular, by corrections of order $\mathcal{O}(L^2_{\text{obs}}/L^2_{\text{fol}})$.

5.4 Exclusion of Strong Fields in Observable Regions

The requirements of preservation of the null cone under a change of observer chart at a point $p \in \Sigma_s$ and under transfer between neighboring slices, (28), (29), for a fixed class of transfer procedures and observations on a region $\Omega \subset \Sigma_s$, imply the existence of an upper bound on the hierarchy parameter:

$$\varepsilon \le \varepsilon_{\max}(\delta_{\text{cone}}, C_{\text{reg}}) \ll 1, \tag{51}$$

where $\varepsilon := L_{\text{field}}/L_{\text{fol}}$, δ_{cone} is the admissible cone-mismatch threshold, and C_{reg} collects the regularity constants for $n_A, h_{ab} \in C^2$ and the corresponding norms $\|K_{ab}\|_{L^\infty(\Omega)}$, $\|D_c K_{ab}\|_{L^\infty(\Omega)}$ (the norms are taken with respect to h_{ab}). The bound (51) is local (within the causally accessible region and for a fixed protocol), rather than a universal constant of nature.

Consequently, regimes with $\varepsilon = \mathcal{O}(1)$ are incompatible with the requirements of causal reconstruction in observable regions (and therefore do not belong to the admissible class $\mathcal{S}$ for a fixed protocol): all corrections to SR/GR kinematics are suppressed by at least $\mathcal{O}(\varepsilon^2)$ when $L_{\text{obs}} \sim L_{\text{field}}$ (see §5.1).

Falsifiability. The observation of reproducible effects requiring $\varepsilon \sim 1$ in a causally accessible region under controlled transfer would contradict the model (in the sense of the requirements of causal reconstruction for the chosen class of protocols).

6 Discussion and Outlook

Main results. It has been shown that

- the Einstein–Hilbert action (36) and the equations (39) arise as the variational realization of the compensation principle following from the requirements of causal reconstruction, under minimal structural assumptions (locality, diffeomorphism invariance, second order, and the correct SR limit);
- Lorentz kinematics for observable transformations M is recovered rigorously in the SR regime (flat foliation and locally $g = \eta$), in agreement with the results of [1];
- the hierarchy of scales $L_{\text{field}} \ll L_{\text{fol}}$ arises as a *consequence* of operational consistency (cone consistency at a point and under transfer) and ensures the Newtonian limit;
- in observable regions there exists an operational upper bound on the hierarchy parameter ε of the form $\varepsilon \leq \varepsilon_{\max} \ll 1$ (see (46), (47), (51)), which renders corrections to the SR/GR regime scale-small and ensures falsifiability;
- the source $T^{\text{eff}}_{\mu\nu}$ is emergent (being defined through the metric variation of the effective sector) and is covariantly conserved on solutions of the equations of motion for ψ: $\nabla^{\mu} T^{\text{eff}}_{\mu\nu} = 0$.

Interpretation. The compensation principle of §3.3 provides the condition $\delta S_{\text{eff}} + \delta S_g = 0$ for admissible variations of the reconstructed data (g, ψ). Thus, $G_{\mu\nu}$ is interpreted as the variational response of the compensating geometric sector to the contribution of the effective degrees of freedom, required for the fulfillment of the requirements of causal reconstruction in the presence of the procedure of joint reconstruction/transfer between neighboring slices. Curvature is associated not with fundamental "matter," but with the condition for the existence of stable operational reconstruction. The energy and momentum of the effective fields arise as Noether charges in the SR regime and are continued covariantly through $T^{\text{eff}}_{\mu\nu}$ in the GR regime.

Domain of applicability and the operational hierarchy. The emergent geometry g is applicable in the regime

$$\varepsilon := \frac{L_{\rm field}}{L_{\rm fol}} \ll 1, \qquad \|K_{ab}\|\, L_{\rm field} \ll 1, \quad \|D_c K_{ab}\|\, L_{\rm field}^2 \ll 1,$$

in which operational cone consistency is preserved both under a change of observer chart at a point and under transfer between neighboring slices (see (28), (29)).

Regimes in which ε approaches the local bound $\varepsilon_{\rm max} < 1$ in (51) mark the limit of applicability of the effective description in terms of g. Beyond this bound, causal reconstruction ceases to be stable, so that the emergent metric g and the associated foliation-based description lose operational meaning; at the fundamental level, only the description in terms of the field Φ remains.

Strong fields: black holes. Solutions of (39) with a horizon describe regions of strong curvature of the effective metric g. Singularities of curvature invariants (for example, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$) pertain to the effective geometry g, whereas the fundamental field Φ satisfies (1) by construction.

It is important, however, that the applicability of the description in terms of g is operational: it is limited to the region where stable causal reconstruction is preserved (in particular, where $\varepsilon \ll 1$ and the conditions of cone consistency hold). If, in some region (near strong-field regimes), these conditions are violated, then the emergent metric g and the associated effective description lose operational meaning, and at the fundamental level only the description in terms of Φ remains.

Strong fields: cosmology. For homogeneous-isotropic foliations, the characteristic scale is $L_{\rm fol} \sim H^{-1}$; when extrapolated back to earlier epochs, $\varepsilon(s)$ increases and may approach the operational bound $\varepsilon_{\rm max} < 1$. In this sense, the FRW singularity is interpreted as a limit of applicability of the emergent effective theory g, rather than as a statement about the behavior of the fundamental field Φ. The effective description is valid in late regimes with $\varepsilon \ll 1$.

Possible origin of the cosmological constant. Within the present framework, the cosmological term Λ may be interpreted as an effective parameter associated not with the fundamental vacuum, but with the boundedness of causal reconstruction for the elliptic Laplace equation. Since the observer receives data only in Ω_0 and reconstructs events/geometry in the reconstruction domain Ω, the stability of the inverse problem deteriorates as the reconstruction distance increases, which leads to a spectral cutoff and a finite horizon of stable reconstruction. At the level of the gravitational sector, this means that the large-scale part of the mismatch functional naturally contains a zeroth-order contribution in derivatives, encoding the deficit of reconstructible information on large scales. The only local diffeomorphism-invariant structure of this type is $\int \sqrt{|g|}$, which corresponds precisely to the cosmological term. The specific relation of Λ to the parameters of the reconstruction protocol requires a separate quantitative analysis and is not considered here.

Testable consequences. Corrections to the Newtonian limit are suppressed as $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$ for experiments of scale L_{obs}. This makes it possible to extract (or constrain) ε and $\varepsilon_{\max}$ from data on weak lensing, binary-system dynamics, time delays, Einstein rings, observations by pulsar timing arrays (PTA), and gravitational-wave detectors.

Quasi-charges and boundary fluxes, in the presence of a variational formulation of the effective sector, are expressed in the standard way through $J^\mu[\xi] = T^{\text{eff}\,\mu}{}_\nu \xi^\nu$ under prescribed conditions on the boundary of the integration region.

Universality. In the present construction, the reconstructed metric $g_{\mu\nu}$ is common to all operationally accessible degrees of freedom; any observable field-dependent splitting of cones/metrics falsifies the model.

Equivalence principle. In GR, local inertiality is formulated as a principle; in the present framework, it follows from the compensation construction and from the geometric interpretation of foliation curvature as a locally eliminable rotation on scales $L_{\text{obs}} \ll L_{\text{fol}}$ (see §3.5).

Relation to alternative approaches. The proposed construction has points of contact with several well-known directions in the foundations of quantum theory and quantum gravity, but differs from them in its underlying formulation. In particular, it differs from entropic and induced schemes in that the geometric sector is derived from conditions of operational consistency (through the compensation principle), rather than from statistical assumptions or coarse-graining; moreover, the SR regime for observable transformations M is preserved exactly in the corresponding limit.

What the present work shares with relational approaches, in particular Rovelli's program, is the rejection of an a priori global event structure and the emphasis on operationally defined relations; however, the starting point here is not a quantum state depending on the relation between systems, but a timeless Euclidean configuration of the fundamental field, from which the causal and metric structure are reconstructed by a localized observer [8]. What this scheme shares with Bohmian approaches and other "timeless" formulations of quantum gravity is the interest in the problem of time and in the non-fundamental status of the standard spacetime picture; however, the present work introduces neither a wave function, nor guiding dynamics, nor a quantum law of evolution [9]. There is only a limited similarity to programs such as causal dynamical triangulations and Euclidean path-integral approaches, namely that Lorentzian geometry is not taken as fundamentally given; unlike those approaches, however, the present work does not consider a sum over geometries or a quantum amplitude construction, but instead employs a single minimal Euclidean model with operational reconstruction of causality and geometry [10, 11].

Thus, the proposed approach occupies a distinct position: it starts from a timeless Euclidean model with a single fundamental field and shows how effective spacetime geometry and gravitational dynamics may arise from the requirements of admissible causal reconstruction, rather than from a quantum superposition of geometries, from statistical assumptions, or from a quantum structure introduced in advance.

Difference in nature between gravity and effective fields. In the proposed construction, gravity and the other effective fields have fundamentally different status. The effective fields ψ_I arise as operational degrees of freedom on reconstructed slices and are described by the sector $S_{\mathrm{eff}}[g, \psi]$, whereas the gravitational sector appears as a compensating geometric structure required for the fulfillment of the requirements of causal reconstruction. In this sense, gravity is not "just another field" of the same type as ψ_I, but is responsible for the consistency of the reconstructed geometry itself. This asymmetry of nature may indicate why standard programs of quantizing gravity, constructed by analogy with the quantization of ordinary fields, encounter fundamental difficulties. A rigorous analysis of this question requires separate consideration and lies beyond the scope of the present paper.

7 Conclusion

This work has shown that, in a timeless Euclidean model with a single fundamental field Φ and a reconstructed foliation structure $(\Sigma_s, h_{ab}, K_{ab})$, the gravitational sector arises not as an independent postulate, but as a consequence of the requirements of causal reconstruction. Under minimal structural assumptions (locality, diffeomorphism invariance, second order in derivatives, and the correct SR limit), the compensation principle

$$\delta S_{\text{eff}} + \delta S_g = 0$$

leads to the Einstein–Hilbert action (36) and, consequently, to the Einstein equations (39). Thus, the reconstructed metric g acquires dynamics compatible with the requirements of operational consistency.

The key conceptual result is that the basic principles of General Relativity are not postulated in the present framework, but derived. The compensation principle arises as the variational form of the admissibility condition following from the requirements of causal reconstruction. The operational form of the equivalence principle arises as a consequence of the local eliminability of the gravitational component associated with foliation curvature in the adiabatic regime (see §3.5). In the SR regime (flat foliation and locally $g = \eta$), the kinematics of observable transformations M takes a strictly Lorentz form, while under slow variation of the foliation the Newtonian limit is recovered with corrections of order $\mathcal{O}(L_{\text{obs}}^2/L_{\text{fol}}^2)$.

Another essential result is that the hierarchy of scales

$$L_{\text{field}} \ll L_{\text{fol}}$$

is not introduced as an external assumption, but follows from the requirements of cone consistency under a change of observer chart and under transfer between neighboring slices. Thus, the weak-field regime and the applicability of the emergent geometry g receive an operational justification. In observable regions, this leads to the local upper bound

$$\varepsilon = \frac{L_{\text{field}}}{L_{\text{fol}}} \leq \varepsilon_{\max} \ll 1,$$

so that regimes with $\varepsilon = \mathcal{O}(1)$ do not belong to the admissible class of reconstructions for a fixed protocol. In this sense, the strong-field regime marks the boundary of applicability of the effective description in terms of g, rather than a fundamental breakdown at the level of the field Φ.

The effective degrees of freedom ψ_I are defined as operational fields on the slices and admit a local variational realization at the working order. The corresponding source of the geometric sector is defined by the metric variation of the effective action,

$$T_{\mu\nu}^{\text{eff}} := -\frac{2}{\sqrt{|g|}} \frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}},$$

and is covariantly conserved on solutions of the equations of motion of the effective sector. Thus, the energy and momentum of the effective fields arise as emergent quantities: in the SR limit, they are consistent with Noether charges, while in the GR regime they are continued in covariant form through $T^{\mathrm{eff}}_{\mu\nu}$.

The proposed framework remains falsifiable. Any observable necessity of regimes with $\varepsilon \sim 1$ in causally accessible regions under controlled transfer, stable violations of cone consistency, or field-dependent splitting of the reconstructed geometric structure would contradict the basic construction of the model. Thus, the work provides not only an internally consistent, but also a fundamentally testable operational reconstruction of gravity.

Among the open questions remain, in particular, the origin and estimation of the cosmological constant Λ within the full model, as well as the limits of applicability of reconstruction: as $\varepsilon \to \varepsilon_{\max}$, the foliation-based description and the metric g lose operational meaning, and only the fundamental description in terms of Φ remains. These directions constitute a natural continuation of the present analysis.

References

[1] A. N. Smirnov, "Special Relativity as an Emergent Structure in a Timeless Euclidean Model," *International Journal of Quantum Foundations* **12** (2026), no. 2, 272–312. Available at: `https://ijqf.org/archives/8065`

[2] D. Lovelock, "The Einstein tensor and its generalizations," *Journal of Mathematical Physics* **12** (1971), 498–501.

[3] R. M. Wald, *General Relativity*, University of Chicago Press, Chicago (1984).

[4] S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity*, Addison-Wesley, San Francisco (2004).

[5] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, W. H. Freeman, San Francisco (1973).

[6] E. Poisson, *A Relativist's Toolkit*, Cambridge University Press, Cambridge (2004).

[7] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, New York (1972).

[8] C. Rovelli, "Relational Quantum Mechanics," *Int. J. Theor. Phys.* **35** (1996) 1637–1678. arXiv:quant-ph/9609002.

[9] S. Goldstein and S. Teufel, "Quantum Spacetime without Observers: Ontological Clarity and the Conceptual Foundations of Quantum Gravity," arXiv:quant-ph/9902018.

[10] J. Ambjørn, J. Jurkiewicz and R. Loll, "Causal Dynamical Triangulations and the Quest for Quantum Gravity," arXiv:1004.0352 [hep-th].

[11] J. B. Hartle and S. W. Hawking, "Wave Function of the Universe," *Phys. Rev. D* **28** (1983) 2960–2975.