On the Positive Integer Solutions and Uniqueness of an Iterative Equation

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Abstract: This paper studies the positive integer solutions of an exponential Diophantine equation of the following form:

$$\frac{3^aX+3^{a-1}+3^{a-2}\times2^{b_1}+3^{a-3}\times2^{b_1+b_2}\dots+3^{a-a}\times2^{b_1+b_2+\cdots+b_{a-1}}}{2^{b_1+b_2+\cdots+b_a}}=\text{X},$$

where a,X,$b_i$are positive integers and$b_i \geq 1$. This equation arises from the analysis of the cyclic structure of a specific iterative sequence. Through rigorous algebraic derivation and inequality analysis, this paper proves that for any given $a \geq 1$ and all sequences of b_i satisfying the conditions, the only positive integer solution to this equation is $X = 1$. This conclusion reveals the uniqueness of the solution for this class of equations.

Keywords: Iterative Sequence; Cyclic Structure; Number Theory; Diophantine Equation

Define the following two transformations (X is any positive integer):

① If X is even, its next step is $\frac{X}{2^{b_1}} =$ an odd number.

② If X is odd, its next step is $\frac{3X+1}{2^{b_1}} =$ an odd number.

Apply these transformations successively (all variables in this paper are positive integers). If a number transforms into itself after a series of steps, we say X forms a cycle.

The reason X forms a cycle is that when an odd number X transforms into X itself, it creates a cycle. For example, suppose X transforms into itself after one step, i.e., the transformation is $X \rightarrow X$. The equation is $\frac{3X+1}{2^{b_1}} = X$.

Based on this idea, consider a cycle of the form: $X \rightarrow$ Odd① $\rightarrow$ Odd② $\rightarrow$ Odd③ $\rightarrow$ Odd④ ……$\rightarrow X$, where X is the starting number and returns to itself after a transformations (a is a positive integer). Theoretically, one needs to write out the equation for every $a \geq 1$ and analyze whether the positive integer solution for X is only 1. However, since there are infinitely many positive integers, it is impossible to enumerate them all. Therefore, it is necessary to use a to derive a general expression for these equations (X is the starting number of the cycle).

First, write the equations for $a \leq 3$ to observe the structure:

For $a = 1$, the transformation is $X \to X$, and the equation is $\frac{3X+1}{2^{b_1}} = X$.

For $a = 2$, the transformation is $X \to$Odd①$\to X$. The derivation process is:

$\frac{3X+1}{2^{b_1}} =$ Odd①, $\frac{\frac{3X+1}{2^{b_1}}\times 3+1}{2^{b_2}} = X$,resulting in the equation: $\frac{3^2X+3+2^{b_1}}{2^{b_1+b_2}} = X$.

For $a = 3$, the transformation is $X \to$ Odd① $\to$ Odd② $\to X$. Following the derivation for $a = 2$, the final equation is: $\frac{3^3X+3^2+3\times 2^{b_1}+2^{b_1+b_2}}{2^{b_1+b_2+b_3}} = X$.

By observation, these equations share common characteristics. According to the definition, the equation for a transformations (a ≥1) is:

$$\frac{3\times\frac{3\times\frac{3\times\frac{3X+1}{2^{b_1}}+1}{2^{b_2}}+1}{2^{b_3}}+1}{\frac{\cdots}{2^{b_a}}}+1 = X$$

It is required to prove that the only positive integer solution to this equation is 1. However, this formula involves many variables and cannot be proven by exhaustion. This equation can first be transformed into:

$$\frac{3^aX+3^{a-1}+3^{a-2}\times 2^{b_1}+3^{a-3}\times 2^{b_1+b_2}\ldots+3^{a-a}\times 2^{b_1+b_2+\cdots+b_{a-1}}}{2^{b_1+b_2+\cdots+b_a}} = \text{X},$$

The numerator is then converted into another form. For $a \geq 3$, note that the last two terms of the numerator $3^1 \times 2^{b_1+b_2+\cdots+b_{a-2}} + 2^{b_1+b_2+\cdots+b_{a-1}}$can be converted into

$$2^{b_1+b_2+\cdots b_{a-2}} \times (2^{b_{a-1}} + 3)$$

Continuing forward, i.e.:

$$3^2 \times 2^{b_1+b_2+\cdots+b_{a-3}}+2^{b_1+b_2+\cdots b_{a-2}} \times (2^{b_{a-1}} + 3)$$

this can be converted into$2^{b_1+b_2+\cdots\ldots b_{a-3}} \times (2^{b_{a-2}} \times (2^{b_{a-1}} + 3) + 3^2)$. Continuing this accumulation, the final numerator is:

$$2^{b_1} \times (\ldots(2^{b_{a-3}} \times (2^{b_{a-2}} \times (2^{b_{a-1}} + 3) + 3^2) + 3^3)\ldots) + 3^{a-1}$$

Finally, the equation becomes:

$$\frac{2^{b_1}\times(\ldots(2^{b_{a-3}}\times(2^{b_{a-2}}\times(2^{b_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-1}}{2^{b_1+b_2+\cdots+b_a}-3^a} = \text{X}.$$

The transformed equation is:

$\cfrac{\cfrac{\cfrac{\frac{(2^{b_1+b_2+\cdots+b_a}-3^a)X-3^{a-1}}{2^{B_1}}-3^{a-2}}{2^{B_2}}-3^{a-3}}{\cdots}\cdots}{2^{B_{a-2}}}-3^1=2^{B_{a-1}}$ $(2^{b_1}=2^{B_1},\ 2^{b_2}=2^{B_2},\ \cdots 2^{b_{a-1}}=2^{B_{a-1}})$, $(2^{b_1+b_2+\cdots+b_a}-3^a)X-3^{a-1}$ can be transformed into $2^{b_1+b_2+\cdots+b_a}X-3^{a-1}(3X+1)$.thus, the equation is: (valid for a $a\geq 3$)

$$\cfrac{\cfrac{\cfrac{\cfrac{2^{b_1+b_2+\cdots+b_a}X-3^{a-1}(3X+1)}{2^{B_1}}-3^{a-2}}{2^{B_2}}-3^{a-3}}{\cdots}\cdots}{2^{B_{a-2}}}-3^1=2^{B_{a-1}}$$

From this equation, it can be understood that if the only positive integer solution for X is 1, then when the positive integer solution for X is not only 1,

$2^{B_1+B_2+\cdots\ldots+B_{a-1}}>2^{b_1+b_2+\cdots b_{a-1}}$, $(2^{B_1+B_2+\cdots\ldots+B_{a-1}}\neq 2^{b_1+b_2+\cdots b_{a-1}})$.

If $2^{B_1+B_2+\cdots\ldots+B_{a-1}}<2^{b_1+b_2+\cdots b_{a-1}}$, since 2^{b_a} can be adjusted, it is equivalent to "=" ($2^{B_1+B_2+\cdots\ldots+B_{a-1}}$is a positive integer). For example: when ($a=3$), the equation is

$\frac{(2^{b_1+b_2+b_3})3-3^2(3\times 3+1)}{2^{B_1}}-3=2^{B_2}$,set $2^{b_1+b_2+b_3}=32$, then the result is $\frac{6}{2^1}-3=0$, 0 is not a positive integer.

Set $2^{b_1+b_2+\cdots b_{a-1}}=K$, $2^{B_1+B_2+\cdots\ldots+B_{a-1}}=Q$.then transform the equation into:

$\frac{(2^{b_1+b_2+\cdots+b_a})X-3^{a-1}\times(3X+1)}{2^{B_1}}=2^{B_2}\times(\ldots(2^{B_{a-3}}\times(2^{B_{a-2}}\times(2^{B_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-2}$,$(2^{B_1}=\frac{2^Q}{2^{B_2+B_3+\cdots+B_{a-1}}})$

Substitute Q and K:

$\frac{K\times 2^aX-3^{a-1}\times(3X+1)}{2^Q}\times 2^{B_2+B_3+\cdots+B_{a-1}}=2^{B_2}\times(\ldots(2^{B_{a-3}}\times(2^{B_{a-2}}\times(2^{B_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-2}$,

Transform into:

$\frac{2^aX}{2^{Q-K}}-\frac{3^{a-1}\times(3X+1)}{2^Q}=\frac{2^{B_2}\times(\ldots(2^{B_{a-3}}\times(2^{B_{a-2}}\times(2^{B_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-2}}{2^{B_2+B_3+\cdots+B_{a-1}}}$,

reduction of fractions to a common denominator:

$$\frac{2^{B_2+B_3+\cdots+B_{a-1}+b_a}\times X}{2^{Q+B_2+B_3+\cdots+B_{a-1}-K}}-\frac{2^{B_2+B_3+\cdots+B_{a-1}-K}\times 3^{a-1}(3X+1)}{2^{Q+B_2+B_3+\cdots+B_{a-1}-K}}$$
$$=\frac{2^{Q-K}\times(2^{B_2}\times(\ldots(2^{B_{a-3}}\times(2^{B_{a-2}}\times(2^{B_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-2})}{2^{Q+B_2+B_3+\cdots+B_{a-1}-K}}$$

After reducing to the lowest common denominator, we can directly analyze the numerator:

$2^{B_2+B_3+\cdots+B_{a-1}+b_a}X-2^{B_2+B_3+\cdots+B_{a-1}-K}\times 3^{a-1}(3X+1)=2^{Q-K}\times(2^{B_2}\times(\ldots(2^{B_{a-3}}\times(2^{B_{a-2}}\times(2^{B_{a-1}}+3)+3^2)+3^3)\ldots)+3^{a-2})$,

When $Q=K$ and $3X+1\neq 2^{B_1}$, it is equivalent to: even - even ≠ odd, indicating that X has no positive integer solution. But when $Q=K$, the subtrahend can be written as:

$\frac{3^{a-1}\times(3X+1)}{2^{B_1}}$It is an odd number only when$3X+1=2^{B_1}$, which is equivalent to even - odd = odd. But why don't $X=5,21,85$ ... form cycles?

This is because when $3X+1=2^{B_1}$, the transformation of X is:

$X\to 1\to 1\to 1$ ...so only when $X=1$ does it form a cycle. This proves that for $a\geq 3$, the only positive integer solution for X is 1. It remains to prove that for $a\leq 2$, the only positive integer solution for X is 1.

For $a=1$, the equation is $\frac{3X+1}{2^{b_1}}=X$, which simplifies to $(2^{b_1}\text{-}3)X=1$, and further to $\frac{1}{2^b-3}=X$. It is known that X is a positive odd number, the only positive integer solution is $X=1, 2^{b_1}=4$.

For $a=2$, the equation is$\frac{3^2X+3+2^{b_1}}{2^{b_1+b_2}}=X$, which simplifies to $\frac{3+2^{b_1}}{2^{b_1+b_2}-3^2}=\mathrm{X}$, and further to $\frac{3+2^{b_1}}{X}=2^{b_1+b_2}-3^2$. First, set $X=1$. If 2^{b_1} is the largest in $2^{b_1+b_2}$ at this point, $\frac{3+2^{b_1}}{1}<2^{b_1+b_2}-3^2$ then for $X>1$ they are also not equal. From $2^{b_1+b_2}-3^2$, we know the minimum value of $2^{b_1+b_2}$ is 16. Thus, we can deduce: $2^{b_1}=2^{b_1+b_2}-3^2-3$, solving gives $2^{b_1}=4, 2^{b_2}=4$. If $\frac{3+2^{b_1}}{1}<2^{b_1+b_2}-3^2$, then $2^{b_1+b_2}-3^2-3-\frac{2^{b_1+b_2}}{2}=$ a positive integer ($\frac{2^{b_1+b_2}}{2}$ represents that 2^{b_1} is the largest in $2^{b_1+b_2}$), which can be simplified to $\frac{2^{b_1+b_2}}{2}-12$ =a positive integer. It can be noted that when $2^{b_1+b_2}>16$, the result is a positive integer. Therefore, the only positive integer solution is $X=1$, $2^{b_1}=4$, $2^{b_2}=4$.

This proves that for $a \geq 1$, the only positive integer solution for X is 1.