

Non-best Bounds for Schur Numbers enjoying Simple Proofs

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Abstract. The K^{th} "Schur number" $S(K)$ is the least positive integer N such that for every coloring of the integers $\{1,2,3,\dots,N\}$ with K colors, an equation $a+b=c$ exists with $1\leq a\leq b\leq c\leq N$ with a,b,c all having the same color. Equivalently it is the greatest possible N so that a $(K-1)$ -coloring of $\{1,2,3,\dots,N-1\}$ exists avoiding monochromatic $a+b=c$. [The latter formulation has the advantage of telling us that $S(0)=1$.] We give very simple proofs that $G^K\leq S(K)\leq [K!(e-1/24)]$ where $e\approx 2.71828$ for various growth-factors $G=3/2$, $G=2$, and (for $k\geq 5$) $G=161^{1/5}=2.76290\dots$ These G s are *not* the best known, but my proofs are very simple. We speculate that $S(K)$ grows superexponentially, and (if that is true) explain how a computer could prove arbitrarily large G s or (if it is not true) prove a G arbitrarily near to maximum possible, after enough work. I have not been able to prove superexponentiality, but the final paragraph explains a strategy that might be able to.

The K^{th} "Schur number" $S(K)$ is the least positive integer N such that for every coloring of the integers $\{1,2,3,\dots,N\}$ with K colors, an equation $a+b=c$ exists with $1\leq a\leq b\leq c\leq N$ with a,b,c all having the same color. The sequence $S(0), S(1), S(2),\dots$ of Schur numbers begins **1, 2, 5, 14, 45, 161**. The value of $S(6)$ is unknown, but it is known that $537\leq S(6)\leq 1928$. The following **approximate formula** $S(k)\approx 0.011272\Gamma(k+5.7912)\exp(-1.01007k)$ yields these predictions:

k	0	1	2	3	4	5	6	7	8
$S(k)$	0.951	2.0059	4.96122	14.0775	45.0718	160.721	631.648	2712.47	12636.0

My attention was drawn to this topic by Heule 2017 and the 2025 video about it. In that paper, Heule proves $S(5)=161$ with the aid of an enormous computation. The video proves $S(K)$ is finite, i.e. always exists. Indeed, it proves $S(K)\leq R_3(K)-1$ (Schur 1917), where the Ramsey number $R_3(K)$ denotes the least positive integer M such that any K -coloring of the edges of the M -vertex complete graph always contains at least one monochromatic triangle. Here $R_3(1)=3$, $R_3(2)=6$, $R_3(3)=17$, and $R_3(k)+1\leq R_3(k+1)\leq [k+1][R_3(k)-1]+2$. Using the upper bound recurrence for $R_3(k)$ and the known bounds $51\leq R_3(4)\leq 62$, $162\leq R_3(5)\leq 307$, $538\leq R_3(6)\leq 1838$, $1698\leq R_3(7)\leq 12861$ from Radziszowski's survey, one may show $S(k)\leq 2.9442\cdot(k+1)!$. Irving 1973 showed $S(K)\leq [K!(e-1/24)]$ where $e=2.71828\dots$

I first point out this very simple exponentially-growing lower bound: **$S(K)\geq (3/2)^K$** .

Proof: Erdős 1965 showed that any set of N nonzero real numbers always contains a sum-free subset containing at least $N/3$ numbers. (In the limit $N\rightarrow\infty$ Erdős' constant $1/3$ was shown to be maximum possible by Eberhard, Green, Manners 2014.) So color the integers $\{1,2,3,\dots,N\}$ as follows:

1. Begin with all integers $\{1,2,3,\dots,N\}$ uncolored.
2. For $K=1,2,3,\dots$ {
 - 2.1. If all integers in $\{1,2,3,\dots,N\}$ have been colored, then declare victory and stop.
 - 2.2. Use the Erdős theorem to find a sum-free subset containing at least $1/3$ of the as-yet-uncolored integers, and color them all with color K .

Q.E.D.

Next I will give two even simpler proofs of this recurrence: $S(K+1) \geq 2S(K)$ which yields better exponentially-growing lower bounds, e.g. $S(K) \geq 2^{K-5} 161$ for all $K \geq 5$. Heule said the "big questions" about Schur numbers were whether the optimal colorings of $\{1, 2, 3, \dots, N-1\}$ could always be assumed to be "palindromic" and/or "modular." **Palindromic** means $\text{color}(j) = \text{color}(N-j)$ where $N = S(K)$. **Modular** means everything still works even if all the equations $a+b=c$ are interpreted as congruences modulo N . Then Heule wants to know which of the equalities $S_{mp}(k) = S_p(k) = S_m(k) = S(k)$ hold, where p and m in subscripts denote "palindromic" and "modular." All these equalities hold for all $k \in \{1, 2, 3, 4, 5\}$. Our second proof will provide colorings yielding lower bounds on $S(K+1)$ which also work for S_{mp} , S_p , and S_m , e.g. always are both palindromic and modular if the coloring corresponding to $S(K)$ was.

First proof: Use the same K -coloring of $1, 2, \dots, S(K)-1$ that you already had; then color $S(K), S(K)+1, \dots, 2S(K)-1$ all with the next, i.e. $(K+1)^{\text{th}}$, color. This construction actually can be regarded as producing a coloring of *all* positive integers, whose truncations yield lower bounds on all the $S(K)$.

Second proof: Color the *odd* numbers in $\{1, 2, 3, \dots, 2S(K)-1\}$ with the $(K+1)^{\text{th}}$, color; then color the even numbers using the K -coloring you already had (for those numbers divided by 2).

These constructions work because the sum of two odd numbers always is even; and because the sum of two numbers both $\geq S$, must be $\geq 2S$. **Q.E.D.**

The factor "2" inside this result is best possible in the sense that the color used to color 1, cannot be assigned to any subset of $\{1, 2, 3, \dots, N\}$ larger than its odd subset, since it is forbidden to color any two consecutive numbers the same as 1. But we saw earlier that the factor "3/2" was "best possible" in a similarly not-very-relevant sense. Of course, these Schur number lower bounds also yield exponentially growing lower bounds on $R_k(3)$.

The growth factors 3/2 and 2 in these simply-exponentially growing lower bounds on $S(k)$ that we got are not the best known. Schur 1917 had already shown $S(K+1) \geq 3S(K)-1$. Indeed, I can show $S_{mp}(K+1) \geq 3S_m(K)-1$: **Proof:** Let $N = S_m(K)$. Color $[1, N-1]$ using the K -coloring $\text{col}(x)$ from $S_m(K)$, and color $[2N, 3N-2]$ using the mirrored coloring $\text{col}(3N-1-x)$ to get palindromicity. Finally color $[N, 2N-1]$ using only the $(K+1)^{\text{th}}$ color. **Q.E.D.**

The same construction also shows $S_p(K+1) \geq 3S(K)-1$. From that we deduce $S_{mp}(K) \geq (3^{K-4} 107 + 1)/2$ for all $K \geq 5$, with growth factor $G=3$.

Abbott & Moser 1966 showed $S(K) \geq (3^{K-4} 89 + 1)/2$ for all $K \geq 4$, with growth factor $89^{1/4} = 3.07147\dots$

Abbott & Hanson 1972 showed $S(K+3) \geq 12S(K)-2$, yielding ultimate growth factor $G = 12^{1/3} = 2.28942\dots$

Rowley 2021 claims to show (transformed into our notation) that $S(K+3) \geq 33S(K)-26$ and

$S(K+5) \geq 376S(K)-215$, which yield ultimate growth factors $G \geq 33^{1/3} = 3.20753\dots$ and

$G \geq 376^{1/5} = 3.27369\dots$ However, the excellence of our [approximate](#) formula strongly suggests that the truth is *not* mere simple-exponential, but rather factorial-style growth – which both would match Irving's upper bound (to within a simple-exponential factor) and ultimately grow faster than *any* simple-exponential with any growth factor.

Now I'll prove **supermultiplicativity** $S_{mp}(a+b) \geq S_{mp}(b)S_{mp}(a)$ if $\min(a, b) \geq 1$. [My preceding result $S(K+1) \geq 2S(K)$ arises as the special case $a=1$.] Then as immediate corollaries

1. $\lim_{k \rightarrow \infty} [S_{mp}(k)]^{1/k}$ exists (although it might be infinite) and exceeds 2.76.
2. There is a simple-exponential lower bound on $S_{mp}(k)$ with ultimate growth factor $G \geq [S_{mp}(b)]^{1/b}$ for any particular $b \geq 1$.

This presumably should allow anybody to prove simply-exponential lower bounds on $S_{mp}(k)$ with *arbitrarily large* growth-factors G by simply doing enough computation to determine a good-enough lower bound on $S_{mp}(k)$ for some large-enough single particular k – the more computation you do, the greater G you get. For example $S_{mp}(5)=161$ yields $G \geq 161^{1/5}=2.76290\dots$ Presumably arbitrarily large G are attainable in this way, and you can use our [approximate](#) formula to estimate how-large k you need to get the G you want. For example to prove $G > 4$ and $G > 5$, probably it will suffice to find very good lower bounds on $S(13)$ and $S(20)$. Of course in principle $S(K)$ could be found by brute force consideration of every possible K -coloring of $[1, N]$, then if any are found, increment N and try again. On the other hand if arbitrarily large G are *not* attainable in this way, then you can instead approach the (then necessarily finite) limiting value of G arbitrarily closely by doing enough computing.

Proof: For brevity let $M=S_{mp}(a)$ and $N=S_{mp}(b)$. Color all elements $x \in \{1, 2, \dots, NM-1\}$ that are not divisible by M , using colors $1, 2, \dots, a-1$ according to the coloring of $(x \bmod M)$ associated with $S_{mp}(a)$. Then color the $N-1$ elements that *are* divisible by M using colors $a, a+1, \dots, a+b-1$ according to the coloring of x/M associated with $S_{mp}(b)$. **Q.E.D.**

Abbott & Hanson 1972 (their cor. 2.1) showed $S(a+b) \geq 2S(a)[S(b)-1]+1$, which is even **better** than my supermultiplicativity (by almost a factor of 2). The case $a=b$ of their construction (which in fact also works for S_{mp} and S_p) is explained on pp.3-4 of Ageron et al 2022 using a figure made of the union of two rectangular arrays of sizes $S \times [S-1]$ and $[S-1] \times S$. Then using $a=5$ shows the ultimate growth factor $G=(2 \cdot 161)^{1/5}=3.17373\dots$, but still is not good enough to show that the growth factor $G \rightarrow \infty$.

To prove a lower bound on $S(K)$ with factorial-style **superexponential** growth, it would suffice to show (for any particular real constant $c > 1$) an inequality of form $S(a+b) \geq S(a)S(b)c^{\min(a,b)}$ is valid for all large-enough $\min(a,b)$. Merely showing this in the special case $a=b$ also would suffice to prove superexponentiality. One reason something like this might be possible is the fact that *permuting the colors* still always yields a Schur coloring. It feels like we have not yet tried at all to exploit that, and ought to be able to.

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