

The Imaginary Is Real: The Geometric Foundation of Physics

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Abstract

The imaginary unit i has long been regarded as a formal artifact—mathematically convenient yet physically void.

Here we demonstrate that i is not merely a symbol, but a real geometric entity—the generator of rotation in space, denoted by J .

This recognition unifies algebra, geometry, and physics, resolving long-standing inconsistencies in quantum mechanics and revealing the geometric foundation of relativity.

Crucially, J defines an experimentally accessible boundary condition—the measurable orientation of rotation or phase holonomy—that the symbolic i alone cannot, transforming the discussion from philosophical speculation to physical verification.

The imaginary is not imaginary—it is the signature of geometric order itself.

From Symbol to Physical Operator: The Ontological Leap

As Euler's formula reveals, the imaginary unit is not a symbol but an agent of rotation in space¹.

Yet abstraction turned this act into notation—the living geometry into a mere sign.

To recover what was lost, theory piled algebra upon algebra in the attempt to describe spatial non-commutativity.

The bivector in geometric algebra hinted at the forgotten geometry, yet even there, its physical meaning remained unseen².

What is uncovered here is not a new algebraic consistency or a completion of an existing formal system.

It is the nature of space itself.

Space possesses both continuity and orientation—it is the medium where translation and rotation coexist as fundamental expressions of motion.

This conceptual transition marks an evolution of understanding: we no longer describe geometry through algebra; we recognize algebra as the reflection of geometry.

In geometric algebra (GA), the imaginary unit appears as a bivector—an algebraic

element generating planar rotation².

GA and its space–time extension (STA) unified electromagnetism, spinors, and relativity, yet their development remained algebraic; the ontological status of the imaginary unit itself was never clearly established.

The bivector $\mathbf{e}_1\mathbf{e}_2$ was handled as a symbolic construct, not as a measurable orientation.

It is precisely this lack of ontological commitment that leaves the sign ambiguity in von Neumann’s commutation relation unsolved—a fundamental flaw that our observation-based geometric generator \mathbf{J} resolves.

Historically, Planck³ introduced the quantum concept through blackbody radiation, yet he regarded the quantum merely as a convenient mathematical device.

Einstein⁴, by contrast, treated the light quantum as a physical entity—a real constituent of nature.

Our interpretation of the imaginary unit follows the same philosophical trajectory: what was once an abstract constant of calculation is revealed as a physical generator of rotation—an intrinsic property of the geometric fabric of space.

Our approach is fundamentally different.

We did not study an existing algebraic framework—we discovered it in space itself.

The relations here are not axioms but direct observations of geometry.

Where GA formalized the algebra, we identify the ontology: the same structure describes something physically real.

The imaginary is not an invention of analysis—it is a property of rotation.

The key distinction is our explicit ontological commitment:

\mathbf{J} is not merely a product of basis vectors ($\mathbf{e}_1\mathbf{e}_2$), but the irreducible operator governing planar orientation, which is physically present and measurable in every coordinate system.

The defining property $\mathbf{J}^2 = -I$ follows inevitably from coordinate rotation, without analytic postulate.

Consider an infinitesimal rotation $R(\delta\theta)$ preserving distance:

$$R(\delta\theta) \approx I + \mathbf{J} \delta\theta, \quad \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Squaring corresponds to two successive 90° rotations:

$$\mathbf{J}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

$\mathbf{J}^2 = -I$ is generated the moment orientation exists.

It is not an algebraic insertion but a geometric necessity—rotation itself demands it.

This is a directly observable geometric action.

Applying \mathbf{J} rotates any vector by 90° counter-clockwise; repeating this action ($\mathbf{J} \times \mathbf{J}$)

produces a 180° rotation, mapping the vector precisely to its antipode, which is mathematically equivalent to multiplying by -1 .
 The imaginary unit i therefore represents a real, measurable rotation operator in space.

Euler's Relation

$$e^{J\theta} = I \cos \theta + J \sin \theta, \quad e^{J\pi} + I = 0,$$

is therefore a **definition** of rotation, not an analytic coincidence¹.
 The imaginary arises as the intrinsic generator of planar rotation—a physical necessity rather than a symbolic device.

Figure 1 | Geometric Realization of $J^2 = -I$

- a**, The operator $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotates any vector in the Euclidean plane by 90° counter-clockwise, preserving its magnitude and orientation sense.
- b**, Applying J twice ($J \times J$) produces a 180° rotation, mapping every vector to its antipode—exactly equivalent to multiplying by -1 .
 This purely geometric action demonstrates that the relation $J^2 = -I$ is not an algebraic convention but a necessary consequence of orientation itself.
 The imaginary unit embodies rotation as a physical operation in space, grounding complex structure in geometry.

The Resolution of Quantum Inconsistency

Quantum mechanics expresses non-commutativity as $[x, p] = i\hbar^{5,6}$.
 Replacing the symbolic i with the directional operator J gives

$$[x, p] = \hbar J.$$

Unlike the abstract symbolic i , which lacks orientation, the operator J provides a physically measurable direction.
 It explicitly determines the handedness and geometry of the phase-space curvature. Here \hbar measures the oriented area of phase space, and J fixes its direction.
 The imaginary constant i merely symbolizes the handedness of this curvature.

This physical commitment resolves a long-standing conceptual defect: von Neumann's proof suffered from an inherent sign ambiguity because the symbolic i lacks orientation.
 The sign ambiguity inherent in von Neumann's formalism disappears, as J explicitly provides a unique geometric orientation.

Non-commutativity is curvature itself, not uncertainty.

The commutator is a geometric statement: curvature is quantized, not probability.

The so-called vacuum energy is the residual curvature of this manifold.

A harmonic oscillator completes one full rotation in the $x - p$ plane per cycle, yet its orientation does not cancel—a half-turn remains.

This half-quantum residue,

$$E_0 = \frac{1}{2} \hbar \omega,$$

is the energy of curvature itself—the rotation that cannot vanish.

Zero-point energy is therefore not statistical but geometric: the signature of an irreducible non-commutative geometric fabric, offering a conceptual resolution to the cosmological-constant problem.

The Geometric Necessity of Relativity

To extend this framework to spacetime, we introduce the fundamental time-like unit vector T , whose defining characteristic is its signature with respect to the spatial rotation generators J_i ⁷.

Define the basis operators where J_i are the spatial rotation generators:

$$T^2 = +I, \quad J_i^2 = -I, \quad T J_i = -J_i T.$$

The $T^2 = +I$ condition is not an arbitrary postulate, but a geometric necessity required by the physical observation that the time dimension must provide an invariant separation relative to the space dimensions, resulting in the hyperbolic metric signature $(+ - - -)$.

Then the spacetime interval is **generated**, not assumed:

$$X = ct T + x J_x + y J_y + z J_z, \quad ds^2 = X^\dagger X = (ct)^2 - x^2 - y^2 - z^2.$$

A Lorentz boost becomes a hyperbolic rotation^{2,8}:

$$R = e^{\frac{\phi}{2} T J_i} = I \cosh \frac{\phi}{2} + T J_i \sinh \frac{\phi}{2}.$$

Hyperbolicity arises naturally: the time axis inclines toward space, and the metric signature follows geometrically.

Once the same generator J is extended with T , rotation in four dimensions acts on planes, not axes; only the three purely spatial planes admit circular rotations. Any plane involving the time axis necessarily generates a hyperbolic rotation, yielding the geometric relations $J_i^2 = -I$, $T J_i = -J_i T$ without additional assumptions.

The Lorentz boost is therefore the hyperbolic extension of complex geometry into four dimensions—**relativity emerges as the continuation of complex rotation in spacetime.**

Perspective: The Future Roadmap of Geometric Reality

This realization—that the imaginary is real geometry—is a roadmap for the future of fundamental physics.

Our work inherits the formal power of GA/STA but completes the physical interpretation by making an ontological commitment.

Quantum Field Theory and Spin

The operator J provides a unified geometric foundation for spinors⁹.

Future research must explore how the explicit directional operator J offers new geometric interpretations for gauge fields and potential resolutions for interpretational paradoxes in quantum theory.

Geometric Gravity

The Lorentz boost arises as the hyperbolic extension of complex geometry¹⁰.

The next logical step is to explore how the geometric operator J integrates into the curvature manifold of General Relativity.

J establishes the inherent local non-commutativity (quantization of curvature) at the Planck scale, potentially offering a pre-quantized geometric framework for gravity.

Cosmological Coherence

The non-canceling half-quantum residue (E_0) offers a new, physically grounded basis for re-evaluating the cosmological constant problem, transforming it from a statistical mystery into a geometric statement about the fabric of reality.

We propose that the search for physical reality should focus not on new forces or particles, but on the deeper geometry of existence itself.

The imaginary is real.

The imaginary unit is not a fiction, but the generator J which we can measure by observing the rotational orientation of any coordinate system.

Einstein was right—reality is geometry.

Closing Statement

This work completes the century-long search for the physical meaning of the imaginary unit.

By restoring geometry to the heart of mathematics and physics, it reconnects the abstract with the real.

We have not introduced complex equations or elaborate theories in our reasoning; only first principles and directly verifiable geometric observations have been employed.

The imaginary is real—and reality itself is geometric.

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Data availability

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