STRUCTURES OF FINITE AND INFINITE TYPES OF NON-COMPACT RIEMANN SURFACES VIA FUNDAMENTAL GROUPS

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ABSTRACT. The purpose of this article is to introduce the notion of constructing any arbitrary finite and infinite types of non-compact hyperbolic Riemann surfaces via (non-abelian) fundamental groups equipped with various types of classical Schottky structures, with limit sets as uncountable sets (but not necessarily Cantor sets), emphasising the cases in which the surfaces are of infinite hyperbolic areas. In particular, in this paper, the primary goal is to fabricate three different types of caconical non-compact infinite area Fuchsian polygons in the hyperbolic plane endowed with various kinds of classical Schottky structures. After that, we have initiated a structure of an arbitrary finite type non-compact hyperbolic Riemann surface with genus, conformal holes, cusps, and funnel ends by using the canonical Fuchsian Schottky polygons. Furthermore, in this manuscript, we have proposed the ideas of infinite types conformally compact and semi-conformally compact hyperbolic Riemann surfaces. Indeed, we have introduced four new and interesting types of infinite type hyperbolic Riemann surfaces (we call generalized flute surfaces) that are constructed from infinite sequences of infinite area hyperbolic pair of pants, each glued to the next along a common geodesic boundary with certain strategies.

1. INTRODUCTION

The aim of this article is threefold.

(1) In [9], Keen has constructed all possible canonical polygons for finitely generated Fuchsian groups that give rise to compact hyperbolic Riemann surfaces. In this paper, our initial goal is to extend her work [9] from compact to non-compact hyperbolic Riemann surfaces. In particular, we have fabricated three types of canonical non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures within the Poincaré disk model.

(2) Secondly, we have formulated a strategy for how to construct any finite type arbitrary non-compact hyperbolic Riemann surfaces via fundamental (Fuchsian) groups. In fact, this investigation generalizes the paper of Keen [10] from compact Riemann surfaces to non-compact ones, although the procedure is completely different from her work in [10].

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(3) Finally, we have introduced four new types of infinite type hyperbolic surfaces, which we named as generalized flute surfaces (flute surface is first originated in a paper of Basmajian [5]), linked to various types of infinite dimensional Teichmüller spaces. Of course, the way flute surface was structured by Basmajian ([4], [5]), the flute surfaces and generalized flute surfaces are almost equivalent, but from the point of view of the infinite dimensional Teichmüller theory (in particular, various types of metrices, viz., length spectrum metric, Fenchel-Nielsen metric, bi-Lipschitz metric, and quasiconformal metric, see [2], [3], [1], and [11]), these two kinds of infinite type surfaces (i.e., flute surface and generalized flute surface) are not analogous at all. In particular, we have constructed four kinds of infinite type generalized non-tight flute surfaces and four types of infinite type generalized tight flute surfaces, with limit sets that are Cantor sets and the whole circle at infinity, respectively. In fact, Basmajian [4] introduced the notion of infinitely generated Fuchsian groups from the point of view of the hyperbolic Riemann surface theory. After that, Schwartz [15] initiated the idea of infinitely generated Schottky groups equipped with the notion of limit sets. Although these two works were developed from completely different perspectives. In this paper, the infinitely generated groups that we aim to construct are basically Fuchsian groups with Schottky flavor, i.e., Fuchsian groups equipped with classical Schottky structures. More precisely, the infinitely generated groups that Basmajian [4] formulated, although the associated hyperbolic surface is of infinite area, but each block is non-compact and finite area. On the other hand, Schwartz's [15] constructed infinitely generated group gives rise to compact hyperbolic Riemann surface. But, in this study, we are dealing with infinitely generated groups that lead to non-compact Riemann surfaces, additionally, each block may carry an infinite hyperbolic area. Apart from that, it is well known that the limit sets of any non-trivial finitely generated and infinitely generated Schottky groups are usually Cantor sets. In this investigation, we have proposed some finitely generated and infinitely generated Schottky groups whose limit sets are uncountable sets, but not necessarily Cantor sets, rather the whole circle at infinity. In essence, the study of infinitely generated Fuchsian groups of signatures $(0, 0, \infty, 0, 0)$, $(0, 0, 0, \infty, 0)$, and $(0, 0, \infty, \infty, 0)$ is somewhat known in the literature. In this article, we have introduced the notion of infinitely generated Fuchsian groups of signatures $(0, 0, 0, 0, \infty)$, $(0, 0, 0, \infty, \infty)$, $(0, 0, \infty, 0, \infty)$, and $(0, 0, \infty, \infty, \infty)$.

In this paper, we have proved the following theorems in this regard. Indeed, the following Theorem 1.1 supplies the construction of any non-compact Fuchsian polygons with infinite hyperbolic areas. On the other hand, the subsequent Theorem 1.2 provides the structure of an arbitrary finite type non-compact Fuchsian (surface) group with Schottky settings, whereas

Theorem 1.3 indicates the construction of infinite types non-compact Fuchsian groups of signatures $(0, 0, 0, 0, \infty)$, $(0, 0, 0, \infty, \infty)$, $(0, 0, \infty, 0, \infty)$, and $(0, 0, \infty, \infty, \infty)$. In particular, Theorem 1.1 is a generalization of [[9], Theorems 6 and 7], from compact hyperbolic Riemann surfaces to non-compact ones. On the other hand, Theorem 1.2 generalized the work [[10], Theorem 5] from compact hyperbolic Riemann surfaces to non-compact (hyperbolic) Riemann surfaces. Furthermore, Theorem 1.3 extend the result [[10], Theorem 5] from finitely generated Fuchsian groups to infinitely generated ones.

Theorem 1.1. *For any non-compact infinite area hyperbolic Riemann surface, $\exists$ a canonical Fuchsian polygon with Schottky settings.*

Theorem 1.2. *Let Ω be an arbitrary finite type Fuchsian surface group of signature $(2, 0, 1, 2, 8)$. Then, by utilizing Klein-Maskit combination theorem as a tool, we have proved that $\Omega = < \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_6^*, \Omega_8, \Omega_7^* >$, where Ω_i, $i = 1, 2, ..., 8$ are non-compact Fuchsian groups with some specified signatures of finite types and the symbol '*' indicates the number of operation act in that group.*
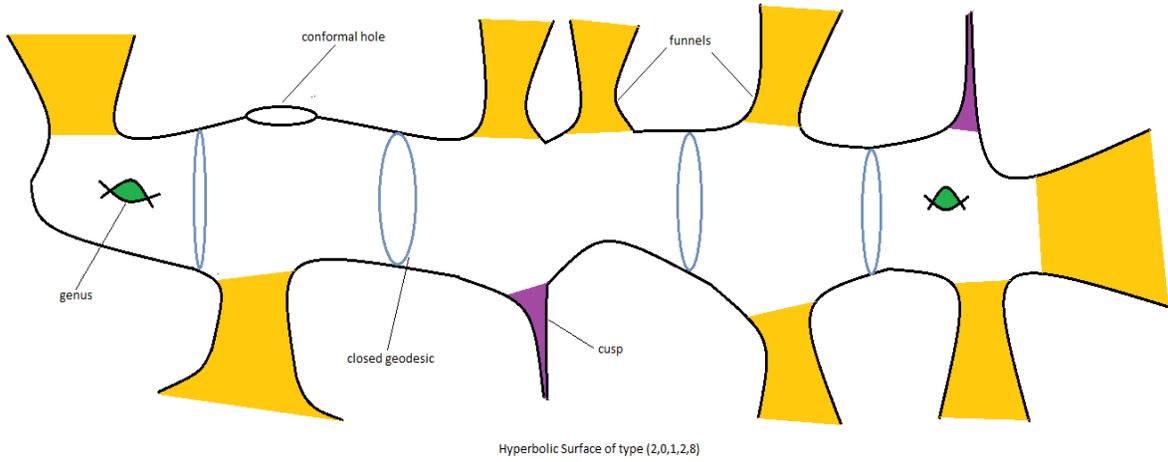

Figure : 1.1

Theorem 1.3. *Suppose Γ, Δ, Ω, Φ, Ξ, Ψ, Σ, and Υ are Fuchsian groups, where '*' represents the number of operation act in that group. Then, by using the Klein-Maskit combination theorem as a tool, we have proved the following results:*

(a) Let Γ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 4)$, $i \in 2\mathbb{N} \cup \{1\}$. Further, we assume that Γ_{3+i} is a Fuchsian group of signature $(0, 0, 2, 0, 2)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Gamma =$

$< \Gamma_1^*, \Gamma_3, \Gamma_2^{**}, \Gamma_5, \Gamma_4^{**}, \Gamma_7, \Gamma_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, 0, 0, \infty)$ *(see Figure* 8 *(b) in Section* 8*).*

(b) *Let* Δ_i *be Fuchsian groups with signatures* $(0, 0, 0, 0, 3)$*,* $i \in 2\mathbb{N} \cup \{1\}$*. Furthermore, suppose that* Δ_{3+i} *is a Fuchsian group of signature* $(0, 0, 2, 0, 1)$*, where* $i \in 2\mathbb{N} \cup \{0\}$*. Then, the group* $\Delta = < \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, 0, 0, \infty)$ *(see Figure* 8 *(c)).*

(c) *Let* Ω_i *be Fuchsian groups with signatures* $(0, 0, 0, 0, 3)$*,* $i \in 2\mathbb{N} \cup \{1\}$*. Additionally, let* Ω_{3+i} *is a Fuchsian group of signature* $(0, 0, 3, 0, 0)$*, where* $i \in 2\mathbb{N} \cup \{0\}$*. Then, the group* $\Omega = < \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, \infty, 0, \infty)$ *(see Figure* 8 *(d)).*

(d) *Let* Φ_i *be Fuchsian groups with signatures* $(0, 0, 1, 0, 3)$*,* $i \in 2\mathbb{N} \cup \{1\}$*. Moreover, we assume that,* Φ_{3+i} *is a Fuchsian group of signature* $(0, 0, 3, 0, 1)$*, where* $i \in 2\mathbb{N} \cup \{0\}$*. Then, the group* $\Pi = < \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, \infty, 0, \infty)$ *(see Figure* 8 *(e)).*

(e) *Let* Ξ_i *be Fuchsian groups with signatures* $(0, 0, 0, 1, 3)$*,* $i \in 2\mathbb{N} \cup \{1\}$*. In addition, suppose that* Γ_{3+i} *is a Fuchsian group of signature* $(0, 0, 2, 1, 1)$*, where* $i \in 2\mathbb{N} \cup \{0\}$*. Then, the group* $\Xi = < \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, 0, \infty, \infty)$ *(see Figure* 8 *(f)).*

(f) *Let* Ψ_i *be Fuchsian groups with signatures* $(0, 0, 0, 0, 3)$*,* $i \in 2\mathbb{N} \cup \{1\}$*. Furthermore, let* Π_{3+i} *is a Fuchsian group of signature* $(0, 0, 2, 1, 0)$*, where* $i \in 2\mathbb{N} \cup \{0\}$*. Then, the group* $\Psi = < \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, 0, \infty, \infty)$ *(see Figure* 8 *(g)).*

(g) *Let* Σ_i*,* Π_i*, and* Υ_i *be Fuchsian groups with signatures* $(0, 0, 0, 0, 3)$*,* $(0, 0, 1, 0, 2)$*, and* $(0, 0, 0, 1, 2)$ *respectively,* $i \in \mathbb{N}$*. Then, the group* $\Upsilon = < \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, \infty, \infty, \infty)$ *(see Figure* 8 *(h)).*

(h) *Let* Λ_i*,* Ω_i*, and* Θ_i *be Fuchsian groups with signatures* $(0, 0, 0, 0, 4)$*,* $(0, 0, 0, 1, 3)$*, and* $(0, 0, 3, 0, 1)$ *respectively,* $i \in \mathbb{N}$*. Then, the group* $\Phi = < \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, ... >$ *is an infinitely generated Fuchsian group of type* $(0, 0, \infty, \infty, \infty)$ *(see Figure* 8 *(i)).*

It is well known that, unlike the case of hyperbolic surface of finite type, $\exists$ various types of Teichmüller spaces corresponding to an infinite type hyperbolic surface, and each of these Teichmüller spaces (heavily) depends on the choice of a basepoint for that Teichmüller space. Moreover, even if someone fix a base point, Teichmüller space depends set theoretically on a distance function that one can put on that space. In fact, there are various types of distances that

we can use in the Teichmüller space. In this point of view, the infinite type surfaces coming out from the Fuchsian groups of signatures $(0, 0, 0, 0, \infty)$ (Theorem 1.2 (a) and (b)), $(0, 0, \infty, 0, \infty)$ (Theorem 1.2 (c) and (d)), $(0, 0, 0, \infty, \infty)$ (Theorem 1.2 (e) and (f)), and $(0, 0, \infty, \infty, \infty)$ (Theorem 1.2 (g) and (h)) are not analogous. In particular, both the infinitely generated groups, Γ and Δ with signatures $(0, 0, 0, 0, \infty)$ provide infinite type surfaces, where each block is of infinite hyperbolic area, but each block of Δ contains one funnel end (apart from the first one), whereas for the group Γ, two funnels appear in each block. Furthermore, both the infinitely generated groups Ω and Π of signature $(0, 0, \infty, 0, \infty)$ supply infinite type surfaces, but they are different from the point of view of the infinite dimensional Teichmüller spaces. In fact, for the group Ω, 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic areas, but 2nd, 4th, 6th, 8th, ... blocks are of finite areas (i.e., π), whereas for the group Π, each block consists of infinite hyperbolic area. Moreover, the infinitely generated groups Ξ and Ψ of signature $(0, 0, 0, \infty, \infty)$, the associated hyperbolic surfaces, for the group Ξ, each block is of infinite hyperbolic area, on the other hand, for Ψ, 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic areas, but 2nd, 4th, 6th, 8th, ... blocks are of finite areas. Finally, the infinitely generated groups Υ and Φ of signature $(0, 0, \infty, \infty, \infty)$, the corresponding infinite type surfaces, for the group Υ, 1st, 4th, 7th, ... blocks are non-compact hyperbolic areas, whereas 2nd, 5th, 8th, ... blocks are compact finite areas, on the other hand, 3rd, 6th, 9th, ... blocks are non-compact but finite areas. However, for the group Φ, each block carries an infinite hyperbolic area.

Outline of this paper. The rest of this article is arranged as follows: In Section 2, we have discussed some preliminaries that will be needed throughout this paper. In Section 3, we have proved Theorem 1.1 by constructing three types of non-compact infinite area Fuchsian polygons with various kinds of classical Schottky structures in the Poincaré disk model. To prove Theorems 1.2 and 1.3, we have utilized Sections 4 to 7 as the technical core. In particular, in Section 4, we have initiated the strategy of gluing two hyperbolic surfaces of type $(0, 0, 0, 0, f)$ via fundamental groups (see Lemma 4.1). After that, we have approached the technique of attaching two hyperbolic surfaces of type $(0, 0, 0, c, f)$ in Section 5 (see Lemma 5.1). In Section 6, we have investigated the situations when the non-compact infinite area hyperbolic surface contains conformal holes (see Lemma 6.1). Furthermore, in Section 7, we have studied the circumstances when non-compact hyperbolic surfaces carry conformal holes but of finite area (see Lemma 7.1). Finally, in Section 8, we have proved Theorems 1.2 and 1.3. At the end, we proposed a brief notion about the limit sets of our constructed infinitely generated groups.

2. Preliminaries

A conformally compact Riemann surface is basically a non-compact hyperbolic surface.

Definition 2.1. A hyperbolic Riemann surface is said to be conformally compact if it is of infinite hyperbolic area endowed with a metric of Gaussian curvature -1, providing a non-abelian finitely generated fundamental group, and without a cusp end.

So, a conformally compact hyperbolic surface can contain only funnel ends in the boundary of its compact core. On the other hand, a semi-conformally compact hyperbolic Riemann surface is also a non-compact hyperbolic surface, but the notion is slightly general compared to a conformally compact hyperbolic surface. In particular, a semi-conformally compact hyperbolic surface may also contain cusp ends.

Definition 2.2. A semi-conformally compact hyperbolic Riemann surface is of type (g, p, h, c, f), where g is the number of genus, p and h are the distinguished points and conformal disks (that are removed from its compact core), c and f are the numbers of cusp and funnel ends respectively.

We assert that g, p, h, and $c \geq 0$, but $f > 0$ always. Moreover, if $p = p_1, p_2, ..., p_m$, and $h = h_1, h_2, ..., h_n$, $p_i \neq p_j$, for $i \neq j$, and $h_i \cap h_j = \phi$, $i \neq j$, with $p_i \notin h_j$. Now, if the genus $g = 0$ and $h + c + f = 3$, we say that the quotient surface is a pair of pants (see [13] and [14] for more details). In this article, we have explored torsion-free Fuchsian groups, so there is no elliptic element in the group. In fact, Keen ([9] and [10]) carried out the situations for containing elliptic elements in Fuchsian groups. So, we have exempted that case here. More precisely, in this manuscript, we are mainly interested in the type $(g, 0, h, c, f)$.

Definition 2.3. A flute surface is a special type of infinite type hyperbolic surface that looks like an infinite chain of (finite area) pair of pants glued end to end.

Topologically, a flute surface is homeomorphic to a sphere removing a Cantor set of points. Although, geometrically, a flute surface carries a hyperbolic metric.

Definition 2.4. We define a flute surface is tight if it contains at least one (hence countably infinite number) cusp end. If an infinite type flute surface doesn't contain any cusp end, we call non-tight.

For the basics of a hyperbolic manifold, funnel, cusp, and pair of pants, the readers may go through the expositions, for instance, [13], [6], [7], [5], and [12].
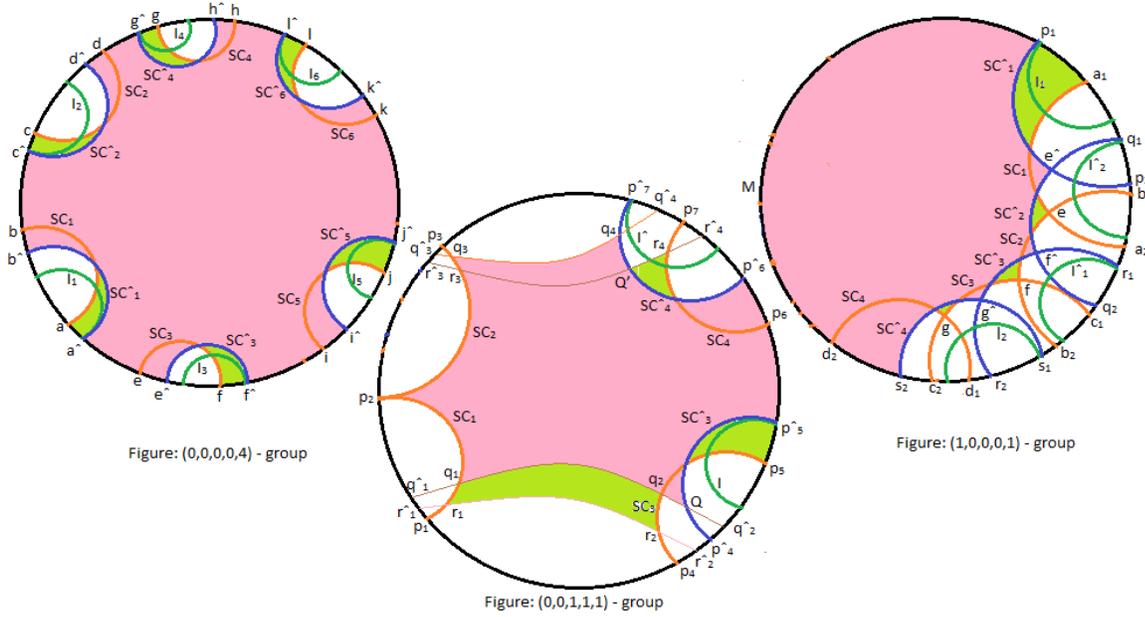
3. Proof of the first Main Theorem

Figure: (0,0,0,0,4) - group

Figure: (0,0,1,1,1) - group

Figure: (1,0,0,0,1) - group

$Figure : 3.1$

Proof of Theorem 1.1. To prove this theorem, we have constructed three different types of non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures in the Poincaré disk model. In particular, we initiated the canonical polygons of the groups $(0,0,0,0,4)$, $(0,0,1,1,1)$, and $(1,0,0,0,1)$ which we have presented as follows.

Constructing canonical Fuchsian Schottky polygon 1.

Step 1. Let us consider six semi-circles SC_1, SC_2, SC_3, SC_4, SC_5, and SC_6 within the Poincaré disk model with end points $a, b; c, d; e, f; g, h; i, j; k, l$ respectively lying on the boundary of the disk. SC_1 is paired with SC_2, SC_3 twins with SC_4, and SC_5 is linked with SC_6 by three hyperbolic Möbius transformations. Now, the open region (looks like a polygon) bounded by these six semi-circles with arcs lying on the boundary of the disk represents the open non-compact polygon of a $(0,0,0,0,4)$ group. In particular, the area surrounded by the non-Euclidean lines joining the points $a, e; f, i; j, k; l, h; g, d, c; d,$ and the semi-circles SC_1, SC_2, SC_3, SC_4, SC_5, and SC_6 exhibits the fundamental domain for the group $(0,0,0,0,4)$. Now, for a $(0,0,0,0,4)$ group, to make the non-compact polygon canonical, we have utilized the procedure as follows:

Step 2. Let, $\Omega^1 = \langle g_1, g_2, g_3 \rangle$ be a group with signature $(0,0,0,0,4)$. Note that all three generators g_1, g_2, and g_3 are hyperbolic Möbius transformations. Now, we consider the three

isometric circles I_1, I_2, and I_3 corresponding to the right side of the three axes of the hyperbolic elements g_1, g_2, and g_3 respectively (see [8] for the notion of isometric circles). Suppose the end points of the isometric circles I_1, I_2, and I_3 are $\hat{a}$ (between the points a and e), $\hat{f}$ (between the points f and i), and $\hat{j}$ (between the points j and k) respectively. Let, $g_1(\hat{a}) = \hat{d}$, $g_2(\hat{f}) = \hat{h}$, and $g_3(\hat{j}) = \hat{k}$. Now, to fabricate a canonical polygon for the group $(0,0,0,0,4)$, we draw six semi-circles in the following way. We translate six semi-circles SC_1 to $\hat{SC_1}$ with end point at $\hat{a}$ (by the hyperbolic curve length $(\hat{a} - a)$), SC_2 with end point at $\hat{d}$ (by the hyperbolic curve length $(\hat{d} - d)$), SC_3 to $\hat{SC_3}$ with end point at $\hat{f}$ (by the hyperbolic curve length $(\hat{f} - f)$), SC_4 to $\hat{SC_4}$ with end point at $\hat{h}$ (by the hyperbolic curve length $(\hat{h} - h)$), SC_5 to $\hat{SC_5}$ with end point at $\hat{j}$ (by the hyperbolic curve length $(\hat{j} - j)$), and SC_6 to $\hat{SC_6}$ with end point at $\hat{k}$ (by the hyperbolic curve length $(\hat{k} - k)$). Let $\hat{b}$, $\hat{c}$, $\hat{e}$, $\hat{g}$, $\hat{i}$, $\hat{l}$ be the starting points of the semi-circles $\hat{SC_1}$, $\hat{SC_2}$, $\hat{SC_3}$, $\hat{SC_4}$, $\hat{SC_5}$, $\hat{SC_6}$ respectively. Now, the region bounded by the semi-circles $\hat{SC_1}$, $\hat{SC_2}$, $\hat{SC_3}$, $\hat{SC_4}$, $\hat{SC_5}$, $\hat{SC_6}$, and the arcs joining by the points $\hat{a}$ to $\hat{e}$, $\hat{f}$ to $\hat{i}$, $\hat{j}$ to $\hat{k}$, $\hat{l}$ to $\hat{h}$, $\hat{g}$ to $\hat{d}$, $\hat{c}$ to $\hat{b}$ lying on the boundary of the disk represents the canonical polygon of the $(0,0,0,0,4)$ group.

Constructing canonical Fuchsian Schottky polygon 2.

Step 1. Let us consider four semi-circles SC_1, SC_2, SC_3, and SC_4 within the Poincaré disk model with end points $p_1, p_2; p_2, p_3; p_4, p_5; p_6, p_7$ respectively lying on the boundary of the disk. SC_1 is paired with SC_2 by a parabolic Möbius transformation p, and SC_3 is linked with SC_4 by a hyperbolic Möbius transformation h. Now, the open region (looks like a polygon) bounded by these four semi-circles SC_1, SC_2, SC_3, and SC_4 represents an open non-compact polygon of a $(0,0,0,1,2)$ group. In particular, the area surrounded by the non-Euclidean lines joining the points $p_1, p_4; p_5, p_6; p_7, p_3$ (all lying on the boundary of the disk) and the semi-circles SC_1, SC_2, SC_3, and SC_4 exhibits the fundamental domain for the group $(0,0,0,1,2)$. Now, we consider two non-Euclidean lines; one intersects at r_1 in SC_1 and r_2 in SC_3; the other one intersects at r_3 in SC_2 and r_4 in SC_4. We produce r_2r_1 to $\hat{r_1}$, r_1r_2 to $\hat{r_2}$, r_4r_3 to $\hat{r_3}$, and r_3r_4 to $\hat{r_4}$ in the boundary of the disk. The region surrounded by the arcs joining the points $p_2, r_1, r_2, p_5, p_6, r_4, r_3$, and p_2 represents the fundamental polygon of a $(0,0,1,1,1)$ group. Now, to make this polygon canonical, we have used the procedure as follows:

Step 2. Let $\Omega^2 = \ <p, h>$ be a group with signature $(0,0,1,1,1)$. Here, p is parabolic, whereas h is a hyperbolic Möbius transformation. Now, let us consider the isometric circle I corresponding to the right side of the axis of the hyperbolic element h. Assume that, the end point of the isometric circle I is $\hat{p_5}$ (between the points p_5 and p_6) on the boundary of

the Poincäre disk. Let, $h(\hat{p_5}) = \hat{p_6}$. Now, to construct a canonical polygon for the group $(0, 0, 1, 1, 1)$, we draw two semi-circles I, and $\hat{I}$ in the subsequent way. We translate these two semi-circles SC_3 to $\hat{SC_3}$ with end point at $\hat{p_4}$ (by the hyperbolic curve length $(\hat{p_5} - p_5)$), SC_4 with end point at $\hat{p_7}$ (by the hyperbolic curve length $(\hat{p_6} - p_6)$. Let $\hat{p_5}$ and $\hat{p_6}$ be the starting points of the semi-circles $\hat{SC_3}$ and $\hat{SC_4}$, respectively. Now, on the boundary of the disk, we take four points $\hat{q_1}$ (between p_2 and $\hat{r_1}$), $\hat{q_2}$ (between $\hat{p_5}$ and $\hat{p_4}$), $\hat{q_3}$ (between $\hat{r_3}$ and p_3), $\hat{q_4}$ (between $\hat{p_7}$ and $\hat{r_4}$). We produce the non-Euclidean arcs $\hat{q_1}\hat{q_2}$ (intersecting SC_1 at q_1, and $\hat{SC_3}$ at Q), and $\hat{q_3}\hat{q_4}$ (intersecting SC_2 at q_3, and $\hat{SC_4}$ at Q'). The region bounded by the non-Euclidean arcs joining by the points p_2 to q_1, q_1 to q_2, q_2 to Q, Q to $\hat{p_5}$, $\hat{p_5}$ to p_6, p_6 to $\hat{p_6}$, $\hat{p_6}$ to Q', Q' to q_4, q_4 to q_3, q_3 to r_3, and r_3 to p_2 represents the canonical polygon for the group $(0, 0, 1, 1, 1)$.

Constructing canonical Fuchsian Schottky polygon 3.

Step 1. Let us consider four semi-circles SC_1, SC_2, SC_3, and SC_4 within the Poincaré disk model with end points $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ respectively that are lying on the boundary of the disk. SC_1 is paired with SC_3, and SC_4 twins with SC_2 by two hyperbolic Möbius transformations h_1 and h_2, respectively. The open region (looks like a polygon) encircled by these four semi-circles equipped with the arc lying on the boundary of the disk represents the non-compact polygon for a $(1, 0, 0, 0, 1)$ group. Let M be a point situated on the boundary of the disk at the point $(-1, 0)$. Furthermore, we assume that the semi-circles SC_1 and SC_2 intersect at the point e, SC_2 and SC_3 meet at f, and SC_3 and SC_4 intersect at the point g. Note that all the points e, f, and g are lying within the disk. Now, the area surrounded by the non-Euclidean lines joining the points $d_2, g; g, f; f, e; e, a_1; a_1, M; M, d_2$ exhibits the fundamental domain for the group $(1, 0, 0, 0, 1)$. Now, our goal is to produce the canonical polygon of the group $(1, 0, 0, 0, 1)$.

Step 2. Let $\Omega^3 = <h_1, h_2>$ be a group with signature $(1, 0, 0, 0, 1)$. Here, both the generators h_1 and h_2 are hyperbolic Möbius transformations. Now, we consider two isometric circles I_1 and I_2, corresponding to the two axes of the hyperbolic transformations h_1 and h_2 respectively. Suppose, the end points of the isometric circles I_1 and I_2 are p_1 (between the points a_1 and M) and s_1 (between the points d_1 and b_2) respectively. Let, $h_1(p_1) = r_1$, and $h_2(s_1) = q_1$. Now, to fabricate a canonical polygon for the group $(1, 0, 0, 0, 1)$, we draw four semi-circles in the following way. We translate four semi-circles SC_1 to $\hat{SC_1}$ with end point at p_1 (by the hyperbolic curve length $(p_1 - a_1)$), SC_2 to $\hat{SC_2}$ with end point at q_1 (by the hyperbolic curve length $(q_1 - b_1)$, SC_3 to $\hat{SC_3}$ with end point at r_1 (by the hyperbolic curve length $(r_1 - c_1)$), SC_4 to $\hat{SC_4}$ with end point at s_1 (by the hyperbolic curve length $(s_2 - d_1)$). Let, $p_2, q_2, r_2,$

and s_2 are the starting points of the semi-circles $\hat{SC_1}$, $\hat{SC_2}$, $\hat{SC_3}$, and $\hat{SC_4}$ respectively. Let the semi-circles $\hat{SC_1}$ and $\hat{SC_2}$ intersect at the point $\hat{e}$, $\hat{SC_2}$ and $\hat{SC_3}$ meet at $\hat{f}$, and finally, $\hat{SC_3}$ and $\hat{SC_4}$ intersect at the point $\hat{g}$. Now, the region bounded by the arcs joining by the points s_2 to $\hat{g}$, $\hat{g}$ to $\hat{f}$, $\hat{f}$ to $\hat{e}$, $\hat{e}$ to p_1, p_1 to M, M to s_2 represents the canonical polygon of the $(1, 0, 0, 0, 1)$ group.

In this way, by utilizing the above three constructions, one can originate any canonical non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures in the hyperbolic plane. This completes the proof of Theorem 1.1. $\square$

In the subsequent Sections 4, 5, 6, and 7, we have described a general method for constructing new non-compact finite and infinite area hyperbolic surfaces by gluing along infinite funnel boundaries with certain methodologies. The essential tool we employ here is the (Klein-Maskit) combination theorem (see [10], Theorem 1 for an exposition of the combination theorem) that allows us to amalgamate two non-compact Fuchsian groups to form a larger discrete group. In particular, Sections 4 to 6 deal with infinite area hyperbolic surfaces, whereas Section 7 concerns with finite area surfaces.

4. Gluing non-compact hyperbolic surfaces of types $(0,0,0,0,f)$ via fundamental groups

Lemma 4.1. *(Construction* 1.) *Let,* Ω_1 *be a Fuchsian group of signature* $(0, 0, 0, 0, 3)$ *with neck lengths* κ_1, κ_2, $2\kappa_3$, *and* Ω_2 *be a Fuchsian group of signature* $(0, 0, 0, 0, 3)$ *with neck lengths* κ_4, κ_5, $2\kappa_6$. *Suppose both the corresponding surfaces possess different neck lengths* κ_i, $i = 1, 2, ..., 6$, *where* $\kappa_i \neq \kappa_j$, *for* $i \neq j$. *Then:*

(1) For a group Π_3 *of signature* $(0, 0, 2, 0, 1)$ *consisting of the lengths of their closed geodesics (of the conformal holes)* κ_2 *and* κ_4, *with neck length* $2\kappa_7$, *we prove that* $\Omega^1 = <\Omega_1^*, \Pi_3, \Omega_2^*>$, *where '*'* indicates the number of operation act on that group.*

(2) Gluing two surfaces of types $(0, 0, 0, 0, 3)$ *yields a hyperbolic surface of type* $(0, 0, 0, 0, 5)$. *In particular,* Ω^1 *is a hyperbolic surface of five funnels ends with neck lengths* κ_1, $2\kappa_3$, $2\kappa_7$, $2\kappa_6$, *and* κ_5.

Proof. Let D_1, D_2, D_3, and D_4 be four semi-circles within the Poincaré disk model. In particular, D_1 is paired with D_2 by the hyperbolic Möbius transformation h_1, whereas D_3 twins with D_4 by the hyperbolic Möbius transformation h_2. Suppose these four semi-circles meet the boundary of the disk at the points $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ respectively. Now, on the boundary of the disk, we take eight points, two between a_1 and a_2, viz., $\tilde{e}$ (near the point a_1) and $\tilde{g}$ (near the point a_2),

two between b_1 and b_2, viz., $\tilde{h}$ (near the point b_1) and $\tilde{i}$ (near the point b_2), two between c_1 and c_2, viz., $\tilde{j}$ (near the point c_1) and $\tilde{k}$ (near the point c_2), and finally two, between the points d_1 and d_2, viz., $\tilde{l}$ (near the point d_1) and $\tilde{f}$ (near the point d_2). We draw the non-Euclidean curves lines by joining the points $\tilde{e}$ to $\tilde{f}$, $\tilde{g}$ to $\tilde{h}$, $\tilde{i}$ to $\tilde{j}$, and $\tilde{k}$ to $\tilde{l}$. Note that, these four curves intersect in the four semi-circles D_1, D_2, D_3, and D_4 at the points $e, g; h, i; j, k; l$, and f. Assume that the lengths of the curve line $\overline{gh}$ is κ_1, $\overline{lk}$ is κ_2, and $\overline{ef}$ (or $\overline{ij}$) is κ_3. Now, the area (denoted by R_0) joining the points $a_1, d_2, f, l, d_1, c_2, k, j, c_1, b_2, i, h, b_1, a_2, g, e$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_1 = <h_1, h_2>$) with signature $(0,0,0,0,3)$. So, Ω_1 is a hyperbolic Riemann surface with three funnel ends consisting of neck lengths κ_1, κ_2, and $2\kappa_3$. However, the region (denoted by R_1) joining the points $a_1, d_2, f, l, k, j, c_1, b_2, i, h, b_1, a_2, g, e$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by Ω_1^*) with signature $(0,0,1,0,2)$ with neck lengths κ_1, and $2\kappa_3$ consisting a closed geodesic boundary with length κ_2. Now, we consider another Fuchsian group Ω_2 with signature $(0,0,0,0,3)$ with neck lengths κ_4, κ_5, and $2\kappa_6$. We proceed in a similar situation as in the above formation. Now, the area (denoted by R_2) joining the points $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, f_1, e_2, j_1, i_1$, and e_1 represents a fundamental domain of a Fuchsian group (denoted by $\Omega_2 = <h_3, h_4>$) with signature $(0,0,0,0,3)$. On the other hand, the region (denoted by R_3) joining the points $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, j_1, i_1$, and e_1 represents a fundamental domain of a Fuchsian group (denoted by Ω_2^*) with signature $(0,0,1,0,2)$ consisting one closed geodesic boundary of length κ_4, and two funnels with neck lengths κ_5, and $2\kappa_6$. Let us consider a group Π_3 with signature $(0,0,2,0,1)$ with one funnel end of neck length $2\kappa_7$. Then, from Figure 4.1 one can easily see that, the region (denoted by R_4) joining the points $o, m, a, h, n, t, s, r, e, d, q, p$, and o represents the fundamental domain of a group Π_3 with signature $(0,0,2,0,1)$ with neck length $2\kappa_7$ and two boundary geodesic with lengths κ_2 and κ_4.

To glue two groups Ω_1 and Ω_2 with signatures $(0,0,0,0,3)$, we cut the two corresponding surfaces, one funnel end each, along the neck of their funnels. After this operation, it reduces two groups, one is Ω_1^* and the other is Ω_2^*. Then, adding a group Π_3 in such a way that the neck length can match with the groups Ω_1^* and Ω_2^*. Note that, when we paste with Ω_1^* to Π_3 the curve lines (i.e., $\overline{kl}$ in Ω_1^* and $\overline{op}$ in Π_3) can match, i.e, lengths are the same and their directions are opposite to each other. Now, by using the combination theorem, one can get a new group $\Omega_{13} = <\Omega_1^*, \Pi_3>$ with signature $(0,0,1,0,3)$. After that, we will attach the groups Ω_{13} and Ω_2^* on the analogy of the technique that we have mentioned just before (i.e., matching the curve lines $\overline{st}$ in Ω_{13} and $\overline{j_1 j_2}$ in Ω_2^*). Consequently, one can again utilize the combination theorem to get a new

group $\Omega_{132} = <\Omega_{13}, \Omega_2^* > (= \Omega^1$, say) with signature $(0,0,0,0,5)$. Therefore, the fundamental domain of the arising group Ω_{132} is the area bounded by the non-Euclidean lines joining the points $a_1, d_2, f, m, a, h, n, i_1, e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, r, e, d, q, j, c_1, b_2, i, h, b_1, a_2, g, e,$ and a_1. So, the proof of Lemma 4.1 is completed. □
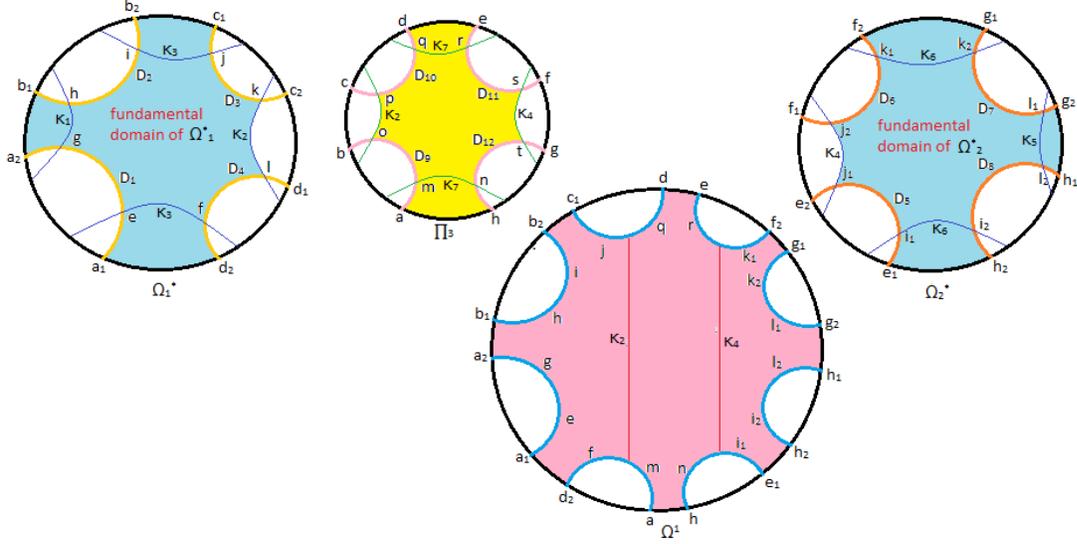

Figure : 4.1

Remark 1. Note that the polygon of a Fuchsian group with signature $(0,0,3,0,0)$ that Keen [10] introduced for compact Riemann surface, is completely different from our constructed polygon for Fuchsian group equipped with classical Schottky structures of signature $(0,0,3,0,0)$ in non-compact settings.

Remark 2. A pair of pants with signature $(0,0,0,0,3)$ consists of three funnel components whose two (out of three) neck lengths are coming from the axes of two hyperbolic elements, and the third one arises from the product of these two hyperbolic elements. On the other hand, for a pair of pants with signature $(0,0,0,1,2)$, one neck length (out of two) is coming from the hyperbolic element, cusp arises from the parabolic element, and finally, the other funnel's neck emerges from the product of parabolic and hyperbolic elements.

5. Attaching surfaces of types $(0,0,0,c,f)$ via fundamental groups

Lemma 5.1. *(Construction 2.) Let, Ω_3 be a Fuchsian group of signature $(0,0,0,1,2)$ with neck lengths $2\kappa_1$, κ_2, and Ω_4 be a Fuchsian group of signature $(0,0,0,1,2)$ with neck lengths $2\kappa_3$,*

κ_4. *Assume that both the corresponding surfaces possess different neck lengths* κ_i, $i = 1, 2, ..., 4$, *where* $\kappa_i \neq \kappa_j$, *for* $i \neq j$. *Then:*

(1) For a group Π_5 *of signature* $(0, 0, 2, 0, 1)$ *consisting of the lengths of their closed geodesics (of the conformal holes)* κ_2 *and* κ_4, *with neck length* $2\kappa_5$, *we prove that* $\Omega^2 = \,< \Omega_3^*, \Pi_5, \Omega_4^* >$, *where '*'* *represents the number of operations acting on that group.*

(2) Gluing two surfaces of types $(0, 0, 0, 1, 2)$ *yields a hyperbolic surface of type* $(0, 0, 0, 2, 3)$. *In particular,* Ω^2 *represents a hyperbolic surface of two cusps and three funnel ends with neck lengths* $2\kappa_1$, $2\kappa_5$, *and* $2\kappa_3$.
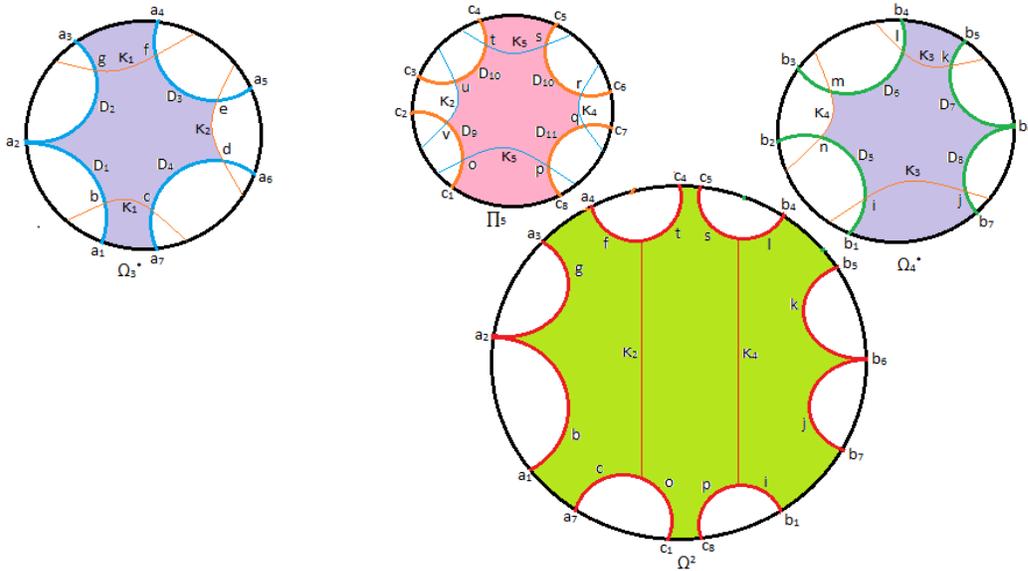

Figure : 5.1

Proof. Let D_1, D_2, D_3, and D_4 be four semi-circles within the Poincaré disk model. In fact, D_1 is paired with D_2 by the parabolic Möbius transformation p_1, whereas D_3 is linked with D_4 by the hyperbolic Möbius transformation h_1. Assume that these four semi-circles meet the boundary of the disk at the points $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$ respectively. Now, on the boundary of the disk, we take six points, two between a_7 and a_6, viz., $\tilde{c}$ (near the point a_7) and $\tilde{d}$ (near the point a_6), two between a_5 and a_4, viz., $\tilde{e}$ (near the point a_5) and $\tilde{f}$ (near the point a_4), one between a_1 and a_2, viz., $\tilde{b}$ (near the point a_1), and finally one, between the points a_2 and a_3, viz., $\tilde{g}$ (near the point a_3). We draw the non-Euclidean curves lines by joining the points $\tilde{b}$ to $\tilde{c}$, $\tilde{d}$ to $\tilde{e}$, and $\tilde{f}$ to $\tilde{g}$. Notice that these three curves intersect in the four semi-circles D_1, D_2, D_3, and D_4 at the points $b; g; f, e; d$, and c. Suppose the lengths of the

curve lines $\overline{bc}$ (or $\overline{fg}$) is κ_1, and $\overline{de}$ is κ_2. Now, the region (denoted by R_5) joining the points $a_1, a_7, c, d, a_6, a_5, e, f, a_4, a_3, g, a_2, b$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_3 = < p_1, h_1 >$) with signature $(0, 0, 0, 1, 2)$. Consequently, the group Ω_3 is a hyperbolic Riemann surface with one cusp and two funnel ends consisting of neck lengths $2\kappa_1$, and κ_2. However, the region (denoted by R_6) joining the points $a_1, a_7, c, d, e, f, a_4, a_3, g, a_2, b$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by Ω_3^*) with signature $(0, 0, 1, 1, 1)$ with one funnel of neck length $2\kappa_1$ consisting a closed geodesic boundary with length κ_2. Now, we consider another group Ω_4 with signature $(0, 0, 0, 1, 2)$ with one cusp end and two funnels of neck lengths $2\kappa_3$, and κ_4. We proceed in a similar situation to the one we did in the above construction. Now, the domain (denoted by R_7) joining the points $b_1, b_7, j, b_6, k, b_5, b_4, l, m, b_3, b_2, n, i$, and b_1 represents a fundamental region of a Fuchsian group (denoted by $\Omega_4 = < h_2, p_2 >$) with signature $(0, 0, 0, 1, 2)$. However, the area (denoted by R_7) joining the points $b_1, b_7, j, b_6, k, b_5, b_4, l, m, n, i$, and b_1 represents a fundamental domain of a Fuchsian group (denoted by Ω_4^*) with signature $(0, 0, 1, 1, 1)$ consisting one closed geodesic boundary of length κ_4, and one funnel with neck length $2\kappa_3$. Now, we are considering a group Π_5 with signature $(0, 0, 2, 0, 1)$ consisting of one funnel end of neck length $2\kappa_5$. Then, from the preceding construction, one can easily observe that, the region (denoted by R_8) joining the points $c_1, c_8, p, q, r, s, c_5, c_4, t, u, v, o$, and c_1 represents the fundamental domain of a group Π_5 with signature $(0, 0, 2, 0, 1)$ with one funnel of neck length $2\kappa_5$ and two closed geodesic boundary with lengths κ_2 and κ_4.

To glue two groups Ω_3 and Ω_4 with signatures $(0, 0, 0, 1, 2)$, we cut the two corresponding surfaces, one funnel end each, along the neck of their funnels, where neck lengths are κ_2 and κ_4 respectively. After this operation, it reduces two groups, one is Ω_3^* and the other is Ω_4^*. Then, adding a group Π_5 in such a way that the neck length can match the groups Ω_3^* and Π_5. Note that, when we paste with Ω_3^* to Π_5 the curve lines (i.e., $\overline{de}$ in Ω_3^* and $\overline{uv}$ in Π_5) can match, i.e, their directions are opposite to each other. Now, by using the combination theorem, one can get a new group $\Omega_{35} = < \Omega_3^*, \Pi_5 >$ with signature $(0, 0, 1, 1, 2)$. After that, we will attach the groups Ω_{35} and Ω_4^* on the analogy of the technique that we have mentioned just before (i.e., matching the curve lines $\overline{rq}$ in Ω_{35} and $\overline{nm}$ in Ω_4^*). So, we can utilize the combination theorem to get a new group $\Omega_{354} = < \Omega_{35}, \Omega_4^* > (= \Omega^2$, say) with signature $(0, 0, 0, 2, 3)$. Therefore, the fundamental domain of the arising group Ω_{354} is the area bounded by the non-Euclidean lines joining the points $a_1, a_7, c, o, c_1, c_8, p, i, b_1, b_7, j, b_6, k, b_5, b_4, l, s, c_5, c_4, t, f, a_4, a_3, g, a_2, b$, and a_1. This proves the Lemma 5.1. □

6. Amalgamation of non-compact hyperbolic surfaces of types $(0,0,m,0,f)$ and $(0,0,m,c,f)$ via fundamental groups

Lemma 6.1. *(Construction 3.) Let, Ω_5 be a Fuchsian group of signature $(0,0,1,0,2)$ consisting length of the closed geodesic (of the conformal hole) $2\kappa_3$, with neck lengths κ_1, κ_2 and Ω_6 be a Fuchsian group of signature $(0,0,1,1,1)$ consisting length of the closed geodesic (of the conformal hole) $2\kappa_5$, with neck length κ_4. Suppose the lengths of their necks and closed geodesic boundaries in the conformal holes are all distinct. Then:*

(1) For a group Π_7 of signature $(0,0,2,0,1)$ consisting the lengths of the closed geodesic (of conformal holes) κ_2, κ_4, with neck length $2\kappa_6$,, we prove that $\Omega^3 = <\Omega_5^*, \Pi_7, \Omega_6^*>$, where '*' represents the number of operations acting on that group.

(2) Gluing two surfaces of types $(0,0,1,0,2)$ and $(0,0,1,1,1)$ yields a hyperbolic surface of type $(0,0,2,1,2)$. In particular, Ω^3 represents a hyperbolic surface with two cusps, two conformal holes with lengths of their closed geodesics $2\kappa_3$, $2\kappa_5$, and two funnel ends with neck lengths κ_1 and $2\kappa_6$.

Proof. Let D_1, D_2, D_3, and D_4 be four semi-circles within the Poincaré disk model. More precisely, D_1 twins with D_2 by the hyperbolic Möbius transformation h_1, on the other hand, D_3 is linked with D_4 by the hyperbolic Möbius transformation h_2. Suppose these four semi-circles meet the boundary of the disk at the points $a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8$ respectively. We consider eight points, on the boundary of the disk, two between a_1 and a_2, viz., $\tilde{b}$ (near the point a_1) and $\tilde{i}$ (near the point a_2), two between a_3 and a_4, viz., $\tilde{h}$ (near the point a_3) and $\tilde{g}$ (near the point a_4), two between a_5 and a_6, viz., $\tilde{f}$ (near the point a_5), and finally two, between the points a_7 and a_8, viz., $\tilde{d}$ (near the point a_7) and $\tilde{c}$ (near the point a_8). Now, we draw the non-Euclidean curves lines by joining the points $\tilde{b}$ to $\tilde{c}$, $\tilde{i}$ to $\tilde{h}$, $\tilde{g}$ to $\tilde{f}$, and $\tilde{e}$ to $\tilde{d}$. Observe that these four curves intersect in the four semi-circles D_1, D_2, D_3, and D_4 at the points $b, i; h, g; f, e; d$, and c. Assume that the lengths of the curve lines $\overline{ih}$ is κ_1 $\overline{bc}$ (or $\overline{fg}$) is κ_3, and $\overline{de}$ is κ_2. Note that, the region (denoted by F_1) joining the points $i, b, c, d, a_7, a_6, e, f, g, h, a_3, a_2$, and i represents the fundamental domain of a Fuchsian group (denoted by $\Omega_5 = <h_1, h_2>$) with signature $(0,0,1,0,2)$. Now, the area (denoted by F_2) joining the points $i, b, c, d, e, f, g, h, a_3, a_2$, and i represents the fundamental region of a Fuchsian group (denoted by $\Omega_5^* = <h_1, h_2>$) with signature $(0,0,2,0,1)$. Hence, the group Ω_5^* is a hyperbolic Riemann surface consisting of one funnel end of neck length κ_1 with two conformal holes where the lengths of their boundary components are κ_2 and $2\kappa_3$, respectively. Now, we consider another group Ω_6 with signature $(0,0,1,1,1)$ with one cusp end, one funnel end of neck length κ_4, and one conformal hole

with a length of the closed geodesic boundary is $2\kappa_5$. We proceed in a similar manner as we have arranged in the overhead construction. Observe that, the region (denoted by F_3) joining the points $o, j, k, b_6, l, m, n, b_3, b_2$, and o represents a fundamental domain of a Fuchsian group (denoted by $\Omega_6 = \, < h_3, p_1 >$) with signature $(0, 0, 1, 1, 1)$. However, the diagram (denoted by F_4) joining the points o, j, k, b_6, l, m, n, and o represents a fundamental domain of a Fuchsian group (denoted by Ω_6^*) with signature $(0, 0, 2, 1, 0)$ consisting of two closed geodesic boundaries of lengths κ_4 and $2\kappa_5$ and one cusp end.
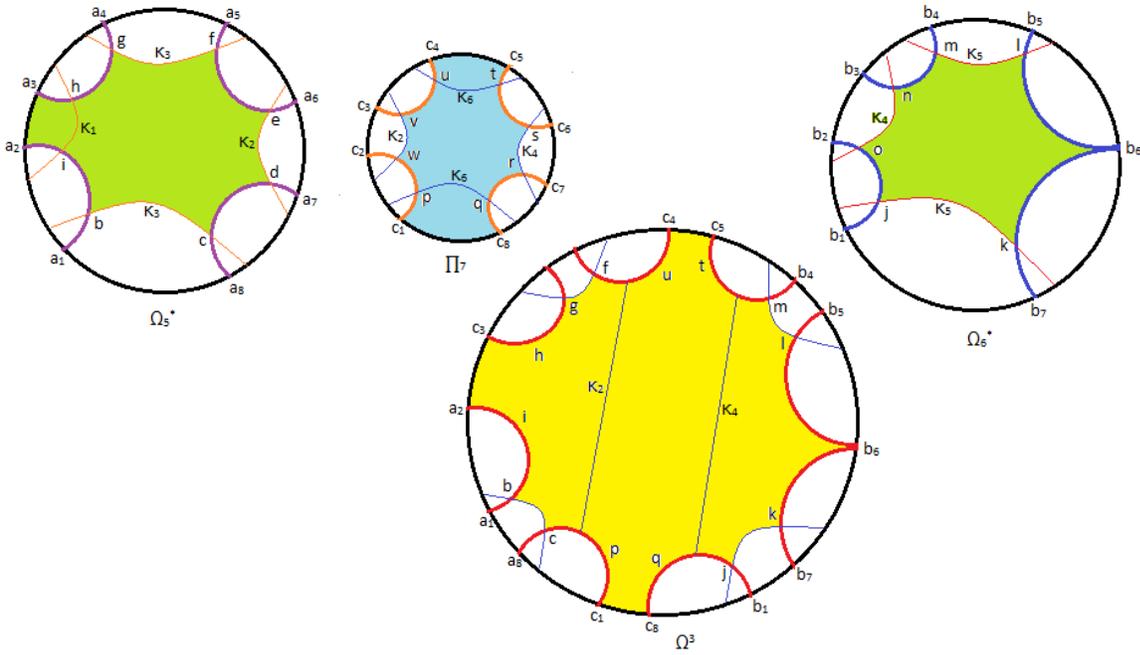

Figure : 6.1

Now, if we want to glue the groups Ω_5^* and Ω_6^*, two cases will occur.

(i) Assume that $\kappa_3 = \kappa_5$. Then, one can easily paste these two groups, Ω_5^* and Ω_6^*, in a direct way. This act gives a Fuchsian group with signature $(0, 0, 0, 1, 3)$. If possible, this will not happen. Then, we will move forward to the second case.

(ii) Suppose $\kappa_3 \neq \kappa_5$. So, in this situation, we can not attach Ω_5^*, and Ω_6^* directly. So, we are going to fabricate a new group in the following way:

Let D_9, D_{10}, D_{11}, and D_{12} be four semi-circles within the Poincaré disk model. More precisely, D_9 is linked with D_{10} by the hyperbolic Möbius transformation h_4, whereas, D_{11} twins with D_{12} by the hyperbolic Möbius transformation h_5. Let, these four semi-circles meet the boundary of

the disk at the points $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$ respectively. Now, on the boundary of the disk, we take eight points, two between c_1 and c_2, viz., $\tilde{p}$ (near the point c_1) and $\tilde{w}$ (near the point c_2), two between c_3 and c_4, viz., $\tilde{v}$ (near the point c_3) and $\tilde{u}$ (near the point c_4), two between c_5 and c_6, viz., $\tilde{t}$ (near the point c_5) and $\tilde{s}$ (near the point c_6), and finally two between c_7 and c_8, viz., $\tilde{r}$ (near the point c_7) and $\tilde{q}$ (near the point c_8). Now, one can draw four non-Euclidean lines by joing the points $\tilde{p}$ to $\tilde{q}$, $\tilde{w}$ to $\tilde{v}$, $\tilde{u}$ to $\tilde{t}$, and $\tilde{s}$ to $\tilde{r}$. Assume that, these four curves intersect in the four semi-circles D_9, D_{10}, D_{11}, and D_{12} at the points $p, w; v, u; t, s; r$, and q. Suppose the lengths of the curve lines $\overline{wv}$, $\overline{rs}$, and $\overline{pq}$ (or $\overline{tu}$) are κ_2 κ_4, $2\kappa_6$ respectively. The region (denoted by F_5) joining the points $w, p, c_1, c_8, q, r, s, t, c_5, c_4, u, v$, and w represents the fundamental domain of the Fuchsian group ($\Pi_7 = \; <h_4, h_5>$) with signature $(0, 0, 2, 0, 1)$.

Now, to glue the groups Ω_5 and Ω_6, we cut their corresponding hyperbolic surfaces one funnel each along the neck of their suitable funnel ends consisting neck lengths κ_2 and κ_4 respectively. After this operation, it reduces two groups, one is Ω_5^*, and other is Ω_6^*. Then, we add the group Π_7 in such a way that the neck length can match with the groups Ω_5^*, and other one Ω_6^*. Note that, when we paste with Ω_5^* to Π_7 the curve lines (i.e., $\overline{de}$ in Ω_5^* and $\overline{vw}$ in Π_7) can match, i.e, their lengths are the same but directions are opposite to each other. Now, by using the combination theorem, one can easily get a new group $\Omega_{57} = \; < \Omega_5^*, \Pi_7 >$ with signature $(0, 0, 2, 0, 2)$. Furthermore, we will attach the groups Ω_{57} and Ω_6^* on the analogy of the technique that we mention just above (i.e., matching the curve lines $\overline{rs}$ in Ω_{57} and $\overline{no}$ in Ω_6^*). Now, again, by utilizing the combination theorem, we get $\Omega_{576} = \; < \Omega_{57}, \Omega_6^* > (= \Omega^3$, say) with signature $(0, 0, 2, 1, 2)$. Therefore, the fundamental domain of the group Ω_{576} ($= \Omega^3$, say) is the region bounded by the non-Euclidean lines joining the points $a_2, i, b, c, p, c_1, c_8, q, j, k, b_6, l, m, t, c_5, c_4, u, f, g, h, a_3$, and a_2. So, Lemma 6.1 is proven.   □

7. Gluing non-compact hyperbolic Riemann surfaces of types $(0,0,m,c,0)$ and $(g,0,m,c,0)$ via fundamental groups

Lemma 7.1. *(Construction 4.) Let, Ω_7 be a Fuchsian group of signature $(0, 0, 2, 1, 0)$ consisting of the lengths of their closed geodesics (of two conformal holes) $2\kappa_1$, κ_2, and Ω_8 be a Fuchsian group of signature $(1, 0, 2, 1, 0)$ with lengths of their closed geodesics (of two conformal holes) κ_3, $2\kappa_4$. Assume that, both the corresponding surfaces posses different lengths of boundaries in their conformal holes, κ_i, $i = 1, 2, ..., 4$, where $\kappa_i \neq \kappa_j$, for $i \neq j$. Then:*

(1) For a group Π_9 of signature $(0, 0, 3, 0, 0)$ with lengths of their closed geodesics (of three conformal holes) κ_2, $2\kappa_5$, κ_3, we prove that $\Omega^4 = \; < \Omega_7, \Pi_9, \Omega_8 >$.

(2) Gluing two surfaces of types $(0,0,2,1,0)$ and $(1,0,2,1,0)$ yields a hyperbolic surface of type $(1,0,3,2,0)$. In particular, Ω^4 represents a hyperbolic surface of genus one, two cusps, and three conformal holes with lengths of their closed geodesics $2\kappa_1$, $2\kappa_5$, and $2\kappa_5$.
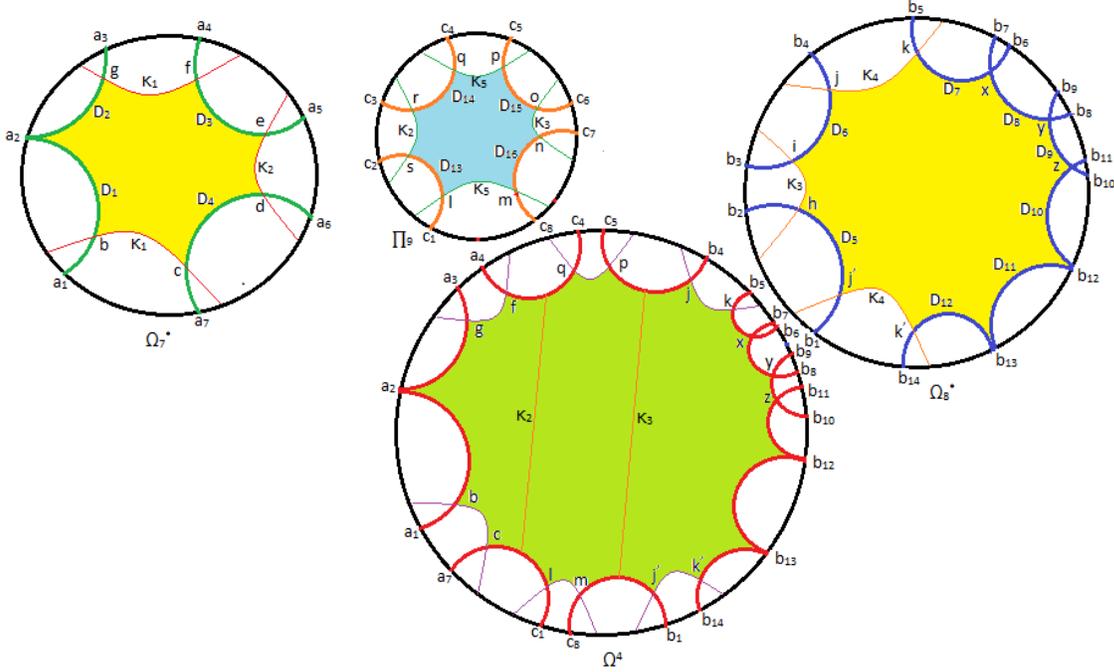

Figure : 7.1

Proof. Let D_1, D_2, D_3, and D_4 be four semi-circles within the Poincaré disk model. In particular, D_1 is paired with D_2 by the parabolic Möbius transformation p_1, whereas D_3 twins with D_4 by the hyperbolic Möbius transformation h_1. Assume that these four semi-circles meet the boundary of the disk at the points $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$ respectively. Now, on the boundary of the disk, we take six points, one between a_1 and a_2, viz., $\tilde{b}$ (near the point a_1), one between a_2 and a_3, viz., $\tilde{g}$ (near the point a_3), two between a_4 and a_5, viz., $\tilde{f}$ (near the point a_4) and $\tilde{e}$ (near the point a_5), and finally two, between the points a_6 and a_7, viz., $\tilde{d}$ (near the point a_6) and $\tilde{c}$ (near the point a_7). We draw the non-Euclidean curve lines by joining the points $\tilde{b}$ to $\tilde{c}$, $\tilde{g}$ to $\tilde{f}$, and $\tilde{e}$ to $\tilde{d}$. Notice that, these four curves intersect in the four semi-circles D_1, D_2, D_3, and D_4 at the points $b; g; f, e; d$, and c respectively. Suppose the length of the curve line $\overline{bc}$ (or $\overline{fg}$) is κ_1, and $\overline{de}$ is κ_2. Observe that the region (denoted by F_6) joining the points a_2, b, c, d, e, f, g, and a_2 represents the fundamental domain of a Fuchsian group ($\Omega_7 = < p_1, h_1 >$) with signature $(0,0,2,1,0)$. In particular, the Fuchsian group Ω_7 provides a hyperbolic Riemann surface with

one cusp and two conformal holes where the lengths of the closed geodesic boundaries are $2\kappa_1$ and κ_2. Now, we aim to consider another group Ω_8 of signature $(1, 0, 2, 1, 0)$ in the following way.

Let D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}, and D_{12} be eight semi-circles within the Poincaré disk model. In fact, D_5 is linked with D_6 by the hyperbolic Möbius transformation h_2, whereas D_7 twins with D_9 by the hyperbolic Möbius transformation h_4, and D_{10} is paired with D_8 by the hyperbolic Möbius transformation h_3; On the other hand, D_{12} is linked with D_{11} by the parabolic Möbius transformation p_2. Suppose these eight semi-circles D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11} and D_{12} meet the boundary of the disk at the points $b_1, b_2; b_3, b_4; b_5, b_6; b_7, b_8$, $b_9, b_{10}; b_{11}, b_{12}; b_{12}, b_{13}; b_{13}, b_{14}$ respectively. Now, on the boundary of the disk, we take six points, one between b_{14} and b_{13}, viz., $\tilde{k}'$ (near the point b_{14}), one between b_5 and b_6, viz., $\tilde{k}$ (near the point b_5), two between b_1 and b_2, viz., $\tilde{j}'$ (near the point b_1) and $\tilde{h}$ (near the point b_2), and finally two, between the points b_3 and b_4, viz., $\tilde{i}$ (near the point b_3) and $\tilde{j}$ (near the point b_4). We draw three non-Euclidean curve lines by joining the points $\tilde{k}'$ to $\tilde{j}'$, $\tilde{h}$ to $\tilde{i}$, and $\tilde{j}$ to $\tilde{k}$. Clearly, these three curves intersect in the four semi-circles D_{12}, D_5, D_6, and D_7 at the points $k'; j', h; i, j;$ and k respectively. We assume that the length of the curve line $\overline{j'k'}$ (or $\overline{jk}$) is $2\kappa_4$, and $\overline{hi}$ is κ_3. Furthermore, we suppose that the semi-circles D_7 and D_8 intersect within the poincaré disk at the point 'x', whereas D_8 and D_9 meet at 'y'; on the other hand, D_9 and D_{10} intersect at the point 'z'. Now, we assert that the region (denoted by F_7) joining the points $h, j', k', b_{13}, b_{12}, z, y, x, k, j, i$, and h represents the fundamental domain of a Fuchsian group $(\Omega_8 = < h_2, h_3, h_4, p_2 >)$ with signature $(1, 0, 2, 1, 0)$. In particular, the group Ω_8 provides a hyperbolic Riemann surface with one genus, one cusp end, and two conformal holes where the lengths of the closed geodesic boundaries are $2\kappa_4$ and κ_3.

Now, if we wish to attach the groups, Ω_7 and Ω_8, two cases will arise:

(i) If possible, let $\kappa_2 = \kappa_3$. Then, one can easily glue these two groups Ω_7 and Ω_8 in a direct way. This act indicates a Fuchsian group with signature $(1, 0, 2, 2, 0)$. Suppose this will not happen. Then, one can move forward to the second case.

(ii) Assume that $\kappa_2 \neq \kappa_3$. So, in this process, we can not attach Ω_7 and Ω_8 directly. Now, to glue these two groups, we are going to fabricate a new group in the subsequent way:

Let D_{13}, D_{14}, D_{15}, and D_{16} be four semi-circles within the Poincaré disk model. In fact, D_{13} is linked with D_{14} by the hyperbolic Möbius transformation h_4, on the other hand, D_{16} is paired with D_{15} by the hyperbolic Möbius transformation h_5. Suppose these four semi-circles D_{13}, D_{14}, D_{15}, and D_{16} meet the boundary of the disk at the points $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$

respectively. Now, on the boundary of the disk, we take eight points, two between c_1 and c_2, viz., $\tilde{l}$ (near the point c_1) and $\tilde{s}$ (near the point c_2), two between c_3 and c_4, viz., $\tilde{r}$ (near the point c_3) and $\tilde{q}$ (near the point c_4), two between c_5 and c_6, viz., $\tilde{p}$ (near the point c_5) and $\tilde{o}$ (near the point c_6), and finally two between c_7 and c_8, viz., $\tilde{n}$ (near the point c_7) and $\tilde{m}$ (near the point c_8). Now, we draw four non-Euclidean lines by joining the points $\tilde{l}$ to $\tilde{m}$, $\tilde{s}$ to $\tilde{r}$, $\tilde{q}$ to $\tilde{p}$, and $\tilde{o}$ to $\tilde{n}$. Suppose these four curves intersect in the four semi-circles D_{13}, D_{14}, D_{15}, and D_{16} at the points $l, s; r, q; p, o; n$, and m. Let the lengths of the curve lines $\overline{rs}$, $\overline{on}$, and $\overline{lm}$ (or $\overline{pq}$) be κ_2 κ_3, $2\kappa_4$ respectively. The region (denoted by F_8) joining the points s, l, m, n, o, p, q, r, and s represents the fundamental domain of the Fuchsian group ($\Pi_9 = <h_4, h_5>$) with signature $(0, 0, 3, 0, 0)$.

Now, to glue the two groups Ω_7 and Ω_8, we just need to introduce another group Π_9 in the scenario. First, we take two groups Ω_7 and Π_{10} in such a way that the curve lines (i.e., $\overline{de}$ in Ω_7 and $\overline{rs}$ in Π_9) can match, i.e, their lengths are the same (i.e., κ_2) but directions are opposite to each other. After this operation, clearly, by utilizing the combination theorem, we get a new group $<\Omega_7, \Pi_9> = \Gamma_1$ with signature $(0, 0, 3, 1, 0)$. Furthermore, we will proceed to glue the groups Γ_1 and Ω_8 on the analogy of the technique that we have just carried out in the above (i.e., by matching the curve lines $\overline{on}$ in Γ_1 and $\overline{hi}$ in Ω_8). Now, again, by using the combination theorem, we achieve a new group $\Gamma_2 = <\Gamma_1, \Omega_8>$ $(= \Omega^4$, say$)$ with signature $(1, 0, 3, 2, 0)$. Therefore, the fundamental region of the group Γ_2 $(= \Omega^4$, say$)$ is the domain bounded by the non-Euclidean lines joining the points $a_2, b, c, l, m, j', k', b_{13}, b_{12}, z, y, x, k, j, p, q, f, g$, and a_2. This completes the proof of Lemma 7.1.   □

8. Proof of the second and third Main Theorems

In this section, we have formulated a general strategy for constructing a finite type Riemann surface of type $(2, 0, 1, 2, 8)$ by proving Theorem 1.2. After that, we have initiated the structures of non-compact hyperbolic Riemann surfaces of types $(0, 0, 0, 0, \infty)$, $(0, 0, 0, \infty, \infty)$, $(0, 0, \infty, 0, \infty)$, and $(0, 0, \infty, \infty, \infty)$ via fundamental groups by establishing Theorem 1.3.

Proof of Theorem 1.2. Step 1. Let, Ω_1 and Ω_2 be two Fuchsian groups with signatures $(1, 0, 0, 0, 2)$ and $(0, 0, 1, 0, 2)$ respectively. Let, SC_1, SC_2, SC_3, SC_4, SC_5, SC_6, SC_7, and SC_8 be eight semi-circles within the Poincaré disk model. In particular, SC_1 is paired with SC_3 by the hyperbolic Möbius transformation h_1, SC_4 is linked with SC_2 by the hyperbolic Möbius transformation h_2, SC_8 twins with SC_5 by the hyperbolic Möbius transformation h_3, and SC_7 is paired with SC_6 by the hyperbolic Möbius transformation h_4. We assume that these eight semi-circles meet the boundary of the disk at the points

$a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8; a_8, a_9; a_{10}, a_{11}; a_{12}, a_{13}; a_{14}, a_1$ respectively. Suppose, within the poincaré disk, SC_1 meets SC_2 at the point x_1, SC_2 intersects SC_3 at x_2, and SC_3 touches SC_4 at the point x_3. Now, on the boundary of the disk, we take two points, one between a_{12} and a_{13}, viz., $\tilde{A}_1$ (near the point a_{12}) and one between a_{10} and a_{11}, viz., $\tilde{A}_2$ (near the point a_{11}). We draw the non-Euclidean curve line by joining the points $\tilde{A}_1$ to $\tilde{A}_2$. The curve line $\overline{\tilde{A}_1\tilde{A}_2}$ intersects in the two semi-circles SC_6 and SC_7 at the points A_1 and A_2. Let the lengths of the curve line $\overline{\tilde{A}_1\tilde{A}_2}$ be κ_1. Furthermore, we assume that the axis of the hyperbolic element h_3 intersects at the point x_4 at SC_8 and x_5 at SC_5. Now, the region (denoted by R_1) joining the points $a_1, x_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{11}, a_{12}, A_1, a_{13}, a_{14}, x_4$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_1 = < h_1, h_2, h_3, h_4 >$) with signature $(1, 0, 0, 0, 2)$. Now, we consider the other group Ω_2 with signature $(0, 0, 1, 0, 2)$. We proceed in a similar way as we have performed in the above construction. Now, the region (denoted by R_2) joining the points $a_{16}, A_3, A_7, A_8, A_6, a_{21}, a_{20}, A_5, A_{10}, A_9, A_4, a_{17}$, and a_{16} represents a fundamental domain of a Fuchsian group (denoted by $\Omega_2 = < h_5, h_6 >$) with signature $(0, 0, 1, 0, 2)$. Note that Ω_2 is a Fuchsian group with signature $(0, 0, 1, 0, 2)$ consisting of one closed geodesic boundary of length $2\kappa_2\kappa_3$, and two funnels with neck lengths κ_2, and κ_3. Let us consider a group Ω_3 with signature $(0, 0, 2, 0, 1)$ consisting one funnel end of neck length $2\kappa_1\kappa_2$ and two conformal holes with lengths of their closed geodesic boundaries κ_1 and κ_2. Now, to glue Ω_1 and Ω_2 with signatures $(1, 0, 0, 0, 2)$ and $(0, 0, 1, 0, 2)$, we cut the two corresponding surfaces one funnel end each along the neck of their funnels. After this operation, it reduces two groups, one is Ω_1^* and the other is Ω_2^*. Then, adding a group Ω_3 in such a way that the neck length can match the groups Ω_1^* and Ω_2^*. Note that, when we paste with Ω_1^* to Ω_3 the curve lines (i.e., $\overline{A_1A_2}$ in Ω_1^* and $\overline{A_{12}A_{11}}$ in Ω_3) can match, i.e, their lengths are the same and directions are opposite to each other. Now, by using the combination theorem, we obtain a Fuchsian group $\Omega_{13} = < \Omega_1^*, \Omega_3 >$ with signature $(1, 0, 1, 0, 2)$. After that, we will paste the groups Ω_{13} and Ω_2^* on the analogy of the technique that we have mentioned just before (i.e., matching the curve lines $\overline{A_{13}A_{14}}$ in Ω_{13} and $\overline{A_3A_4}$ in Ω_2). Consequently, again, one can utilize the combination theorem to get a Fuchsian group $\Omega_{132} = < \Omega_{13}, \Omega_2^* >$ with signature $(1, 0, 1, 0, 3)$. Therefore, the fundamental domain of the arising group Ω_{132} is the region bounded by the non-Euclidean lines joining the points $a_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{26}, a_{27}, A_4, A_9, A_{10}, a_{20}, a_{21}, A_8, A_7, A_3, a_{30}, a_{23}, A_1, a_{13}, a_{14}, x_4$, and a_1.

Step 2. Let us consider two Fuchsian groups, Ω_{132} and Ω_4 with signatures $(1, 0, 1, 0, 3)$ and $(0, 0, 1, 1, 2)$ respectively. To glue these two groups, we apply the same technique as we have

operated in the preceding, i.e., cutting one funnel end each for the groups Ω_{132} and Ω_4 along the neck of their corresponding funnels and adding a new group Ω_5 with signature $(0, 0, 2, 0, 1)$. Note that, the region joining the points $B_1, B_1^*, b_1, b_8, B_2^{**}, B_4, B_3, B^{**}, b_5, b_4, B^*, B_2$, and B_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_5 = <h_9, h_{10}>$) with signature $(0, 0, 2, 0, 1)$. The domain joining the points $B_5, B_7', b_9, b_{18}, B_8', b_{17}, b_{16}, B_{10}, B_9, B_8, b_{13}, b_{12}, B_7, B_6$ and B_5 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_4^* = <h_{11}, h_{12}>$) with signature $(0, 0, 2, 1, 1)$. Now, to glue two Fuchsian groups, Ω_{132} and Ω_4 with signatures $(1, 0, 1, 0, 3)$ and $(0, 0, 1, 1, 2)$, we cut the two Fuchsian surface groups, one funnel end each, along the neck of their funnels. After this action, it reduces two groups, one is Ω_{132}^* and the other is Ω_4^*. Then adding a Fuchsian group Ω_5 in such a way that the neck length can match the groups Ω_{132}^* and Ω_4^*. Observe that, when we attach Ω_{132}^* to Ω_5, the curve lines (i.e., $\overline{A_5 A_6}$ in Ω_{132}^* and $\overline{B_1 B_2}$ in Ω_4^*) can match, i.e, their lengths are the same and directions are opposite to each other. Now, we use the combination theorem to obtain a Fuchsian group $\Omega_{1325} = <\Omega_{132}^*, \Omega_5>$ with signature $(1, 0, 2, 0, 3)$. After that, we will glue the groups Ω_{1325} and Ω_4^* on the harmony of the technique that we have mentioned just before (i.e., the direction of the curve lines $\overline{B_3 B_4}$ in Ω_{1325} and $\overline{B_5 B_6}$ in Ω_4^*). Again, one can utilize the combination theorem to get a Fuchsian group $\Omega_{13254} = <\Omega_{1325}, \Omega_4^*>$ with signature $(1, 0, 2, 1, 4)$. Therefore, the fundamental domain of the arising group Ω_{13254} is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B_7', b_9, b_{18}, B_8', b_{17}, B^*, b_{16}, B_{10}, B_9, b_{13},$ $b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$, and a_1.

Step 3. Furthermore, we take two Fuchsian groups, one is our constructed group, i.e., Ω_{13254}, and the other is Ω_6 with signature $(0, 0, 1, 0, 3)$ respectively. To glue these two groups, we simply apply a direct procedure, keeping in mind that the attaching sides can match. In fact, we take the group Ω_6 in such a way that this operation can happen. Clearly, from Figure 8(a), the region joining the points $c_1, c_{12}, C_3, C_5', c_{11}, c_{10}, C_6', C_8, c_9, C_8, C_6, c_7, c_6, C_5, C_4, c_5, c_4, C_2, C_1$, and c_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_6 = <h_{13}, h_{14}, h_{15}>$) with signature $(0, 0, 1, 0, 3)$. Now, to glue Ω_{13254} and Ω_6 with signatures $(1, 0, 2, 1, 4)$ and $(0, 0, 1, 0, 3)$, we need to take care of the lengths of the closed geodesic boundaries of the corresponding conformal holes, i.e., the curve lines $\overline{C_1 C_2}$ in Ω_6 and $\overline{B_9 B_{10}}$ in Ω_{13254} can match. Now, after utilizing the combination theorem, we get a group $\Omega_{132546} = <\Omega_{13254}, \Omega_6>$ with signature $(1, 0, 1, 1, 7)$. So, the fundamental domain of the arising group Ω_{132546} is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B_7', b_9, b_{18}, B_8', b_{17}, B^*, b_{16}, C_1, c_1, c_{12},$

$C_3, C'_5, c_{11}, c_{10}, C'_6, C_8, c_9, c_8, C_6, c_7, c_6, C_5, C_4, c_5, c_4, B_9, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26},$
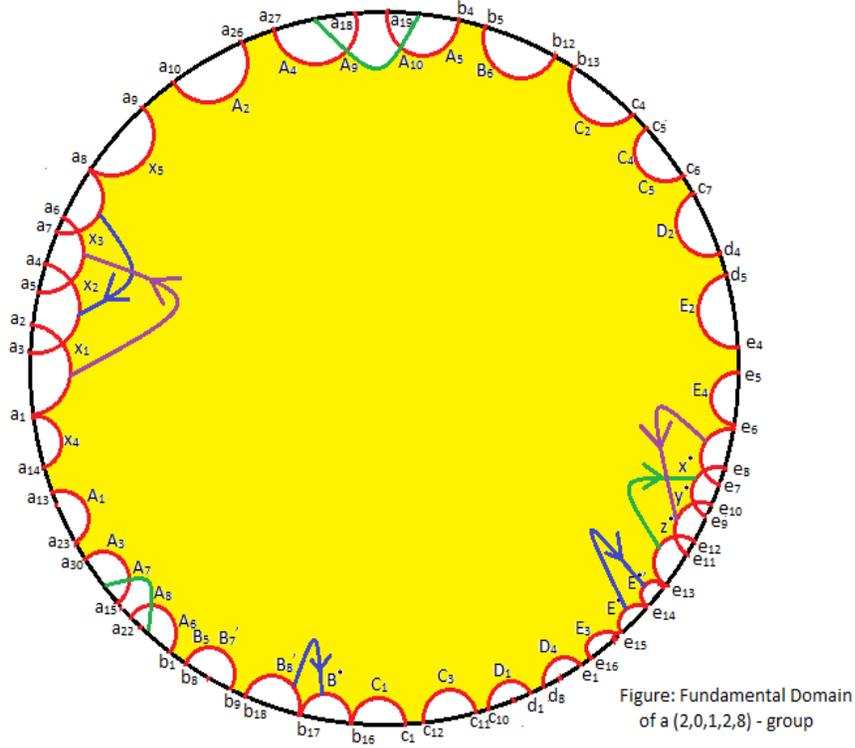$A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1,$ and a_1.

Figure: Fundamental Domain of a (2,0,1,2,8) - group

Figure : 8(a)

Step 4. Finally, we take two Fuchsian groups, Ω_{132546} and Ω_7 with signatures $(1, 0, 1, 1, 7)$ and $(1, 0, 0, 1, 2)$ respectively. To attach these two groups, we apply the same technique as we have performed in the overhead construction, i.e., cutting one funnel end each for the groups Ω_{132546} and Ω_7 along the neck of their corresponding funnels and adding a new group Ω_8 with signature $(0, 0, 2, 0, 1)$. We take the region joining the points $D_1, d_1, d_8, D_4, D_3, d_5, d_4, D_2,$ and D_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_8 = <h_{16}, h_{17}>$) with signature $(0, 0, 2, 0, 1)$. Now, the domain joining the points $e_2, E_1, e_1, e_{16}, E_3, e_{15}, e_{14}, e_{13}, z^*, y^*, x^*, e_6,$ $E_4, e_5, e_4, E_2, e_3,$ and e_2 represents a fundamental domain of a Fuchsian group (denotes $\Omega_7 = <h_{18}, h_{19}, h_{20}, h_{21}>$) with signature $(1, 0, 0, 1, 2)$. Now, to glue two Fuchsian groups Ω_{132546} and Ω_7 with signatures $(1, 0, 1, 1, 7)$ and $(1, 0, 0, 1, 2)$, we cut the two Fuchsian surface groups at one funnel end each, along the neck of their funnels. After this operation, it reduces two groups, one is Ω^*_{132546} and the other is Ω^*_7. Then, adding a Fuchsian group Ω_8 in such a way that the neck length can match the groups Ω^*_{132546} and Ω^*_7. Note that, when we paste the group Ω^*_{132546}

with Ω_8 the curve lines (i.e., $\overline{C_7C_8}$ in Ω^*_{132546} and $\overline{D_1D_2}$ in Ω^*_8) can match. Now, we again use the combination theorem to obtain a Fuchsian group $\Omega_{1325468} = <\Omega^*_{132546}, \Omega^*_8>$ with signature $(1, 0, 1, 1, 8)$. After that, we will attach the groups $\Omega_{1325468}$ and Ω^*_7 on the harmony of the technique that we have mentioned in the above construction (i.e., the direction of the curve lines $\overline{D_3D_4}$ in $\Omega_{1325468}$ and $\overline{E_1E_2}$ in Ω^*_7). Again, by utilizing the combination theorem, we get a Fuchsian group $\Omega_{13254687} = <\Omega_{1325468}, \Omega^*_7>$ (say, Ω^*) with signature $(2, 0, 1, 2, 8)$. Hence, the fundamental domain of the arising group $\Omega_{13254687}$ is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B'_7, b_9, b_{18}, B'_8, b_{17}, B^*, b_{16}, C_1, c_1, c_{12},$ $C_3, c_{11}, c_{10}, D_1, d_1, d_8, E_1, e_1, e_{16}, E_3, E_{15}, E^*, e_{14}, E^{*'}, e_{13}, z^*, y^*, x^*, e_6, E_4, e_5, e_4, E_2, d_5, d_4, D_2, c_7,$ $c_6, C_5, C_4, c_5, c_4, C_2, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1,$ and a_1.

So, Theorem 1.1 is proven. This establishes the second main theorem of this article.

Now, our goal is to construct four kinds of infinite type generalized non-tight flute surfaces and four kinds of infinite type generalized tight flute surfaces via fundamental groups. At first, we look into non-tight surfaces. After that, we have investigated the tight ones.

Proof of Theorem 1.3 (a). **(**Constructing infinite type generalized non-tight flute surface: Type 1, see Figure 8 (b)**)**.

Let, Γ_1 $(=<h_1, h_2, h_3>)$ and Γ_2 $(=<h_4, h_5, h_6>)$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 4)$ consisting of neck lengths κ_1, κ_2, κ_3, κ_4, and κ_5, κ_6, κ_7, κ_8 respectively, where h_i's are hyperbolic elements. Now, we wish to attach these two groups. For this purpose, we cut the two corresponding surfaces of the groups Γ_1 and Γ_2 (reduce to Γ^*_1 and Γ^*_2), one funnel end each along the neck of their funnels where the lengths of their necks are κ_4 and κ_5 respectively. After that, we add a group $\Gamma_3 = <h_7, h_8, h_9>$ of signature $(0, 0, 2, 0, 2)$ consisting of two funnel ends of neck lengths κ_9, κ_{10}, and two conformal holes with lengths of their closed geodesic boundaries are κ_4 and κ_5. In particular, we take Γ_3 in such a way that it can match with the groups Γ^*_1 and Γ^*_2. So, taking the notion of gluing Fuchsian groups from the above sections, we use the combination theorem to get a Fuchsian group $\Gamma_{132} = <\Gamma^*_1, \Gamma_3, \Gamma^*_2>$ of signature $(0, 0, 0, 0, 8)$. In the second step, we consider two groups, one is Γ_{132} and the other is $\Gamma_4 = <h_{10}, h_{11}, h_{12}>$ of signature $(0, 0, 0, 0, 4)$, consisting of four funnel ends of neck lengths δ_{11}, δ_{12}, δ_{13}, and δ_{14}. Then, we proceeded in a similar way as we have carried out in the preceding construction, i.e., cut the two corresponding surfaces of the groups Γ_{132} and Γ_4 (transform to Γ^*_{132} and Γ^*_4), one funnel end each along the neck of their funnels where the lengths of their necks are δ_7 and δ_{11} respectively. Furthermore, we add a group $\Gamma_5 = <h_{13}, h_{14}, h_{15}>$ of signature

$(0, 0, 2, 0, 2)$ with neck lengths δ_{15}, δ_{16}, and closed geodesic boundary lengths are δ_7 and δ_{11}. Now, we again apply the combination theorem to obtain a group $\Gamma_{13254} = < \Gamma^*_{132}, \Gamma_5, \Gamma^*_4 >$ of signature $(0, 0, 0, 0, 12)$. So, in this way, one can continue this process for an infinite number of times to get an infinitely generated group $\Gamma_{13254\ldots} = < \Gamma^*_1, \Gamma_3, \Gamma^{**}_2, \Gamma_5, \Gamma^{**}_4, \Gamma_7, \Gamma^{**}_6, \ldots > = < h_1, h_2, h_3, h_4, \ldots >$. This infinitely generated group is of type $(0, 0, 0, 0, \infty)$ which provides an infinite type generalized non-tight flute surface where each block carries an infinite hyperbolic area.

Proof of Theorem 1.3 (b). **(**Constructing infinite type generalized non-tight flute surface: Type 2, see Figure 8 (c)**)**.

Let, $\Delta_1 = < h_1, h_2 >$ and $\Delta_2 = < h_3, h_4 >$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 3)$ consisting of neck lengths α_1, α_2, α_3, and α_4, α_5, α_6 respectively, where h_i's are hyperbolic elements. Now, we want to glue these two groups. So, we cut the two corresponding surfaces of the groups Δ_1 and Δ_2 (changing to Δ^*_1 and Δ^*_2), one funnel end each, along the neck of their funnels, where the lengths of their necks are α_3 and α_4 respectively. After that, we add a group $\Delta_3 = < h_5, h_6 >$ of signature $(0, 0, 2, 0, 1)$ consisting of one funnel end of neck length α_7, and two conformal holes with lengths of their closed geodesic boundaries are α_3 and α_4. More precisely, we consider Δ_3 in such a way that it can match with the groups Δ^*_1 and Δ^*_2. Now, we use the combination theorem to obtain a group $\Delta_{132} = < \Delta^*_1, \Delta_3, \Delta^*_2 >$ of signature $(0, 0, 0, 0, 5)$. In the second stage, we take two groups, one is Δ_{132} and the other is Δ_4 of signature $(0, 0, 0, 0, 3) = < h_7, h_8 >$. After that, we move forward in an analogous way as we have carried out earlier, i.e., cut the two corresponding surfaces of the groups Δ_{132} and Δ_4 (transforming to Δ^*_{132} and Δ^*_4), one funnel end each along the neck of their funnels, where the lengths of their necks are α_6 and α_8 respectively. Further, we add a group $\Delta_5 = < h_9, h_{10} >$ of signature $(0, 0, 2, 0, 1)$ with neck length α_{11}, and closed geodesic boundaries lengths are α_6 and α_8. Now, we again utilize the combination theorem to obtain $\Delta_{13254} = < \Delta^*_{132}, \Delta_5, \Delta^*_4 >$ of signature $(0, 0, 0, 0, 7)$. Therefore, in this way, one can continue this process for an infinite number of times to get an infinitely generated group $\Delta_{13254\ldots} = < \Delta^*_1, \Delta_3, \Delta^{**}_2, \Delta_5, \Delta^{**}_4, \Delta_7, \Delta^{**}_6, \ldots > = < h_1, h_2, h_3, h_4, \ldots >$. This infinitely generated group is also of type $(0, 0, 0, 0, \infty)$ that supplies an infinite type of generalized non-tight flute surfaces, where each block is of infinite hyperbolic area, but each block of the Fuchsian surface group $\Delta_{132547698\ldots}$ contains one funnel end (apart from the first one), whereas for the group $\Gamma_{13254\ldots}$, two funnels appear in each block.

Proof of Theorem 1.3 (c). **(**Constructing infinite type generalized non-tight flute surface: Type 3, see Figure 8(d)**)**.

Let $\Omega_1 = <h_1, h_2>$ and $\Omega_2 = <h_3, h_4>$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 3)$ consisting neck lengths β_1, β_2, β_3, and β_4, β_5, β_6 respectively. Now, to glue these two groups Ω_1 and Ω_2, we cut the two corresponding surfaces (reduce to Ω_1^* and Ω_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are β_3 and β_4 respectively. After this operation, we add a group $\Omega_3 = <h_5, h_6>$ of signature $(0, 0, 3, 0, 0)$ consisting three conformal holes with lengths of their closed geodesic boundaries being β_3, β_4, and β_7. In fact, we take Ω_3 in such a way that it can match with the groups Ω_1^* and Ω_2^*. After that, we use the combination theorem to obtain a new Fuchsian group $\Omega_{132} = <\Omega_1^*, \Omega_3, \Omega_2^*>$ of signature $(0, 0, 1, 0, 4)$. In the next stage, we take two groups, one is Ω_{132} and the other is $\Omega_4 = <h_7, h_8>$ of signature $(0, 0, 0, 0, 3)$. Then, we procced a similar path as we have done in the earlier construction, i.e., cut the two corresponding surfaces of the groups Ω_{132} and Ω_4 (transform to Ω_{132}^* and Ω_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are β_6 and β_8 respectively. After that, we add a group $\Omega_5 = <h_9, h_{10}>$ of signature $(0, 0, 3, 0, 0)$ with closed geodesic boundaries lengths are β_6, β_8, and β_{11}. So, again utilizing the combination theorem, we obtain $\Omega_{13254} = <\Omega_{132}^*, \Omega_5, \Omega_4^*>$ of signature $(0, 0, 2, 0, 5)$. Consequently, in this fashion, one can continue this process for an infinite number of times to get an infinitely generated group $\Omega_{13254...} = <\Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, ...> = <h_1, h_2, h_3, h_4, ...>$. This infinitely generated group is of type $(0, 0, \infty, 0, \infty)$ that indicates an infinite type generalized non-tight flute surface, where 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area, but, 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e., π), h_i's are hyperbolic elements, $i \in \mathbb{N}$.

Proof of Theorem 1.3 (d). (Constructing infinite type generalized non-tight flute surface: Type 4, see Figure 8(e)).

Let, Φ_1 $(= <h_1, h_2, h_3>)$ and Φ_2 $(= <h_4, h_5, h_6>)$ be two Fuchsian surface groups with signature $(0, 0, 1, 0, 3)$ consisting one conformal hole each with lengths of their closed geodesics being γ_4 and γ_8 respectively and three funnel ends (each) of neck lengths γ_1, γ_2, γ_3, and γ_5, γ_6, γ_7 respectively. Now, we wish to glue these two groups, but assume that $\gamma_4 \neq \gamma_8$. So, we cut the two corresponding surfaces of the groups Φ_1 and Φ_2 (changing to Φ_1^* and Φ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are γ_3 and γ_5 respectively. After that, we add a group Φ_3 $(= <h_7, h_8, h_9>)$ of signature $(0, 0, 3, 0, 1)$ consisting one funnel end of neck length γ_9, and three conformal holes with lengths of their closed geodesic boundaries are γ_3, γ_5, and γ_{10}. In particular, we take Φ_3 in such a way that it can match with the groups Φ_1^* and Φ_2^*. So, by using the combination theorem, we get Π_{132}

$= < \Phi_1^*, \Phi_3, \Phi_2^* >$ of signature $(0,0,3,0,5)$. In the second stage, we consider two groups one is Π_{132} and other is Φ_4 $(=< h_{10}, h_{11}, h_{12} >)$ of signature $(0,0,1,0,3)$ with neck length γ_{12}, and lengths of the closed geodesic boundaries are γ_{11}, γ_{13}, and γ_{14}. Then, we procced in a similar way as we have operated in the earlier construction, i.e., cut the two corresponding surfaces of the groups Π_{132} and Φ_4 (reduce to Π_{132}^* and Φ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_7 and γ_{11} respectively. After that, we add Φ_5 $(=< h_{13}, h_{14}, h_{15} >)$ of signature $(0,0,3,0,1)$ with neck length γ_{15}, and closed geodesic boundaries lengths are γ_7, γ_{11}, and γ_{16}. Now, again utilizing the combination theorem, we obtain a Fuchsian group $\Pi_{13254} = < \Pi_{132}^*, \Phi_5, \Phi_4^* >$ of signature $(0,0,5,0,7)$. Therefore, in this way, we can continue this process for an infinite number of times to get an infinitely generated group $\Pi_{13254...} = < \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, ... > = < h_1, h_2, h_3, h_4, ... >$. This infinitely generated group is also of type $(0,0,\infty,0,\infty)$ that provides an infinite type generalized non-tight flute surface, but, here each block carries an infinite hyperbolic area, h_i's are hyperbolic elements, $i \in \mathbb{N}$.

In the following, we have constructed four new kinds of infinite type generalized tight flute surfaces via fundamental (non-compact Fuchsian) groups.

Proof of Theorem 1.3 (e). **(**Constructing infinite type generalized tight flute surface: Type 1, see Figure 8(f)**).**

Let, Ξ_1 $(=< h_1, h_2, p_1 >)$ and Ξ_2 $(=< h_3, h_4, p_2 >)$ be two Fuchsian surface groups with signature $(0,0,0,1,3)$ consisting neck lengths α_1, α_2, α_3, and α_4, α_5, α_6, where h_i's are hyperbolic elements and where p_i's are parabolic elements respectively. Now, to obtain a new surface we will glue these two groups. For this reason, we cut the two corresponding surfaces of the groups Ξ_1 and Ξ_2 (reduce to Ξ_1^* and Ξ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are α_3 and α_4 respectively. After cutting, we add Ξ_3 $(=< h_5, h_6, p_3 >)$ of signature $(0,0,2,1,1)$ consisting one funnel end of neck length α_7, one cusp, and two conformal holes with lengths of their closed geodesic boundaries are α_3 and α_4. In particular, we consider Ξ_3 in such a way that it helps to match with the groups Ξ_1^* and Ξ_2^*. Now, we can use the combination theorem to obtain a group $\Xi_{132} = < \Xi_1^*, \Xi_3, \Xi_2^* >$ of signature $(0,0,0,3,5)$. In the next stage, we take two groups one is Ξ_{132} and other is Ξ_4 $(=< h_7, h_8, p_4 >)$ of signature $(0,0,0,1,3)$ with neck lengths α_8, α_9, and α_{10}. After that, we procced a similar way as we have approached in the construction just above, i.e., cut the two corresponding surfaces of the groups Ξ_{132} and Ξ_4 (reduce to Ξ_{132}^* and Ξ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are α_6 and α_8 respectively.

Now, we move forward in a similar path as we have operated in the preceding framework, i.e., cutting the corresponding surfaces arise from the groups Ξ_{132} and Ξ_4 (reduce to Ξ_{132}^* and Ξ_4^*), one funnel end each along the neck of their funnels, where the lengths of their necks are α_6 and α_8 respectively. After that, we add Ξ_5 ($=< h_9, h_{10}, p_5 >$) of signature $(0,0,2,1,1)$ with neck length α_{11}, and closed geodesic boundaries lengths are α_6 and α_8. Now, again applying the combination theorem, we obtain $\Xi_{13254} = < \Xi_{132}^*, \Xi_5, \Xi_4^* >$ of signature $(0,0,5,0,7)$. So, in this way, we can continue this process for an infinite number of times to get an infinitely generated group $\Xi_{13254...} = < \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, ... > = < h_1, h_2, h_3, ...; p_1, p_2, p_3, ... >$. This infinitely generated group is of type $(0,0,0,\infty,\infty)$ that gives rise to an infinite type generalized tight flute surface, where each block is of infinite hyperbolic area, h_i's and p_i's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.3 (f). **(**Constructing infinite type generalized tight flute surface: Type 2, see Figure 8(g)**)**.

Let, Ψ_1 ($=< h_1, h_2 >$) and Ψ_2 ($=< h_3, h_4 >$) be two Fuchsian surface groups with signature $(0,0,0,0,3)$ consisting neck lengths β_1, β_2, β_3, and β_4, β_5, β_6 respectively. Now, we aim to glue these two groups to obtain a new surface. So, we cut the two corresponding surfaces of the groups Ψ_1 and Ψ_2 (reduce to Ψ_1^* and Ψ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are β_3 and β_4 respectively. After this operation, we add a Fuchsian group Π_3 ($=< h_5, p_1 >$) of signature $(0,0,2,1,0)$ consisting one cusp end with two conformal holes where lengths of their closed geodesic boundaries are β_3 and β_4. In fact, we take Π_3 in such a way that it wil fit for the groups Ψ_1^* and Ψ_2^*. So, applying the combination theorem, we get $\Psi_{132} = < \Psi_1^*, \Pi_3, \Psi_2^* >$ of signature $(0,0,0,1,4)$. In the second stage, we take two groups one is Ψ_{132} and other is Ψ_4 ($=< h_6, h_7 >$) of signature $(0,0,0,0,3)$ of neck lengths β_7, β_8, and β_9. Now, we can continue in an analogous way as we have performed in the preceding formation, i.e., cutting the corresponding surfaces arise from the groups Ψ_{132} and Ψ_4 (reduce to Ψ_{132}^* and Ψ_4^*), one funnel end each along the neck of their funnels, where the lengths of their necks are β_6 and β_7 respectively. Then, we add a group Π_5 ($=< h_8, p_2 >$) of signature $(0,0,2,1,0)$, where the closed geodesic boundaries lengths are β_6 and β_7. Now, again we use the combination theorem and we obtain $\Psi_{13254} = < \Psi_{132}^*, \Pi_5, \Psi_4^* >$ of signature $(0,0,0,2,5)$. Hence, in this way, one can continue this process for an infinite number of times to get an infinitely generated group $\Psi_{13254...} = < \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, ... > = < h_1, h_2, h_3, ...; p_1, p_2, p_3, ... >$. This infinitely generated group also is of type $(0,0,0,\infty,\infty)$ which produces an infinite type generalized tight flute surface, but, unlike the previous case, here 1st, 3rd, 5th, 7th, ... blocks

are of infinite hyperbolic area, whereas 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e., π), h_i's and p_i's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.3 (g). **(**Constructing infinite type generalized tight flute surface: Type 3, see Figure 8(h)**)**.

Let, Σ_1 $(=< h_1, h_2 >)$ and Υ_1 $(=< h_3, p_1 >)$ be two Fuchsian surface groups with signatures $(0, 0, 0, 0, 3)$ and $(0, 0, 0, 1, 2)$ consisting neck lengths $\gamma_1, \gamma_2, \gamma_3$, and γ_4, γ_5 respectively. Now, our goal is to attach these two groups to obtain a new group. Now, we cut the two corresponding surfaces of the groups Σ_1 and Υ_1 (reduce to Σ_1^* and Υ_1^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_3 and γ_4 respectively. After cutting these surfaces, we add a Π_1 $(=< h_4, h_5 >)$ of signature $(0, 0, 3, 0, 0)$ consisting three conformal holes where lengths of their closed geodesic boundaries are γ_3, γ_4, and γ_6. In particular, we take Π_1 in such a way that it can match with the groups Σ_1^* and Υ_1^*. So, by utilizing the combination theorem, we get $\Omega_{111} = < \Sigma_1^*, \Pi_1, \Upsilon_1^* >$ of signature $(0, 0, 1, 1, 3)$. In the second step, we consider two groups one is Ω_{111} and other is Π_2 $(=< h_6, h_7 >)$ of signature $(0, 0, 1, 0, 2)$ consisting one conformal hole with length of the closed geodesic is γ_9, and two funnels end of neck length γ_7 and γ_8. After that, we will continue in a similar fashion as we have performed in the overhead construction, i.e., cut the two corresponding surfaces of the groups Ω_{111} and Π_2 (transform to Ω_{111}^* and Π_2^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_5 and γ_7 respectively. Furthermore, we add Σ_2 $(=< h_8, h_9 >)$ of signature $(0, 0, 2, 0, 1)$ with neck length γ_{10}, and the lengths of closed geodesic boundaries are γ_5 and γ_7. Now, by applying the combination theorem, we get $\Omega_{11122} = < \Omega_{111}^*, \Sigma_2, \Pi_2^* >$ of signature $(0, 0, 2, 1, 4)$. In the third step, we further take two groups one is Ω_{11122} and other is Σ_3 $(=< h_{10}, h_{11} >)$ of signature $(0, 0, 0, 0, 3)$ consisting three funnel ends with neck lengths γ_{11}, γ_{12}, and γ_{13}. Furthermore, we procced in a similar way as we have carried out in the preceeding formation, i.e., cut the two corresponding surfaces of the groups Ω_{11122} and Σ_3 (convert to Ω_{11122}^* and Σ_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_8 and γ_{11} respectively. In this situation, we again add a group Υ_2 $(=< h_{12}, p_2 >)$ of signature $(0, 0, 2, 1, 0)$, where length of the closed geodesic boundaries are γ_8 and γ_{11}. So, we can again utilize the combination theorem to get a new group $\Omega_{1112223} = < \Omega_{11122}^*, \Upsilon_2, \Sigma_3^* >$ of signature $(0, 0, 2, 2, 5)$. In the fourth step, we will again consider two groups one is $\Omega_{1112223}$ and other is Υ_3 $(=< h_{13}, p_3 >)$ of signature $(0, 0, 0, 1, 2)$ consisting one cusp end and two funnel ends with neck lengths γ_{14} and γ_{15}. Furthermore, we cut the two corresponding surfaces of the groups $\Omega_{1112223}$ and Υ_3 (reduce to $\Omega_{1112223}^*$ and Υ_3^*), one funnel end each along the neck of

their funnels where the lengths of their necks are γ_{13} and γ_{14} respectively to glue them to each other. Finally, we again add Π_3 $(=< h_{14}, h_{15} >)$ of signature $(0, 0, 2, 0, 1)$ with neck length γ_{16}, and the lengths of their closed geodesic boundaries are γ_{13}, γ_{14}, and γ_{16}. Hence, by using the combination theorem, we get $\Omega_{1112223333} = < \Omega_{1112223}, \Pi_3, \Upsilon_3^* >$ of signature $(0, 0, 3, 3, 5)$. Note that, this fourth step is basically a repetition of our previous first step. Therefore, in this way, one can continue this process for an infinite number of times to get an infinitely generated group $\Upsilon = \Omega_{111222333...} = < \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, ... > = < h_1, h_2, h_3, ...; p_1, p_2, p_3, ... >$. This group is of type $(0, 0, \infty, \infty, \infty)$ which provides an infinite type generalized tight flute surface, where 2nd, 5th, 8th, ... blocks are of non-compact infinite hyperbolic areas, whereas 2nd, 4th, 6th, ... blocks are of compact and hence finite areas (more precisely, π), on the other hand, 3rd, 6th, 9th, ... blocks are non-compact but finite hyperbolic areas, h_i's and h_i's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.3 (h). (Constructing infinite type generalized tight flute surface: Type 4, see 8(i)).

Let, Λ_1 $(=< h_1, h_2, h_3 >)$ and Ω_1 $(=< h_4, h_5, p_1 >)$ be two Fuchsian surface groups with signatures $(0, 0, 0, 0, 4)$ and $(0, 0, 0, 1, 3)$ consisting neck lengths δ_1, δ_2, δ_3, δ_4, and δ_5, δ_6, δ_7 respectively. Now, we aim to glue these two groups to obtain a new group. So, we cut the two corresponding surfaces of the groups Λ_1 and Ω_1 (reduce to Λ_1^* and Ω_1^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_4 and δ_5 respectively. After cutting these surfaces, we add Θ_1 of signature $(0, 0, 3, 0, 1)$ consisting three conformal holes where lengths of their closed geodesic boundaries are δ_4, δ_5, and δ_8. In particular, we take Θ_1 $(=< h_5, h_6, h_7 >)$ in such a way that it can match with the groups Λ_1 and Ω_2. So, by utilizing the combination theorem, we get $\Pi_{132} = < \Lambda_1^*, \Theta_1, \Omega_1^* >$ of signature $(0, 0, 1, 1, 6)$. In the second step, we consider two groups one is Π_{111} and other is Θ_2 $(=< h_8, h_9, h_{10} >)$ of signature $(0, 0, 3, 0, 1)$ consisting three conformal holes with length of their closed geodesics are δ_{10}, δ_{12}, δ_{13} and one funnel end of neck length δ_{11}. After that, we procced in a similar way as we have carried out in the preceding construction, i.e., cut the two corresponding surfaces of the groups Π_{111} and Θ_2 (convert to Π_{111}^* and Θ_2^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_7 and δ_{10} respectively. Furthermore, we add Λ_5 $(=< h_{11}, h_{12}, h_{13} >)$ of signature $(0, 0, 2, 0, 2)$ with neck lengths δ_{14}, δ_{15}, and closed geodesic boundaries lengths are δ_7 and δ_{10}. Now, we again use the combination theorem to obtain a new group $\Pi_{11122} = < \Pi_{111}^*, \Lambda_2, \Theta_2^* >$ of signature $(0, 0, 2, 1, 9)$. In the third step, we further take two groups one is Π_{11122} and other is Λ_3 $(=< h_{14}, h_{15}, h_{16} >)$ of signature $(0, 0, 0, 0, 4)$

consisting four funnel ends with neck lengths δ_{16}, δ_{17}, δ_{18}, and δ_{19}. Furthermore, we procced in a similar way as we have done in the overhead formation, i.e., cut the two corresponding surfaces of the groups Π_{11122} and Λ_3 (transform to Π_{11122}^* and Λ_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_{12} and δ_{16} respectively. Now, again, we add a group Ω_2 ($=< h_{17}, h_{18}, p_2 >$) of signature $(0, 0, 2, 1, 1)$ with neck length δ_{20}, and closed geodesic boundaries lengths are δ_{12} and δ_{16}. So, one can again utilize the combination theorem to get a new group $\Pi_{1112223} = < \Pi_{11122}^*, \Omega_2, \Lambda_3^* >$ of signature $(0, 0, 2, 2, 12)$. In the fourth step, we again consider two groups one is $\Pi_{1112223}$ and other is Ω_3 ($=< h_{19}, h_{20}, p_3 >$) of signature $(0, 0, 0, 1, 3)$ consisting one cusp end and three funnel ends with neck lengths δ_{21}, δ_{22}, and δ_{23}. Further, we cut the two corresponding surfaces of the groups $\Pi_{1112223}$ and Ω_3 (reduce to $\Pi_{1112223}^*$ and Ω_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_{18} and δ_{21} respectively, to glue each other. Finally, again, we add a group Θ_3 ($=< h_{21}, h_{22}, h_{23} >$) of signature $(0, 0, 3, 0, 1)$ with neck length δ_{24}, and closed geodesic boundaries lengths are δ_{18}, δ_{21}, and δ_{25}. Therefore, by the combination theorem, we obtain $\Pi_{111222333} = < \Pi_{111222333}^*, \Theta_3, \Omega_3^* >$ of signature $(0, 0, 3, 3, 14)$. Note that, this fourth step is basically a repetition of our previous first step. Therefore, in this way, one can continue this process for an infinitum numbers of times to get an infinitely generated group $\Psi_{111222333...} = < \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, ... > = < h_1, h_2, h_3, ...; p_1, p_2, p_3, ... >$. This group also is of type $(0, 0, \infty, \infty, \infty)$, but unlike the previous case, each block is of infinite hyperbolic area, h_i's and h_i's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

This completes the proof of Theorem 1.3. In particular, our third main theorem of this article is established.

Remark 3. By using the procedure carried out in Theorems 1.2 and 1.3, one can easily construct any infinitely generated Fuchsian groups of signatures $(\infty, 0, 0, 0, \infty)$, $(\infty, 0, 0, \infty, \infty)$, $(\infty, 0, \infty, 0, \infty)$, and $(\infty, 0, \infty, \infty, \infty)$.
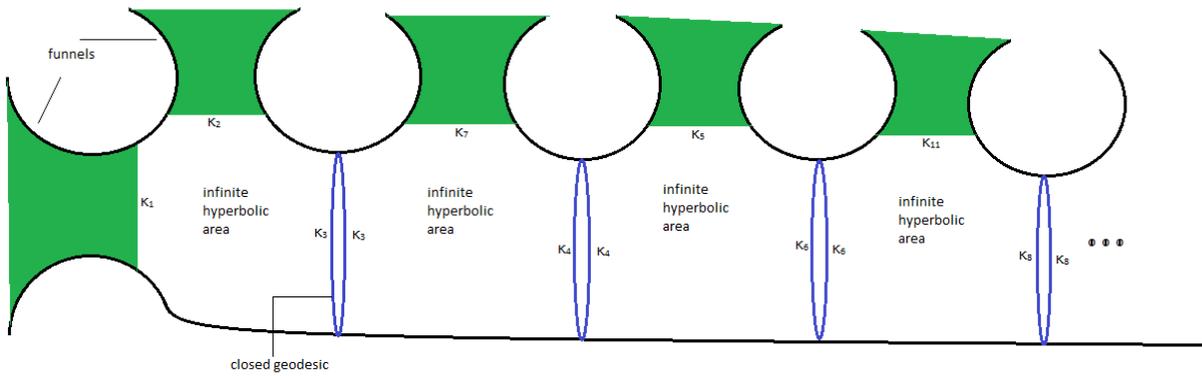
Notion of limit sets. Limit sets of our aforementioned constructed infinitely generated groups of types $(0, 0, 0, 0, \infty)$ and $(0, 0, \infty, 0, \infty)$ that give rise to infinite type generalized non-tight flute surfaces are Cantor sets, whereas for the types $(0, 0, 0, \infty, \infty)$ and $(0, 0, \infty, \infty, \infty)$ lead to infinite types generalized tight flute surfaces, are the whole circle at infinity. Indeed, this type of situation occurred due to the existence of parabolic elements in the infinitely generated SGs of signatures $(0, 0, 0, \infty, \infty)$ and $(0, 0, \infty, \infty, \infty)$. In essence, the infinitely generated groups of signatures $(0, 0, 0, 0, \infty)$ and $(0, 0, \infty, 0, \infty)$ are of Fuchsian groups of the second kind, whereas

the infinitely generated groups of signatures $(0, 0, 0, \infty, \infty)$ and $(0, 0, \infty, \infty, \infty)$ are of Fuchsian groups of the first kind.
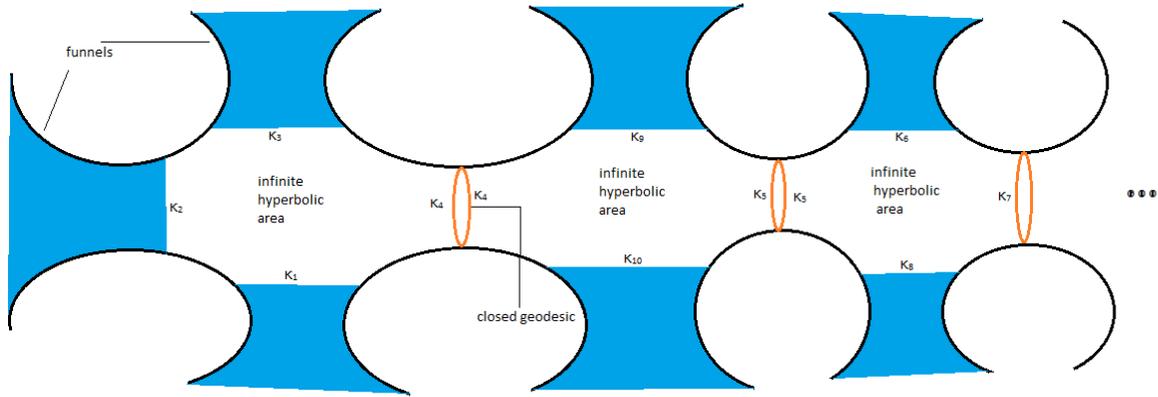
Infinitely generated Fuchsian group of signature (0,0,0,0,∞): Type 1.

$Figure : 8(b)$

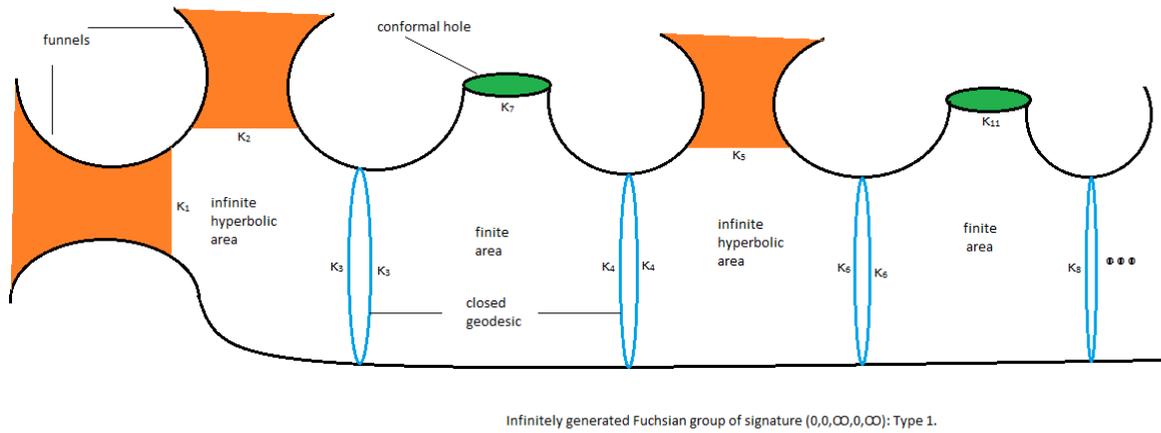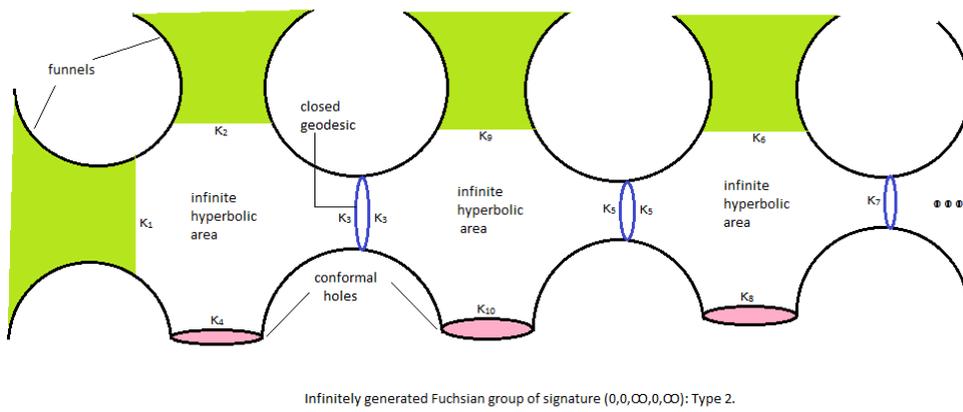Infinitely generated Fuchsian group of signature (0,0,0,0,∞): Type 2.

$Figure : 8(c)$
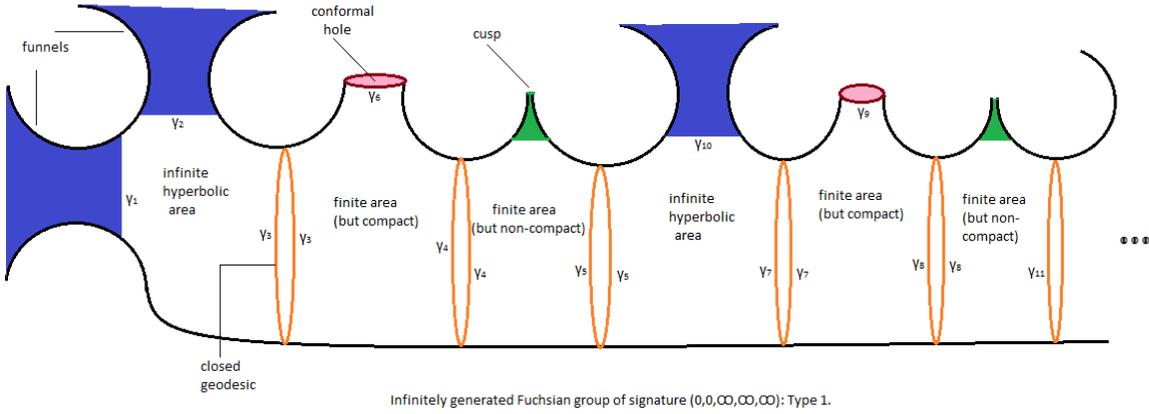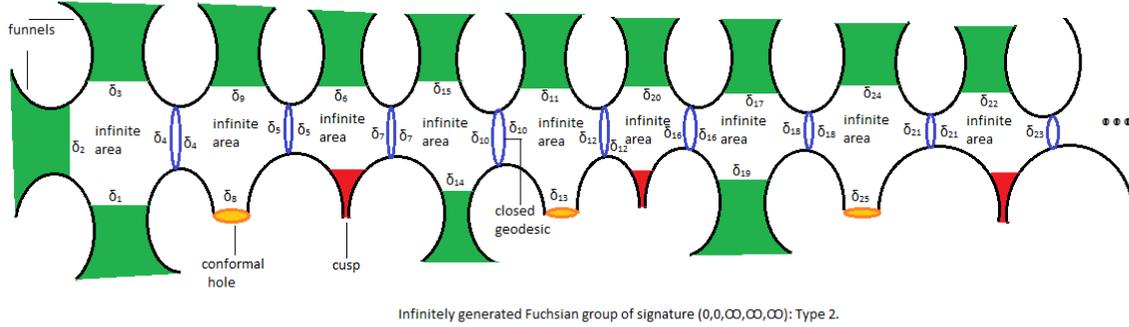
Infinitely generated Fuchsian group of signature (0,0,∞,0,∞): Type 1.

Figure : 8(d)

Infinitely generated Fuchsian group of signature (0,0,∞,0,∞): Type 2.

Figure : 8(e)

funnels

K_2

K_7

K_5

infinite hyperbolic area

closed geodesic K_3 K_3

infinite hyperbolic area

K_4 K_4

infinite hyperbolic area

K_6

K_1

cusps

Infinitely generated Fuchsian group of signature (0,0,0,∞,∞): Type 1.

$Figure: 8(f)$

funnels

β_2

cusp

β_5

β_8

infinite hyperbolic area

β_1

finite area

infinite hyperbolic area

finite area

infinite hyperbolic area

β_3 β_3

β_4 β_4

β_6 β_6

β_7 β_7

β_9

closed geodesic

Infinitely generated Fuchsian group of signature (0,0,0,∞,∞): Type 2.

$Figure: 8(g)$

Figure : 8(h)

Figure : 8(i)

9. **DATA AVAILABILITY STATEMENT**

Our manuscript has no associated data.

10. **STATEMENTS AND DECLARATIONS**

The authors declare that they have no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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