

STRUCTURES OF FINITE AND INFINITE TYPES OF NON-COMPACT RIEMANN SURFACES VIA FUNDAMENTAL GROUPS

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ABSTRACT. The purpose of this article is to introduce the notion of constructing any arbitrary finite and infinite types of non-compact hyperbolic Riemann surfaces via non-abelian fundamental groups equipped with various types of classical Schottky structures, with limit sets as uncountable sets (but not necessarily Cantor sets), emphasising the cases in which the surfaces are of infinite hyperbolic areas. At first, in this paper, we have fabricated three types of canonical non-compact infinite area Fuchsian polygons in the hyperbolic plane with different kinds of classical Schottky structures. After that, we have initiated a structure of an arbitrary finite type non-compact hyperbolic Riemann surface with genus, conformal holes, cusps, and funnel ends by using the canonical Fuchsian Schottky polygons. Furthermore, in this manuscript, we have proposed the notions of infinite types conformally compact and semi-conformally compact hyperbolic Riemann surfaces. In particular, we have introduced eight new and interesting types of infinite type hyperbolic Riemann surfaces (we call generalized flute surfaces) that are constructed from infinite sequences of infinite area hyperbolic pair of pants, each glued to the next along a common geodesic boundary with certain strategies.

1. INTRODUCTION

The aim of this article is as in the following:

- (1) In [8], Keen has constructed all possible canonical polygons for finitely generated Fuchsian groups that give rise to compact hyperbolic Riemann surfaces. In this paper, our initial goal is to extend her work to non-compact hyperbolic Riemann surfaces. In particular, we have fabricated three types of canonical non-compact infinite area Fuchsian polygons equipped with various kinds of classical Schottky structures within the Poincaré disk model.
- (2) Secondly, in this manuscript, we have formulated a strategy for how to construct any finite type arbitrary non-compact hyperbolic Riemann surfaces via fundamental (Fuchsian) groups. In fact, this investigation generalizes the part of Keen [9] from compact Riemann surfaces to non-compact ones, although the procedure is completely different from her work in [9].

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(3) Finally, we have introduced eight new types of infinite type hyperbolic surfaces, which we named as generalized flute surfaces (flute surface is first originated in a paper of Basmajian [4]), linked to various types of infinite dimensional Teichmüller spaces. Of course, the way flute surface was structured by Basmajian ([3], [4]), the flute surfaces and generalized flute surfaces are almost equivalent, but from the point of view of the infinite dimensional Teichmüller theory (in particular, various types of metrics, viz., length spectrum metric, Fenchel-Nielsen metric, bi-Lipschitz metric, and quasiconformal metric, see [2], [1], and [10]), these two kinds of infinite type surfaces (viz., flute surface and generalized flute surface) are not analogous at all. In particular, in this paper, we have constructed four kinds of infinite type generalized non-tight flute surfaces and four types of infinite type generalized tight flute surfaces, with limit sets that are Cantor sets and the whole circle at infinity, respectively. In fact, Basmajian [3] introduced the notion of infinitely generated Fuchsian groups from the point of view of the (hyperbolic) Riemann surface theory. After that, Schwartz [15] initiated the idea of infinitely generated Schottky groups equipped with the notion of limit sets. Although these two works were developed from completely different perspectives. In this thesis, the infinitely generated groups that we aim to construct are basically Fuchsian groups with Schottky flavor, i.e., Fuchsian groups equipped with classical Schottky structures. More precisely, the infinitely generated groups that Basmajian [3] formulated, although the associated hyperbolic surface is of infinite area, but each block is non-compact and finite area. On the other hand, Schwartz's [15] constructed infinitely generated group gives rise to compact hyperbolic Riemann surface. But, in this study, we are dealing with infinitely generated groups that lead to non-compact Riemann surfaces, additionally, each block may carry an infinite hyperbolic area. Apart from that, it is well known that the limit sets of any non-trivial finitely generated and infinitely generated Schottky groups are usually Cantor sets. In this investigation, we have proposed some finitely generated and infinitely generated Schottky groups whose limit sets are uncountable sets, but not necessarily Cantor sets, rather the whole circle at infinity. In essence, the study of infinitely generated Fuchsian groups of types $(0, \infty, 0, 0, 0)$, $(0, 0, \infty, 0, 0)$, and $(0, 0, \infty, \infty, 0)$ is somewhat known in the literature. In this paper, we have introduced the notion of infinitely generated Fuchsian groups of types $(0, 0, 0, 0, \infty)$, $(0, 0, 0, \infty, \infty)$, $(0, 0, \infty, 0, \infty)$, and $(0, 0, \infty, \infty, \infty)$ in various aspects.

In this article, we have proved the following theorems in this regard. Indeed, the subsequent Theorem 1.1 provides the structure of any non-compact Fuchsian group of finite type, whereas Theorem 1.2 indicates the construction of infinite types non-compact Fuchsian groups

of signatures $(0, 0, 0, 0, \infty)$, $(0, 0, 0, \infty, \infty)$, $(0, 0, \infty, 0, \infty)$, and $(0, 0, \infty, \infty, \infty)$. In particular, Theorem 1.1 generalized the work [[9], Theorem 5] from compact hyperbolic Riemann surfaces to non-compact (hyperbolic) Riemann surfaces. On the other hand, Theorem 1.2 extend the work [[9], Theorem 5] from finitely generated Fuchsian groups to infinitely generated ones.

Theorem 1.1. *Let Ω be an arbitrary finite type Fuchsian group of signature $(2, 0, 1, 2, 8)$. Then, by Klein-Maskit combination theorem, we prove that, $\Omega = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_6^*, \Omega_8, \Omega_7^* \rangle$, where Ω_i , $i = 1, 2, \dots, 8$ are non-compact Fuchsian groups with some specified signatures of finite types and the symbol ‘*’ indicates the number of operation act in that group.*

Theorem 1.2. *Suppose Γ , Δ , Ω , Φ , Ξ , Ψ , Σ , and Υ are Fuchsian groups, where ‘*’ represents the number of operation act in that group. Then, by using the Klein-Maskit combination theorem as a tool, we prove the following results:*

(a) *Let Γ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 4)$, $i \in \mathbb{N}$. Further, we assume that, Γ_{3+i} is a Fuchsian group of signature $(0, 0, 2, 0, 2)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Gamma = \langle \Gamma_1^*, \Gamma_3, \Gamma_2^{**}, \Gamma_5, \Gamma_4^{**}, \Gamma_7, \Gamma_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, 0, 0, \infty)$.*

(b) *Let Δ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 3)$, $i \in \mathbb{N}$. Furthermore, suppose that Δ_{3+i} is a Fuchsian group of signature $(0, 0, 2, 0, 1)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Delta = \langle \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, 0, 0, \infty)$.*

(c) *Let Ω_i be Fuchsian groups with signatures $(0, 0, 0, 0, 3)$, $i \in \mathbb{N}$. Additionally, let Ω_{3+i} is a Fuchsian group of signature $(0, 0, 3, 0, 0)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Omega = \langle \Omega_1^*, \Omega_3, \Omega_2^{**}, \Omega_5, \Omega_4^{**}, \Omega_7, \Omega_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, \infty, 0, \infty)$.*

(d) *Let Φ_i be Fuchsian groups with signatures $(0, 0, 1, 0, 3)$, $i \in \mathbb{N}$. In addition, we assume that, Φ_{3+i} is a Fuchsian group of signature $(0, 0, 3, 0, 1)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Pi = \langle \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, \infty, 0, \infty)$.*

(e) *Let Ξ_i be Fuchsian groups with signatures $(0, 0, 0, 1, 3)$, $i \in \mathbb{N}$. Moreover, suppose that Ξ_{3+i} is a Fuchsian group of signature $(0, 0, 2, 1, 1)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Xi = \langle \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, 0, \infty, \infty)$.*

(f) *Let Ψ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 3)$, $i \in \mathbb{N}$. Furthermore, let Π_{3+i} is a Fuchsian group of signature $(0, 0, 2, 1, 0)$, where $i \in 2\mathbb{N} \cup \{0\}$. Then, the group $\Psi = \langle \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Phi_6^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, 0, \infty, \infty)$.*

(g) Let Σ_i , Π_i , and Υ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 3)$, $(0, 0, 1, 0, 2)$, and $(0, 0, 0, 1, 2)$ respectively, $i \in \mathbb{N}$. Then, the group $\Upsilon = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, \infty, \infty, \infty)$.

(h) Let Λ_i , Ω_i , and Θ_i be Fuchsian groups with signatures $(0, 0, 0, 0, 4)$, $(0, 0, 0, 1, 3)$, and $(0, 0, 3, 0, 1)$ respectively, $i \in \mathbb{N}$. Then, the group $\Phi = \langle \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, \dots \rangle$ is an infinitely generated Fuchsian group of type $(0, 0, \infty, \infty, \infty)$.

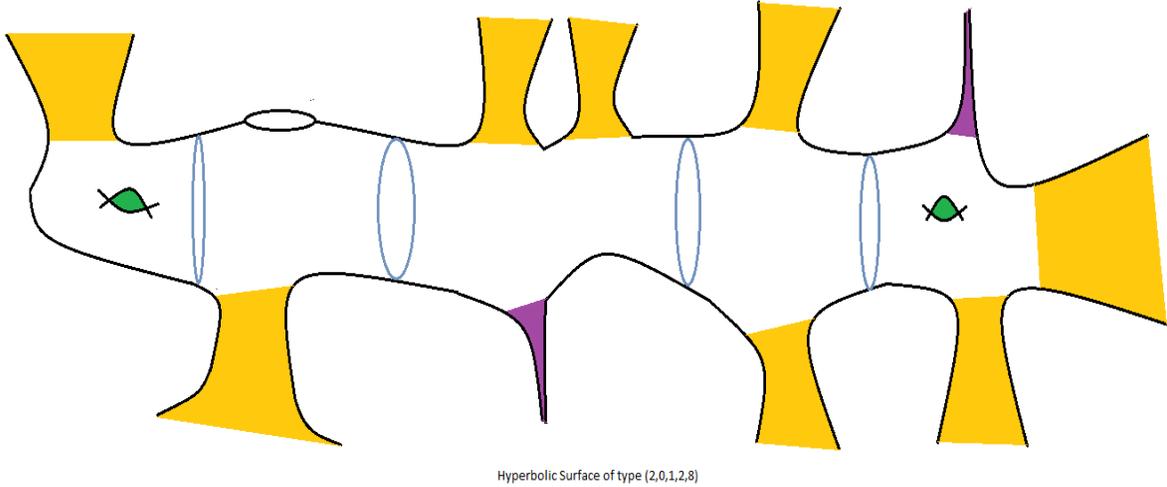


Figure : 2.2.1

It is well known that, unlike the case of hyperbolic surface of finite type, \exists various types of Teichmüller spaces corresponding to an infinite type hyperbolic surface, and each of these Teichmüller spaces (heavily) depending on the choice of a basepoint for that Teichmüller space. Moreover, even if someone fix a base point, Teichmüller space depends set theoretically on a distance function that one can put on that space. In fact, there are various types of distances that we can use in the Teichmüller space. So, the infinite type surfaces coming out from infinitely generated Fuchsian groups of types $(0, 0, 0, 0, \infty)$ (Theorem 1.2 (a) and (b)), $(0, 0, \infty, 0, \infty)$ (Theorem 1.2 (c), (d) and (f)), and $(0, 0, \infty, \infty, \infty)$ (Theorem 1.2 (g) and (h)) are not analogous. In particular, both the infinitely generated groups, Γ and Δ with signatures $(0, 0, 0, 0, \infty)$ provide infinite type surfaces where each block is of infinite hyperbolic area, but, each block of Δ contains one funnel end (apart from the first one), whereas for the group Γ , two funnels appear in each block. Furthermore, both the infinitely generated groups Ω and Π of type $(0, 0, \infty, 0, \infty)$ supply infinite type surfaces, but they are different from the point of view of the infinite dimensional Teichmüller spaces. In fact, for the group Ω , 1st, 3rd, 5th, 7th, ...

blocks are of infinite hyperbolic area and 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e., π), whereas for the group Π , each block is of infinite hyperbolic area. Moreover, the infinitely generated groups Ξ and Ψ of type $(0, 0, 0, \infty, \infty)$, the associated hyperbolic surfaces, for the group Ξ , each block is of infinite hyperbolic area, on the other hand, for Ψ , 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area and 2nd, 4th, 6th, 8th, ... blocks are of finite area. Finally, the infinitely generated groups Υ and Φ of type $(0, 0, \infty, \infty, \infty)$, the corresponding infinite type surfaces, for the group Υ , 1st, 4th, 7th, ... blocks are non-compact hyperbolic area, whereas 2nd, 5th, 8th, ... blocks are compact finite area, on the other hand, 3rd, 6th, 9th, ... blocks are non-compact but finite area. But, for the group Φ , each block is of infinite hyperbolic area.

Outline of this paper. The rest of this article is arranged as follows: In Section 2, we have discussed some preliminaries that will be needed throughout this paper. Sections 3 to 6 are the technical core of this paper. In Section 3, we have constructed three types of non-compact infinite area Fuchsian polygons with various kinds of classical Schottky structures in the Poincaré disk model. In Section 4, we have initiated the strategy of gluing two hyperbolic surfaces of type $(0, 0, 0, 0, f)$ via fundamental groups (see Lemma 4.1). After that, we have approached the technique of attaching two hyperbolic surfaces of type $(0, 0, 0, c, f)$ to each other in Section 5 (see Lemma 5.1). In Section 6, we have investigated the situations when the non-compact infinite area hyperbolic surface contains conformal holes (see Lemma 6.1). Furthermore, we have studied the circumstances when non-compact hyperbolic surfaces carry conformal holes but of finite area (see Lemma 6.2). In Section 7, we have proved the first main Theorem 1.1. Finally, in Section 8, we have established the second main Theorem 1.2. At the end, we have proposed some notions about the limit sets of our constructed infinitely generated Fuchsian groups.

2. Preliminaries

A conformally compact hyperbolic Riemann surface is basically a non-compact surface. More precisely, a hyperbolic Riemann surface is said to be conformally compact if it is of infinite hyperbolic area endowed with a metric of Gaussian curvature -1 , providing a non-abelian finitely generated fundamental group, and without a cusp end. So, a conformally compact hyperbolic surface can contain only funnel ends in the boundary of its compact core. On the other hand, a semi-conformally compact hyperbolic Riemann surface is also a non-compact hyperbolic surface, but the notion is slightly general compared to a conformally compact hyperbolic surface.

In particular, a semi-conformally compact hyperbolic surface may also contain cusp ends. Indeed, a semi-conformally compact hyperbolic Riemann surface is of type (g, p, h, c, f) , where g is the number of genus, p and h are the distinguished points and conformal disks (that are removed from its compact core), c and f are the numbers of cusp and funnel ends respectively. We assert that g, p, h , and $c \geq 0$, but $f > 0$ always. Moreover, if $p = p_1, p_2, \dots, p_m$, and $h = h_1, h_2, \dots, h_n$, $p_i \neq p_j$, for $i \neq j$, and $h_i \cap h_j = \phi$, $i \neq j$, with $p_i \notin h_j$. Now, if the genus $g = 0$ and $h + c + f = 3$, we say that the quotient surface is a pair of pants (see [13] and [14] for more details). In this article, we have explored torsion-free Fuchsian groups, so there is no elliptic element in the group. In fact, Keen ([8] and [9]) carried out the situation for containing elliptic elements in Fuchsian groups. So, we have exempted that case here. More precisely, in this manuscript, we are mainly interested in the type $(g, 0, h, c, f)$. A flute surface is a special type of infinite type hyperbolic surface that looks like an infinite chain of (finite area) pair of pants glued end to end. Topologically, a flute surface is homeomorphic to a sphere removing a Cantor set of points. But, geometrically, a flute surface carries a hyperbolic metric. In this paper, we mean a flute surface is tight if it contains at least one (hence countably infinite number) cusp end. If an infinite type flute surface doesn't contain any cusp end, we call non-tight. For the basics of a hyperbolic manifold, funnel, cusp, and pair of pants the readers may go through the expositions, for instance, [13], [5], [6], [11], [4], and [12].

3. CONSTRUCTING NON-COMPACT INFINITE AREA FUCHSIAN POLYGONS WITH SCHOTTKY SETTINGS

In this section, we are going to construct three types of non-compact infinite area Fuchsian polygons equipped with classical Schottky structures. In particular, these polygons give rise to certain non-compact Fuchsian groups, and these Fuchsian groups will be needed in the following sections to established our main Theorem 1.1, and 1.2. Indeed, the following constructions are the generalizations of [8, Theorems 6 and 7], from compact to non-compact Riemann surfaces.

Constructing canonical Fuchsian Schottky polygon 1.

Step 1. Let us consider six semi-circles $SC_1, SC_2, SC_3, SC_4, SC_5$, and SC_6 within the Poincaré disk model with end points $a, b; c, d; e, f; g, h; i, j; k, l$ respectively lying on the boundary of the disk. SC_1 is paired with SC_2 , SC_3 twins with SC_4 , and SC_5 is linked with SC_6 by three hyperbolic Möbius transformations. Now, the open region (looks like a polygon) bounded by these six semi-circles with arcs lying on the boundary of the disk represents the open non-compact polygon of a $(0, 0, 0, 0, 4)$ group. In particular, the area surrounded by the non-Euclidean lines joining the points $a, e; f, i; j, k; l, h; g, d; c, d$, and the semi-circles SC_1, SC_2 ,

SC_3 , SC_4 , SC_5 , and SC_6 exhibits the fundamental domain for the group $(0, 0, 0, 0, 4)$. Now, the non-compact polygon for a $(0, 0, 0, 0, 4)$ group, to make it canonical we have utilized the procedure as follows:

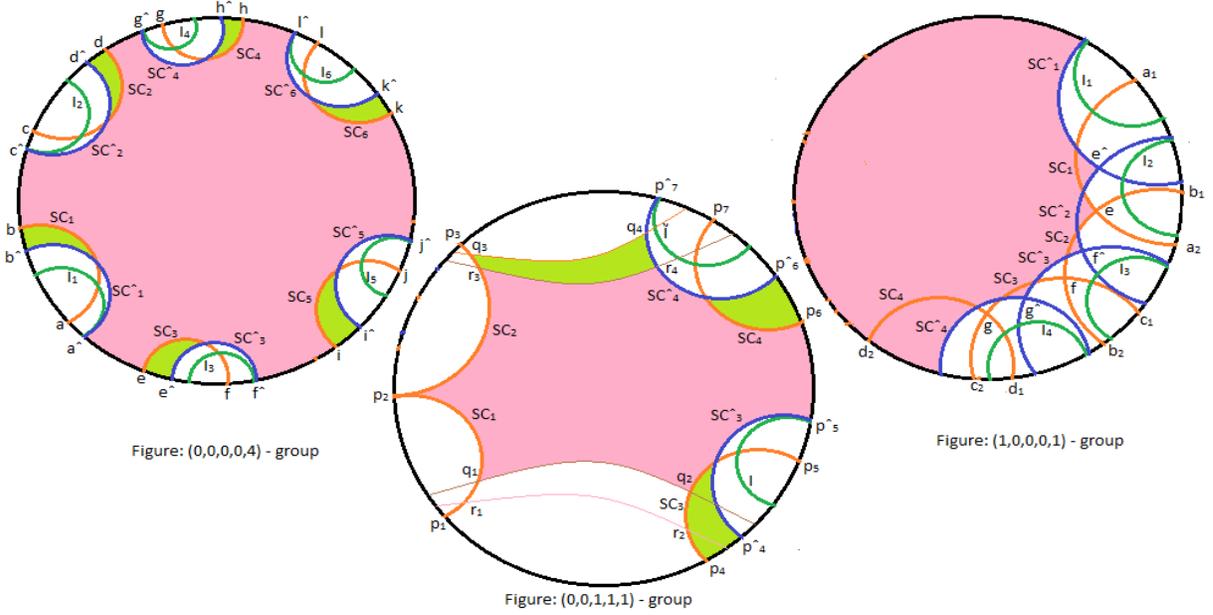


Figure : 2.2.1

Step 2. Let, $\Omega^1 = \langle g_1, g_2, g_3 \rangle$ be a group with signature $(0, 0, 0, 0, 4)$. Note that all three generators g_1 , g_2 , and g_3 are hyperbolic Möbius transformations. Now, we consider the three isometric circles I_1 , I_2 , and I_3 corresponding to the right side of the three axes of the hyperbolic elements g_1 , g_2 , and g_3 respectively (see [7] for the notion of isometric circles). Suppose the end points of the isometric circles I_1 , I_2 , and I_3 are \hat{a} (between the points a and e), \hat{f} (between the points f and i), and \hat{j} (between the points j and k) respectively. Let, $g_1(\hat{a}) = \hat{d}$, $g_2(\hat{f}) = \hat{h}$, and $g_3(\hat{j}) = \hat{k}$. Now, to fabricate a canonical polygon for the group $(0, 0, 0, 0, 4)$, we draw six semi-circles in the following way. We translate six semi-circles SC_1 to \hat{SC}_1 with end point at \hat{a} (by the hyperbolic curve length $(\hat{a} - a)$), SC_2 with end point at \hat{d} (by the hyperbolic curve length $(\hat{d} - d)$), SC_3 to \hat{SC}_3 with end point at \hat{f} (by the hyperbolic curve length $(\hat{f} - f)$), SC_4 to \hat{SC}_4 with end point at \hat{h} (by the hyperbolic curve length $(\hat{h} - h)$), SC_5 to \hat{SC}_5 with end point at \hat{j} (by the hyperbolic curve length $(\hat{j} - j)$), and SC_6 to \hat{SC}_6 with end point at \hat{k} (by the hyperbolic curve length $(\hat{k} - k)$). Let \hat{b} , \hat{c} , \hat{e} , \hat{g} , \hat{i} , \hat{l} are the starting points of the semi-circles \hat{SC}_1 , \hat{SC}_2 , \hat{SC}_3 , \hat{SC}_4 , \hat{SC}_5 , \hat{SC}_6 respectively. Now, the region bounded by the semi-circles \hat{SC}_1 , \hat{SC}_2 , \hat{SC}_3 , \hat{SC}_4 , \hat{SC}_5 , \hat{SC}_6 , and the arcs joining by the points \hat{a} to \hat{e} , \hat{f} to \hat{i} , \hat{j} to \hat{k} , \hat{l} to \hat{h} , \hat{g} to

\hat{d} , \hat{c} to \hat{b} lying on the boundary of the disk represents the canonical polygon of the $(0, 0, 0, 0, 4)$ group.

Constructing canonical Fuchsian Schottky polygon 2.

Step 1. Let us consider four semi-circles SC_1 , SC_2 , SC_3 , and SC_4 within the Poincaré disk model with end points $p_1, p_2; p_2, p_3; p_4, p_5; p_6, p_7$ respectively lying on the boundary of the disk. SC_1 is paired with SC_2 by a parabolic Möbius transformation p , and SC_3 is linked with SC_4 by a hyperbolic Möbius transformation h . Now, the open region (looks like a polygon) bounded by these four semi-circles SC_1 , SC_2 , SC_3 , and SC_4 represents an open non-compact polygon of a $(0, 0, 0, 1, 2)$ group. In particular, the area surrounded by the non-Euclidean lines joining the points $p_1, p_4; p_5, p_6; p_7, p_3$ (all lying on the boundary of the disk) and the semi-circles SC_1 , SC_2 , SC_3 , and SC_4 exhibits the fundamental domain for the group $(0, 0, 0, 1, 2)$. Now, we consider two non-Euclidean lines; one intersects at r_1 in SC_1 and r_2 in SC_3 ; the other one intersects at r_3 in SC_2 and r_4 in SC_4 . We produce r_2r_1 to \hat{r}_1 , r_1r_2 to \hat{r}_2 , r_4r_3 to \hat{r}_3 , and r_3r_4 to \hat{r}_4 in the boundary of the disk. The region surrounded by the arcs joining by the points $p_2, r_1, r_2, p_5, p_6, r_4, r_3$, and p_2 represents the fundamental polygon of a $(0, 0, 1, 1, 1)$ group. Now, to make the polygon canonical, we have used the procedure as follows:

Step 2. Let $\Omega^2 = \langle p, h \rangle$ be a group with signature $(0, 0, 1, 1, 1)$. Here, p is parabolic, whereas h is a hyperbolic Möbius transformation. Now, let us consider the isometric circle I corresponding to the right side of the axis of the hyperbolic element h . Assume that, the end point of the isometric circle I is \hat{p}_5 (between the points p_5 and p_6) on the boundary of the Poincaré disk. Let, $h(\hat{p}_5) = \hat{p}_6$. Now, to construct a canonical polygon for the group $(0, 0, 1, 1, 1)$, we draw two semi-circles I , and \hat{I} in the subsequent way. We translate these two semi-circles SC_3 to $\hat{S}C_3$ with end point at \hat{p}_4 (by the hyperbolic curve length $(\hat{p}_5 - p_5)$), SC_4 with end point at \hat{p}_7 (by the hyperbolic curve length $(\hat{p}_6 - p_6)$). Let \hat{p}_5 and \hat{p}_6 be the starting points of the semi-circles $\hat{S}C_3$ and $\hat{S}C_4$, respectively. Now, on the boundary of the disk, we take four points \hat{q}_1 (between p_2 and \hat{r}_1), \hat{q}_2 (between \hat{p}_5 and \hat{p}_4), \hat{q}_3 (between \hat{r}_3 and p_3), \hat{q}_4 (between \hat{p}_7 and \hat{r}_4). Produce the non-Euclidean arcs $\hat{q}_1\hat{q}_2$ (intersecting SC_1 at q_1 , and $\hat{S}C_3$ at q_2), and $\hat{q}_3\hat{q}_4$ (intersecting SC_2 at q_3 , and $\hat{S}C_4$ at q_4). The region bounded by the non-Euclidean arcs joining by the points p_2 to q_1 , q_1 to q_2 , q_2 to \hat{p}_5 , \hat{p}_6 to q_4 , q_4 to q_3 , and q_3 to p_2 represents the canonical polygon for the group $(0, 0, 1, 1, 1)$.

Constructing canonical Fuchsian Schottky polygon 3.

Step 1. Let us consider four semi-circles SC_1 , SC_2 , SC_3 , and SC_4 within the Poincaré disk model with end points $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ respectively that are lying on the boundary of

the disk. SC_1 is paired with SC_3 , and SC_4 twins with SC_2 by two hyperbolic Möbius transformations h_1 and h_2 , respectively. Roughly speaking, the open region (looks like a polygon) encircled by these four semi-circles with arcs lying on the boundary of the disk represents the non-compact polygon for a $(1, 0, 0, 0, 1)$ group. Let M be a point situated on the boundary of the disk at the point $(1, 0)$. Furthermore, we assume that the semi-circles SC_1 and SC_2 intersect at the point e , SC_2 and SC_3 intersects at f , and SC_3 and SC_4 intersect at the point g . Note that all the points e, f , and g are lying within the disk. Now, the area surrounded by the non-Euclidean lines joining by the points $d_2, g; g, f; f, e; e, a_1; a_1, M; M, d_2$ exhibits the fundamental domain for the group $(1, 0, 0, 0, 1)$. Now, our goal is to produce the polygon that represents the fundamental domain into a canonical polygon.

Step 2. Let $\Omega^3 = \langle h_1, h_2 \rangle$ be a group with signature $(1, 0, 0, 0, 1)$. Here, both the generators h_1 , and h_2 are hyperbolic Möbius transformations. Now, we consider two isometric circles I_1 , and I_2 corresponding to the two axes of the hyperbolic transformations h_1 , and h_2 respectively. Suppose, the end points of the isometric circles I_1 , and I_2 are p_1 (between the points a_1 and M), and s_1 (between the points d_2 and c_2) respectively. Let, $h_1(p_1) = r_1$, and $h_2(s_1) = q_1$. Now, to fabricate a canonical polygon for the group $(1, 0, 0, 0, 1)$, we draw four semi-circles in the following way. We translate four semi-circles SC_1 to \hat{SC}_1 with end point at p_1 (by the hyperbolic curve length $(p_1 - a_1)$), SC_2 to \hat{SC}_2 with end point at q_1 (by the hyperbolic curve length $(q_1 - b_1)$), SC_3 to \hat{SC}_3 with end point at r_1 (by the hyperbolic curve length $(r_1 - c_1)$), SC_4 to \hat{SC}_4 with end point at s_1 (by the hyperbolic curve length $(s_2 - d_1)$). Let, p_2, q_2, r_2 , and s_2 are the starting points of the semi-circles $\hat{SC}_1, \hat{SC}_2, \hat{SC}_3$, and \hat{SC}_4 respectively. Let the semi-circles \hat{SC}_1 and \hat{SC}_2 at the point \hat{e} , \hat{SC}_2 and \hat{SC}_3 intersect at \hat{f} , and finally, \hat{SC}_3 and \hat{SC}_4 intersect at the point \hat{g} . Now, the region bounded by the arcs joining by the points s_1 to \hat{g} , \hat{g} to \hat{f} , \hat{f} to \hat{e} , \hat{e} to p_1 , p_1 to M , M to s_1 represents the canonical polygon of the $(0, 0, 0, 0, 4)$ group.

In the subsequent Sections 4, 5, and 6, we have described a general method for constructing new non-compact finite and infinite area hyperbolic surfaces by gluing along infinite funnel boundaries or conformal holes closed geodesic boundaries. The essential tool here we employ is the (Klein-Maskit) combination theorem (see [9], Theorem 1) that allows us to amalgamate two non-compact Fuchsian groups to form a larger discrete group.

4. GLUING NON-COMPACT HYPERBOLIC SURFACES OF TYPES $(0,0,0,0,f)$

Lemma 4.1. (Construction 1.) *Let, Ω_1 and Ω_2 be two Fuchsian groups of signature $(0, 0, 0, 0, 3)$. Suppose both the corresponding surfaces possesses different neck lengths $\kappa_i, i = 1, 2, \dots, 6$, where $\kappa_i \neq \kappa_j$, for $i \neq j$. Then:*

(1) *for a Fuchsian group Π_3 of signature $(0, 0, 2, 0, 1)$, we prove that $\Omega^1 = \langle \Omega_1^*, \Pi_3, \Omega_2^* \rangle$, where ‘*’ indicates the number of operation act on that group.*

(2) *Gluing two surfaces of types $(0, 0, 0, 0, 3)$ yields a hyperbolic surface of type $(0, 0, 0, 0, 5)$.*

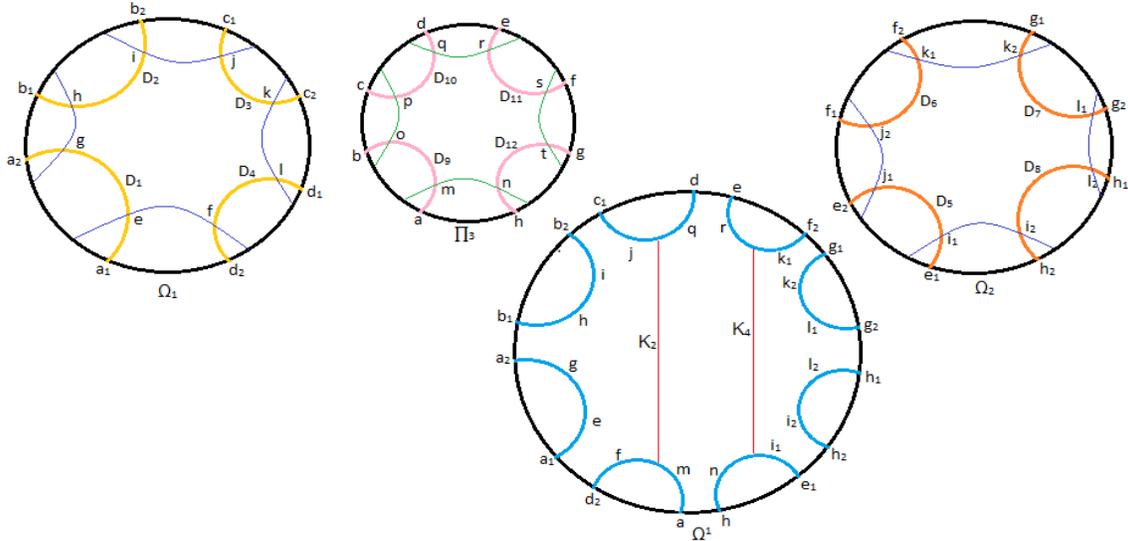


Figure : 2.2.1

Proof. Let D_1, D_2, D_3 , and D_4 be four semi-circles within the Poincaré disk model. In particular, D_1 is paired with D_2 by the hyperbolic Möbius transformation h_1 , whereas D_3 twins with D_4 by the hyperbolic Möbius transformation h_2 . Suppose these four semi-circles meet the boundary of the disk at the points $a_1, a_2; b_1, b_2; c_1, c_2; d_1, d_2$ respectively. Now, on the boundary of the disk, we take eight points, two between a_1 and a_2 , viz., \tilde{e} (near the point a_1) and \tilde{g} (near the point a_2), two between b_1 and b_2 , viz., \tilde{h} (near the point b_1) and \tilde{i} (near the point b_2), two between c_1 and c_2 , viz., \tilde{j} (near the point c_1) and \tilde{k} (near the point c_2), and finally two, between the points d_1 and d_2 , viz., \tilde{l} (near the point d_1) and \tilde{f} (near the point d_2). We draw the non-Euclidean curves lines by joining the points \tilde{e} to \tilde{f} , \tilde{g} to \tilde{h} , \tilde{i} to \tilde{j} , and \tilde{k} to \tilde{l} . Note that, these four curves intersect in the four semi-circles D_1, D_2, D_3 , and D_4 at the points $e, g, h, i; j, k; l,$

and f . Assume that the lengths of the curve line \overline{gh} is κ_1 , \overline{lk} is κ_2 , and \overline{ef} (or \overline{ij}) is κ_3 . Now, the area (denoted by R_0) joining the points $a_1, d_2, f, l, d_1, c_2, k, j, c_1, b_2, i, h, b_1, a_2, g, e$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_1 = \langle h_1, h_2 \rangle$) with signature $(0, 0, 0, 0, 3)$. So, the Fuchsian group Ω_1 is a hyperbolic Riemann surface with three funnel ends consisting of neck lengths κ_1 , κ_2 , and $2\kappa_3$. However, the region (denoted by R_1) joining the points $a_1, d_2, f, l, k, j, c_1, b_2, i, h, b_1, a_2, g, e$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by Π_1) with signature $(0, 0, 1, 0, 2)$ with neck lengths κ_1 , and $2\kappa_3$ consisting a closed geodesic boundary with length κ_2 . Now, we consider another Fuchsian group Ω_2 with signature $(0, 0, 0, 0, 3)$ with neck lengths κ_4 , κ_5 , and $2\kappa_6$. We proceed in a similar situation as in the above formation. Now, the area (denoted by R_2) joining the points $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, f_1, e_2, j_1, i_1$, and e_1 represents a fundamental domain of a Fuchsian group (denoted by $\Omega_2 = \langle h_3, h_4 \rangle$) with signature $(0, 0, 0, 0, 3)$. On the other hand, the figure (denoted by R_3) joining the points $e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, j_2, j_1, i_1$, and e_1 represents a fundamental domain of a Fuchsian group (denoted by Π_2) with signature $(0, 0, 1, 0, 2)$ consisting one closed geodesic boundary of length κ_4 , and two funnels with neck lengths κ_5 , and $2\kappa_6$. Let us consider, a Fuchsian group Π_3 with signature $(0, 0, 2, 0, 1)$ with one funnel end of neck length $2\kappa_7$. Then, from figure one can easily see that, the region (denoted by R_4) joining the points $o, m, a, h, n, t, s, r, e, d, q, p$, and o represents the fundamental domain of a Fuchsian group Π_4 with signature $(0, 0, 2, 0, 1)$ with neck length $2\kappa_7$ and two boundary geodesic with lengths κ_2 and κ_4 .

To glue two Fuchsian groups Ω_1 and Ω_2 with signatures $(0, 0, 0, 0, 3)$, we cut the two corresponding surfaces one funnel end each along the neck of their funnels. After this operation, it reduces two groups, one is Π_1 and other is Π_2 . Then adding a Fuchsian group Π_3 in such a way that the neck length can match with the groups Π_1 and Π_2 . Note that, when we paste with Π_1 to Π_3 the curve lines (i.e., \overline{kl} in Π_1 and \overline{op} in Π_4) can match, i.e, their directions are opposite to each other. Now, by using the combination theorem, one can get a Fuchsian group $\Pi_{13} = \langle \Pi_1, \Pi_3 \rangle$ with signature $(0, 0, 1, 0, 3)$. After that, we will attach the groups Π_{13} and Π_2 on the analogy of the technique that we mentioned just before (i.e., the direction of the curve lines \overline{st} in Π_{13} and $\overline{j_1j_2}$ in Π_2). Consequently, again, one can utilize combination theorem to get a Fuchsian group $\Pi_{132} = \langle \Pi_{13}, \Pi_2 \rangle (= \Omega^1, \text{ say})$ with signature $(0, 0, 0, 0, 5)$. Therefore, the fundamental domain of the arising group Π_{132} is the area bounded by the non-Euclidean lines joining the points $a_1, d_2, f, m, a, h, n, i_1, e_1, h_2, i_2, l_2, h_1, g_2, l_1, k_2, g_1, f_2, k_1, r, e, d, q, j, c_1, b_2, i, h, b_1, a_2, g, e$, and a_1 . So, the proof of Lemma 4.1 is completed.

□

Remarks 1. Note that, the polygon of a Fuchsian group with signature $(0, 0, 3, 0, 0)$ that Keen [9] introduced for compact Riemann surface, is completely different from our constructed polygon for Fuchsian group equipped with classical Schottky structures of signature $(0, 0, 3, 0, 0)$ in non-compact settings.

5. ATTACHING HYPERBOLIC SURFACES OF TYPES $(0,0,0,c,f)$

Lemma 5.1. (Construction 2.) Let, Ω_3 and Ω_4 be two Fuchsian groups of signature $(0, 0, 0, 1, 2)$. Assume that, both the corresponding surfaces possesses different neck lengths κ_i , $i = 1, 2, \dots, 4$, where $\kappa_i \neq \kappa_j$, for $i \neq j$. Then:

(1) for a Fuchsian group Π_5 of signature $(0, 0, 2, 0, 1)$, we prove that $\Omega^2 = \langle \Omega_3^*, \Pi_5, \Omega_4^* \rangle$, where ‘*’ represents the number of operation act on that group.

(2) Gluing two surfaces of types $(0, 0, 0, 1, 2)$ yields a hyperbolic surface of type $(0, 0, 0, 2, 3)$. In particular, Ω^2 represents a hyperbolic surface of two cusps and three funnels of neck lengths

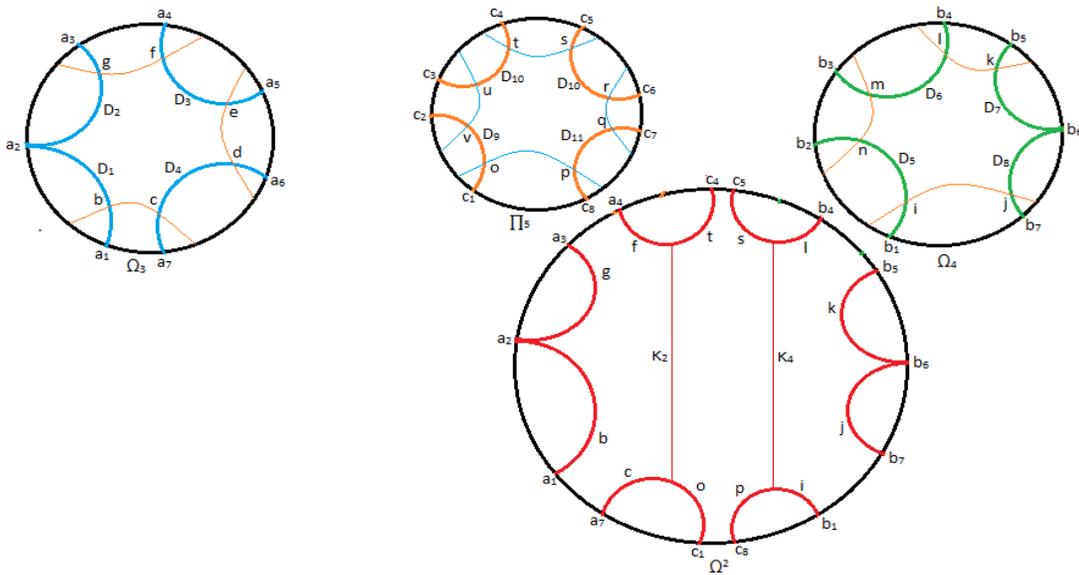


Figure : 2.2.1

Proof. Let D_1, D_2, D_3 , and D_4 be four semi-circles within the Poincaré disk model. In fact, D_1 is paired with D_2 by the parabolic Möbius transformation p_1 , whereas D_3 linked with D_4 by the hyperbolic Möbius transformation h_1 . Assume that these four semi-circles meet

the boundary of the disk at the points $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$ respectively. Now, on the boundary of the disk, we take six points, two between a_7 and a_6 , viz., \tilde{c} (near the point a_7) and \tilde{d} (near the point a_6), two between a_5 and a_4 , viz., \tilde{e} (near the point a_5) and \tilde{f} (near the point a_4), one between a_1 and a_2 , viz., \tilde{b} (near the point a_1), and finally one, between the points a_2 and a_3 , viz., \tilde{g} (near the point a_3). We draw the non-Euclidean curves lines by joining the points \tilde{b} to \tilde{c} , \tilde{d} to \tilde{e} , and \tilde{f} to \tilde{g} . Notice that these three curves intersect in the four semi-circles D_1, D_2, D_3 , and D_4 at the points $b; g; f, e; d$, and c . Suppose the lengths of the curve lines \overline{bc} (or \overline{fg}) is κ_1 , and \overline{de} is κ_2 . Now, the region (denoted by R_5) joining the points $a_1, a_7, c, d, a_6, a_5, e, f, a_4, a_3, g, a_2, b$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_3 = \langle p_1, h_1 \rangle$) with signature $(0, 0, 0, 1, 2)$. Consequently, the Fuchsian group Ω_3 is a hyperbolic Riemann surface with one cusp and two funnel ends consisting neck lengths $2\kappa_1$, and κ_2 . However, the region (denoted by R_6) joining the points $a_1, a_7, c, d, e, f, a_4, a_3, g, a_2, b$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by Π_4) with signature $(0, 0, 1, 1, 1)$ with one funnel of neck length $2\kappa_1$ consisting a closed geodesic boundary with length κ_2 . Now, we consider an another Fuchsian group Ω_4 with signature $(0, 0, 0, 1, 2)$ with one cusp end and two funnels of neck lengths $2\kappa_3$, and κ_4 . We proceed in a similar situation as we did in the above construction. Now, the domain (denoted by R_7) joining the points $b_1, b_7, j, b_6, k, b_5, b_4, l, m, b_3, b_2, n, i$, and b_1 represents a fundamental region of a Fuchsian group (denoted by $\Omega_4 = \langle h_2, p_2 \rangle$) with signature $(0, 0, 0, 1, 2)$. However, the area (denoted by R_7) joining the points $b_1, b_7, j, b_6, k, b_5, b_4, l, m, n, i$, and b_1 represents a fundamental domain of a Fuchsian group (denoted by Π_5) with signature $(0, 0, 1, 1, 1)$ consisting one closed geodesic boundary of length κ_4 , and one funnel with neck length $2\kappa_3$. Now, we are considering a Fuchsian group Π_6 with signature $(0, 0, 2, 0, 1)$ consisting one funnel end of neck length $2\kappa_5$. Then, from the preceding construction, one can easily observe that, the region (denoted by R_8) joining the points $c_1, c_8, p, q, r, s, c_5, c_4, t, u, v, o$, and c_1 represents the fundamental domain of a Fuchsian group Π_6 with signature $(0, 0, 2, 0, 1)$ with one funnel of neck length $2\kappa_5$ and two closed geodesic boundary with lengths κ_2 and κ_4 .

To glue two Fuchsian groups Ω_3 and Ω_4 with signatures $(0, 0, 0, 1, 2)$, we cut the two corresponding surfaces one funnel end each along the neck of their funnels where neck lengths κ_2 and κ_4 respectively. After this operation, it reduces two groups, one is Π_4 and other one Π_5 . Then adding a Fuchsian group Π_6 in such a way that the neck length can match with the groups Π_4 and Π_5 . Note that, when we paste with Π_4 to Π_6 the curve lines (i.e., \overline{de} in Π_4 and \overline{uv} in Π_6) can match, i.e, their directions are opposite to each other. Now,

by using the combination theorem, one can get a Fuchsian group $\Pi_{46} = \langle \Pi_4, \Pi_6 \rangle$ with signature $(0, 0, 1, 1, 2)$. After that, we will attach the groups Π_{46} and Π_5 on the analogy of the technique that we mentioned just before (i.e., the direction of the curve lines $\overline{r\bar{q}}$ in Π_{46} and \overline{nm} in Π_5). Consequently, again, one can utilize the combination theorem to get a Fuchsian group $\Pi_{465} = \langle \Pi_{46}, \Pi_5 \rangle (= \Omega^2, \text{ say})$ with signature $(0, 0, 0, 2, 3)$. Therefore, the fundamental domain of the arising group Π_{465} is the area bounded by the lines joining the points $a_1, a_7, c, o, c_1, c_8, p, i, b_1, b_7, j, b_6, k, b_5, b_4, l, s, c_5, c_4, t, f, a_4, a_3, g, a_2, b$, and a_1 . This proves the Lemma 5.1.

□

6. AMALGAMATION OF NON-COMPACT HYPERBOLIC SURFACES OF TYPES $(0, 0, h, c, f)$ and $(g, 0, h, c, 0)$

Lemma 6.1. *(Construction 3.) Let, Ω_5 and Ω_6 be two Fuchsian groups of signatures $(0, 0, 1, 0, 2)$ and $(0, 0, 1, 1, 1)$. Suppose, all lengths of their necks and closed geodesic boundaries in the conformal holes are distinct. Then:*

- (1) *for a Fuchsian group Π_7 of signature $(0, 0, 2, 0, 1)$, we prove that $\Omega^3 = \langle \Omega_5^*, \Pi_7, \Omega_6^* \rangle$, where ‘*’ represents the number of operation act on that group.*
- (2) *Gluing two surfaces of types $(0, 0, 1, 0, 2)$ and $(0, 0, 1, 1, 1)$ yields a hyperbolic surface of type $(0, 0, 2, 1, 2)$. In particular, Ω^3 represents a hyperbolic surface of two conformal holes, two cusps and two funnels end.*

Proof. Let D_1, D_2, D_3 , and D_4 be four semi-circles within the Poincaré disk model. More precisely, D_1 twins with D_2 by the hyperbolic Möbius transformation h_1 , on the other hand, D_3 linked with D_4 by the hyperbolic Möbius transformation h_2 . Suppose these four semi-circles meet the boundary of the disk at the points $a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8$ respectively. We consider eight points, on the boundary of the disk, two between a_1 and a_2 , viz., \tilde{b} (near the point a_1) and \tilde{i} (near the point a_2), two between a_3 and a_4 , viz., \tilde{h} (near the point a_3) and \tilde{g} (near the point a_4), two between a_5 and a_6 , viz., \tilde{f} (near the point a_5), and finally two, between the points a_7 and a_8 , viz., \tilde{d} (near the point a_7) and \tilde{c} (near the point a_8). Now, we draw the non-Euclidean curves lines by joining the points \tilde{b} to \tilde{c} , \tilde{i} to \tilde{h} , \tilde{g} to \tilde{f} and \tilde{e} to \tilde{d} . Observe that these four curves intersect in the four semi-circles D_1, D_2, D_3 , and D_4 at the points $b, i; h, g; f, e; d$, and c . Assume that the lengths of the curve lines $\overline{i\tilde{h}}$ is κ_1 , \overline{bc} (or \overline{fg}) is κ_3 , and \overline{de} is κ_2 . Note that, the region (denoted by F_1) joining the points $i, b, c, d, a_7, a_6, e, f, g, h, a_3, a_2$, and i represents the fundamental domain of a Fuchsian group (denoted by $\Omega_5 = \langle h_1, h_2 \rangle$) with

signature $(0, 0, 1, 0, 2)$. Now, the area (denoted by F_2) joining the points $i, b, c, d, e, f, g, h, a_3, a_2$, and i represents the fundamental region of a Fuchsian group (denoted by $\Omega_5^* = \langle h_1, h_2 \rangle$) with signature $(0, 0, 2, 0, 1)$. Hence, the Fuchsian group Ω_5^* is a hyperbolic Riemann surface consisting one funnel end of neck length κ_1 with two conformal holes where lengths of their boundary components are κ_2 and $2\kappa_3$, respectively. Now, we consider another Fuchsian group Ω_6 with signature $(0, 0, 1, 1, 1)$ with one cusp end, one funnel end of neck length κ_4 , and one conformal hole with length of the closed geodesic boundary is $2\kappa_5$. We proceed in a similar manner as we have arranged in the overhead construction. Observe that, the region (denoted by F_3) joining the points $o, j, k, b_6, l, m, n, b_3, b_2$, and o represents a fundamental domain of a Fuchsian group (denoted by $\Omega_6 = \langle h_3, p_1 \rangle$) with signature $(0, 0, 1, 1, 1)$. However, the diagram (denoted by F_4) joining the points o, j, k, b_6, l, m, n , and o represents a fundamental domain of a Fuchsian group (denoted by Ω_6^*) with signature $(0, 0, 2, 1, 0)$ consisting two closed geodesic boundaries of lengths κ_4 and $2\kappa_5$ and one cusp end.

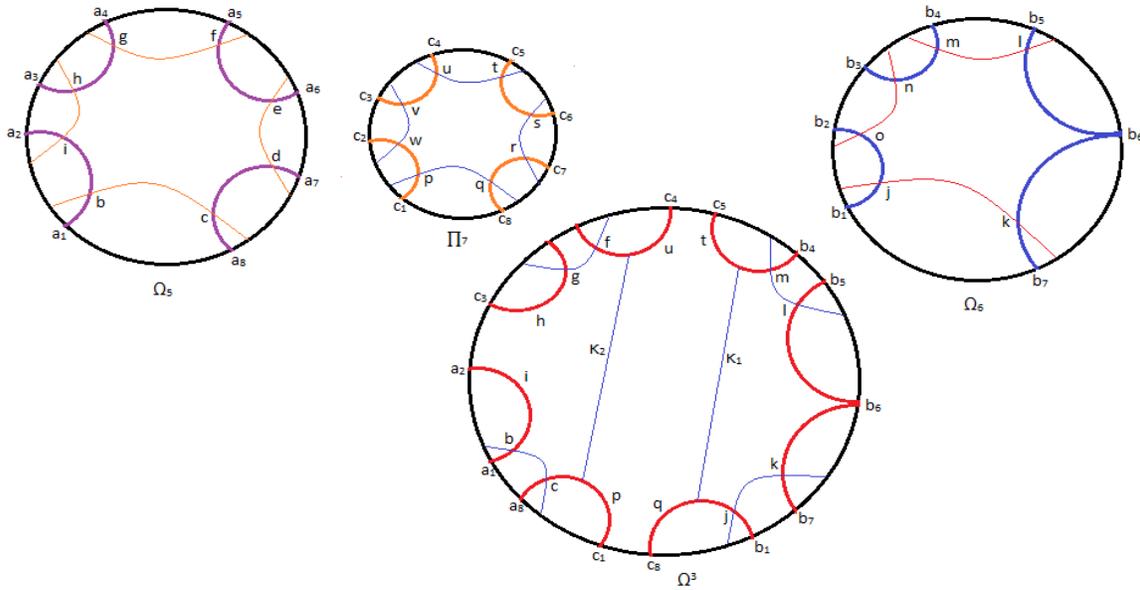


Figure : 2.2.1

Now, if we want to glue the two Fuchsian groups Ω_5^* and Ω_6^* , two cases will occur.
 (i) Assume that $\kappa_3 = \kappa_5$. Then, one can easily paste these two groups Ω_5^* and Ω_6^* in a direct way. This act gives a Fuchsian group with signature $(0, 0, 0, 1, 3)$. If possible this will not happen. Then, we will move forward to the second case.

(ii) Suppose $\kappa_3 \neq \kappa_5$. So, in this situation we can not attach Ω_5^* , and Ω_6^* directly. Now, we are going to fabricate a new group in the following way:

Let D_9, D_{10}, D_{11} , and D_{12} be four semi-circles within the Poincaré disk model. More precisely, D_9 is linked with D_{10} by the hyperbolic Möbius transformation h_4 , whereas, D_{11} twins with D_{12} by the hyperbolic Möbius transformation h_5 . Let, these four semi-circles meet the boundary of the disk at the points $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$ respectively. Now, on the boundary of the disk, we take eight points, two between c_1 and c_2 , viz., \tilde{p} (near the point c_1) and \tilde{w} (near the point c_2), two between c_3 and c_4 , viz., \tilde{v} (near the point c_3) and \tilde{u} (near the point c_4), two between c_5 and c_6 , viz., \tilde{t} (near the point c_5) and \tilde{s} (near the point c_6), and finally two between c_7 and c_8 , viz., \tilde{r} (near the point c_7) and \tilde{q} (near the point c_8). Now, one can draw four non-Euclidean lines by joining the points \tilde{p} to \tilde{q} , \tilde{w} to \tilde{v} , \tilde{u} to \tilde{t} , and \tilde{s} to \tilde{r} . Assume that, these four curves intersect in the four semi-circles D_9, D_{10}, D_{11} , and D_{12} at the points $p, w; v, u; t, s; r$, and q . Suppose the lengths of the curve lines \overline{wv} , \overline{rs} , and \overline{pq} (or \overline{tu}) are $\kappa_2, \kappa_4, 2\kappa_6$ respectively. The region (denoted by F_5) joining the points $w, p, c_1, c_8, q, r, s, t, c_5, c_4, u, v$, and w represents the fundamental domain of the Fuchsian group ($\Pi_7 = \langle h_4, h_5 \rangle$) with signature $(0, 0, 2, 0, 1)$.

Now, to glue the two groups Ω_5 , and Ω_6 , we cut their corresponding hyperbolic surfaces one funnel each along the neck of their suitable funnel ends consisting neck lengths κ_2 and κ_4 respectively. After this operation, it reduces two groups, one is Ω_5^* , and other one Ω_6^* . Then, we add the Fuchsian group Π_7 in such a way that the neck length can match with the groups Ω_5^* , and other one Ω_6^* . Note that, when we paste with Ω_5^* to Π_7 the curve lines (i.e., \overline{de} in Ω_5^* and \overline{vw} in Π_7) can match, i.e, their lengths same but directions are opposite to each other. Now, by using the combination theorem, one can easily get a Fuchsian group $\Omega_{57} = \langle \Omega_5^*, \Pi_7 \rangle$ with signature $(0, 0, 2, 0, 2)$. Furthermore, we will attach the groups Ω_{57} and Ω_6^* on the analogy of the technique that we mention just above (i.e., the direction of the curve lines \overline{rs} in Ω_{57} and \overline{no} in Ω_6^*). Now, again, by utilizing the combination theorem, we get a Fuchsian group $\Omega_{576} = \langle \Omega_{57}, \Omega_6^* \rangle (= \Omega^3, \text{ say})$ with signature $(0, 0, 2, 1, 2)$. Therefore, the fundamental domain of the group $\Omega_{576} (= \Omega^3, \text{ say})$ is the region bounded by the non-Euclidean lines joining the points $a_2, i, b, c, p, c_1, c_8, q, j, k, b_6, l, m, t, c_5, c_4, u, f, g, h, a_3$, and a_2 . So, Lemma 6.1 is proven. □

Lemma 6.2. (Construction 4.) *Let, Ω_7 and Ω_8 be two Fuchsian groups of signatures $(0, 0, 2, 1, 0)$ and $(1, 0, 2, 1, 0)$. Assume that, both the corresponding surfaces possesses different lengths boundaries in their conformal holes, $\kappa_i, i = 1, 2, \dots, 4$, where $\kappa_i \neq \kappa_j$, for $i \neq j$. Then:*

(1) *for a Fuchsian group Π_9 of signature $(0, 0, 3, 0, 0)$, we prove that $\Omega^4 = \langle \Omega_7, \Pi_9, \Omega_8 \rangle$.*

(2) *Gluing two surfaces of types $(0, 0, 0, 1, 2)$ yields a hyperbolic surface of type $(1, 0, 3, 2, 0)$. In particular, Ω^4 represents a hyperbolic surface of genus one, two cusps and three conformal holes.*

Proof. Let $D_1, D_2, D_3,$ and D_4 be four semi-circles within the Poincaré disk model. In particular, D_1 is paired with D_2 by the parabolic Möbius transformation p_1 , whereas D_3 twins with D_4 by the hyperbolic Möbius transformation h_1 . Assume that these four semi-circles meet the boundary of the disk at the points $a_1, a_2; a_2, a_3; a_4, a_5; a_6, a_7$ respectively. Now, on the boundary of the disk, we take six points, one between a_1 and a_2 , viz., \tilde{b} (near the point a_1), one between a_2 and a_3 , viz., \tilde{g} (near the point a_3), two between a_4 and a_5 , viz., \tilde{f} (near the point a_4) and \tilde{e} (near the point a_5), and finally two, between the points a_6 and a_7 , viz., \tilde{d} (near the point a_6) and \tilde{c} (near the point a_7). We draw the non-Euclidean curve lines by joining the points \tilde{b} to \tilde{c} , \tilde{g} to \tilde{f} , and \tilde{e} to \tilde{d} . Notice that, these four curves intersect in the four semi-circles $D_1, D_2, D_3,$ and D_4 at the points $b; g; f, e; d,$ and c respectively. Suppose the lengths of the curve line \overline{bc} (or \overline{fg}) is κ_1 , and \overline{de} is κ_2 . Observe that the region (denoted by F_6) joining the points $a_2, b, c, d, e, f, g,$ and a_2 represents the fundamental domain of a Fuchsian group ($\Omega_7 = \langle p_1, h_1 \rangle$) with signature $(0, 0, 2, 1, 0)$. In particular, the Fuchsian group Ω_7 provides a hyperbolic Riemann surface with one cusp and two conformal holes where the lengths of the closed geodesic boundaries are $2\kappa_1$ and κ_2 . Now, we aim to consider another Fuchsian group Ω_8 of signature $(1, 0, 2, 1, 0)$ in the following way.

Let $D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11},$ and D_{12} be eight semi-circles within the Poincaré disk model. In fact, D_5 is linked with D_6 by the hyperbolic Möbius transformation h_2 , whereas D_7 twins with D_9 by the hyperbolic Möbius transformation h_4 , and D_{10} is paired with D_8 by the hyperbolic Möbius transformation h_3 ; On the other hand, D_{12} is linked with D_{11} by the parabolic Möbius transformation p_2 . Suppose these eight semi-circles $D_5, D_6, D_7, D_8, D_9, D_{10}, D_{11}$ and D_{12} meet the boundary of the disk at the points $b_1, b_2; b_3, b_4; b_5, b_6; b_7, b_8, b_9, b_{10}; b_{11}, b_{12}; b_{12}, b_{13}; b_{13}, b_{14}$ respectively. Now, on the boundary of the disk, we take six points, one between b_{14} and b_{13} , viz., \tilde{k}' (near the point b_{14}), one between b_5 and b_6 , viz., \tilde{k} (near the point b_5), two between b_1 and b_2 , viz., \tilde{j}' (near the point b_1) and \tilde{h} (near the point b_2), and finally two, between the points b_3 and b_4 , viz., \tilde{i} (near the point b_3) and \tilde{j} (near the point b_4). We draw three non-Euclidean curve lines by joining the points \tilde{k}' to \tilde{j}' , \tilde{h} to \tilde{i} , and \tilde{j} to \tilde{k} . Clearly, these three curves intersect in the four semi-circles $D_{12}, D_5, D_6,$ and D_7 at the points $k'; j', h; i, j;$ and k respectively. We assume that the lengths of the curve line $\overline{j'k'}$ (or \overline{jk}) is $2\kappa_4$, and \overline{hi} is κ_3 . Furthermore, we suppose that the semi-circles D_7 and D_8 intersect within the

poincaré disk at the point ‘x’, whereas D_8 and D_9 meet at ‘y’; on the other hand, D_9 and D_{10} intersect at the point ‘z’. Now, we assert that the region (denoted by F_7) joining the points $h, j', k', b_{13}, b_{12}, z, y, x, k, j, i$, and h represents the fundamental domain of a Fuchsian group ($\Omega_8 = \langle h_2, h_3, h_4, p_2 \rangle$) with signature $(1, 0, 2, 1, 0)$. In particular, this Fuchsian group Ω_8 provides a hyperbolic Riemann surface with one genus, one cusp end, and two conformal holes where the lengths of the closed geodesic boundaries are $2\kappa_4$ and κ_3 .

Now, if someone wishes to attach the two Fuchsian groups Ω_7 and Ω_8 , two cases will arise:

(i) If possible, let $\kappa_2 = \kappa_3$. Then, one can easily glue these two groups Ω_7 and Ω_8 in a direct way. This act indicates a Fuchsian group with signature $(1, 0, 2, 2, 0)$. Suppose this will not happen. Then, one can move forward to the second case.

(ii) Assume that $\kappa_2 \neq \kappa_3$. So, in this process, we can not attach Ω_7 , and Ω_8 directly. Now, to glue these two groups, we are going to fabricate a new group in the subsequent way:

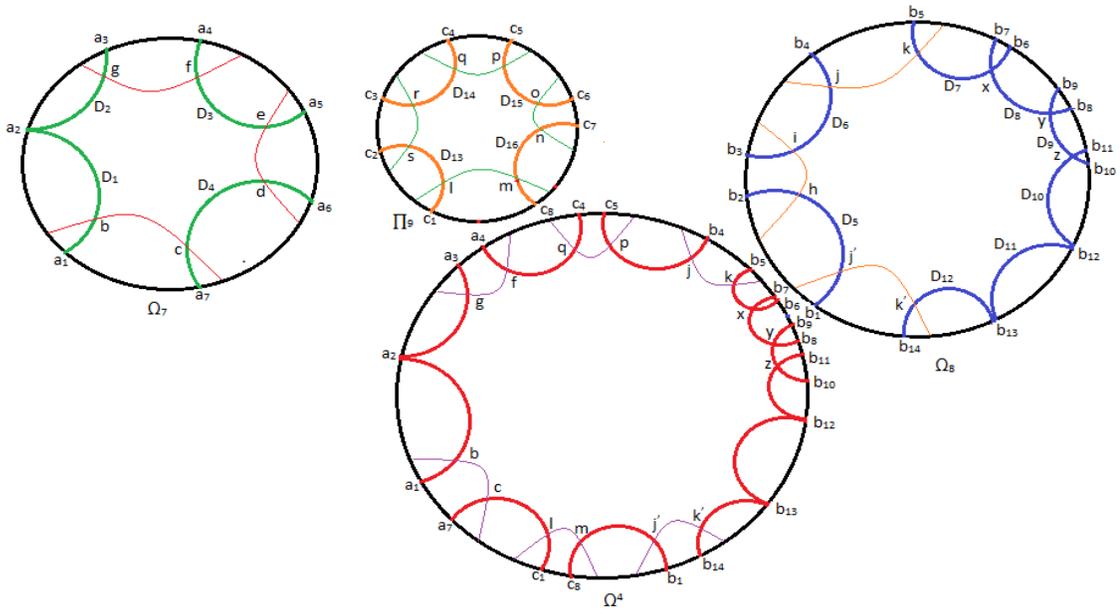


Figure : 2.2.1

Let D_{13}, D_{14}, D_{15} , and D_{16} be four semi-circles within the Poincaré disk model. In fact, D_{13} is linked with D_{14} by the hyperbolic Möbius transformation h_4 , on the other hand, D_{16} is paired with D_{15} by the hyperbolic Möbius transformation h_5 . Suppose these four semi-circles D_{13}, D_{14}, D_{15} , and D_{16} meet the boundary of the disk at the points $c_1, c_2; c_3, c_4; c_5, c_6; c_7, c_8$ respectively. Now, on the boundary of the disk, we take eight points, two between c_1 and c_2 , viz., \tilde{l} (near the point c_1) and \tilde{s} (near the point c_2), two between c_3 and c_4 , viz., \tilde{r} (near the

point c_3) and \tilde{q} (near the point c_4), two between c_5 and c_6 , viz., \tilde{p} (near the point c_5) and \tilde{o} (near the point c_6), and finally two between c_7 and c_8 , viz., \tilde{n} (near the point c_7) and \tilde{m} (near the point c_8). Now, we draw four non-Euclidean lines by joining the points \tilde{l} to \tilde{m} , \tilde{s} to \tilde{r} , \tilde{q} to \tilde{p} , and \tilde{o} to \tilde{n} . Suppose these four curves intersect in the four semi-circles D_{13} , D_{14} , D_{15} , and D_{16} at the points $l, s; r, q; p, o; n, m$. Let the lengths of the curve lines \overline{rs} , \overline{on} , and \overline{lm} (or \overline{pq}) be $\kappa_2, \kappa_3, 2\kappa_4$ respectively. The region (denoted by F_8) joining the points s, l, m, n, o, p, q, r , and s represents the fundamental domain of the Fuchsian group ($\Pi_{10} = \langle h_4, h_5 \rangle$) with signature $(0, 0, 3, 0, 0)$.

Now, to glue the two groups Ω_7 , and Ω_8 , we just need to appear another Fuchsian group Π_{10} in the scenario. First, we take two groups Ω_7 and Π_{10} in such a way that the curve lines (i.e., \overline{de} in Ω_7 and \overline{rs} in Π_{10}) can match, i.e, their lengths are the same (i.e., κ_2) but directions are opposite to each other. such their length of the closed geodesic boundary can match (i.e., length same, i.e., κ_2 , and directions are opposite to each other). After this operation, clearly, by utilizing combination theorem, we get a Fuchsian group $\langle \Omega_7, \Pi_{10} \rangle = \Gamma_1$ with signature $(0, 0, 3, 1, 0)$. Furthermore, we will proceed to glue the groups Γ_1 and Ω_8 on the analogy of the technique that we have just carried out in the above (i.e., the direction of the curve lines \overline{on} in Γ_1 and \overline{hi} in Ω_8). Now, again, by using combination theorem, we achieve a Fuchsian group $\Gamma_2 = \langle \Gamma_1, \Omega_8 \rangle (= \Omega^4, \text{ say})$ with signature $(1, 0, 3, 2, 0)$. Therefore, the fundamental region of the group $\Gamma_1 (= \Omega^4, \text{ say})$ is the domain bounded by the non-Euclidean lines joining by the points $a_2, b, c, l, m, j', k', b_{13}, b_{12}, z, y, x, k, j, p, q, f, g$, and a_2 . This completes the proof of Lemma 6.2.

□

7. PROOF OF THE FIRST MAIN THEOREM

In this Section, we have initiated a general strategy for constructing an arbitrary finite type non-compact hyperbolic Riemann surface via its fundamental groups.

Proof of Theorem 1.1. Step 1. Let, Ω_1 and Ω_2 be two Fuchsian groups with signatures $(1, 0, 0, 0, 2)$ and $(0, 0, 1, 0, 2)$ respectively. Let, $SC_1, SC_2, SC_3, SC_4, SC_5, SC_6, SC_7$, and SC_8 be eight semi-circles within the Poincaré disk model. In particular, SC_1 is paired with SC_3 by the hyperbolic Möbius transformation h_1 , SC_4 is linked with SC_2 by the hyperbolic Möbius transformation h_2 , SC_8 twins with SC_5 by the hyperbolic Möbius transformation h_3 , and SC_7 is paired with SC_6 by the hyperbolic Möbius transformation h_4 . We assume that these eight semi-circles meet the boundary of the disk at the points $a_1, a_2; a_3, a_4; a_5, a_6; a_7, a_8; a_8, a_9; a_{10}, a_{11}; a_{12}, a_{13}; a_{14}, a_1$ respectively. Suppose, within the Poincaré disk, SC_1 meets SC_2 at the point x_1 , SC_2 intersect SC_3 at x_2 , and SC_3 touch SC_4 at

the point x_3 . Now, on the boundary of the disk, we take two points, one between a_{12} and a_{13} , viz., \tilde{A}_1 (near the point a_{12}) and one between a_{10} and a_{11} , viz., \tilde{A}_2 (near the point a_{11}). We draw the non-Euclidean curve line by joining the points \tilde{A}_1 to \tilde{A}_2 . The curve line $\overline{\tilde{A}_1\tilde{A}_2}$ intersects in the two semi-circles SC_6 and SC_7 at the points A_1 and A_2 . Let the lengths of the curve line $\overline{\tilde{A}_1\tilde{A}_2}$ be κ_1 . Furthermore, we assume that the axis of the hyperbolic element h_3 intersects at the point x_4 at SC_8 and x_5 at SC_5 . Now, the region (denoted by R_1) joining the points $a_1, x_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{11}, a_{12}, A_1, a_{13}, a_{14}, x_4$, and a_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_1 = \langle h_1, h_2, h_3, h_4 \rangle$) with signature $(1, 0, 0, 0, 2)$. Now, we consider the other Fuchsian group Ω_2 with signature $(0, 0, 1, 0, 2)$. We proceed in a similar way as we have performed in the above construction. Now, the region (denoted by R_2) joining the points $a_{16}, A_3, A_7, A_8, A_6, a_{21}, a_{20}, A_5, A_{10}, A_9, A_4, a_{17}$ and a_{16} represents a fundamental domain of a Fuchsian group (denoted by $\Omega_2 = \langle h_5, h_6 \rangle$) with signature $(0, 0, 1, 0, 2)$. Note that Ω_2 is a Fuchsian group with signature $(0, 0, 1, 0, 2)$ consisting one closed geodesic boundary of length $2\kappa_2\kappa_3$, and two funnels with neck lengths κ_2 , and κ_3 . Let us consider, a Fuchsian group Ω_3 with signature $(0, 0, 2, 0, 1)$ consisting one funnel end of neck length $2\kappa_1\kappa_2$ and two conformal holes with lengths of their closed geodesic boundaries κ_1 and κ_2 . Now, to glue two Fuchsian groups Ω_1 and Ω_2 with signatures $(1, 0, 0, 0, 2)$ and $(0, 0, 1, 0, 2)$, we cut the two corresponding surfaces one funnel end each along the neck of their funnels. After this operation, it reduces two groups, one is Ω_1^* and other one Ω_2^* . Then adding a Fuchsian group Ω_3 in such a way that the neck length can match with the groups Ω_1^* and Ω_2^* . Note that, when we paste with Ω_1^* to Ω_3 the curve lines (i.e., $\overline{A_1A_2}$ in Ω_1^* and $\overline{A_{12}A_{11}}$ in Ω_3) can match, i.e, their lengths are same and directions are opposite to each other. Now, by using the combination theorem, we obtain a Fuchsian group $\Omega_{13} = \langle \Omega_1^*, \Omega_3 \rangle$ with signature $(1, 0, 1, 0, 2)$. After that, we will paste the groups Ω_{13} and Ω_2^* on the analogy of the technique that we mention just before (i.e., the direction of the curve lines $\overline{A_{13}A_{14}}$ in Ω_{13} and $\overline{A_3A_4}$ in Ω_2^*). Consequently, again, one can utilize combination theorem to get a Fuchsian group $\Omega_{132} = \langle \Omega_{13}, \Omega_2^* \rangle$ with signature $(1, 0, 1, 0, 3)$. Therefore, the fundamental domain of the arising group Ω_{132} is the region bounded by the non-Euclidean lines joining the points $a_1, x_2, x_3, a_8, x_5, a_9, a_{10}, A_2, a_{26}, a_{27}, A_4, A_9, A_{10}, a_{20}, a_{21}, A_8, A_7, A_3, a_{30}, a_{23}, A_1, a_{13}, a_{14}, x_4$, and a_1 .

Step 2. Let us consider two Fuchsian groups Ω_{132} and Ω_4 with signatures $(1, 0, 1, 0, 3)$ and $(0, 0, 1, 1, 2)$ respectively. To glue these two groups, we apply same technique as we have operated in the preceding, i.e., cutting one funnel end each for the groups Ω_{132} and Ω_4 along the neck of their corresponding funnels and adding a new Fuchsian group Ω_5 with signature $(0, 0, 2, 0, 1)$.

Note that, the region joining the points $B_1, B_1^*, b_1, b_8, B_2^*, B_4, B_3, B^{**}, b_5, b_4, B^*, B_2$, and B_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_5 = \langle h_9, h_{10} \rangle$) with signature $(0, 0, 2, 0, 1)$. The domain joining the points $B_5, B_7', b_9, b_{18}, B_8', b_{17}, b_{16}, B_{10}, B_9, B_8, b_{13}, b_{12}, B_7, B_6$ and B_5 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_4^* = \langle h_{11}, h_{12} \rangle$) with signature $(0, 0, 2, 1, 1)$. Now, to glue two Fuchsian groups Ω_{132} and Ω_4 with signatures $(1, 0, 1, 0, 3)$ and $(0, 0, 1, 1, 2)$, we cut the two Fuchsian surface groups one funnel end each along the neck of their funnels. After this action, it reduces two groups, one is Ω_{132}^* and other one Ω_4^* . Then adding a Fuchsian group Ω_5 in such a way that the neck length can match with the groups Ω_{132}^* and Ω_4^* . Observe that, when we attach with Ω_{132}^* to Ω_5 the curve lines (i.e., $\overline{A_5A_6}$ in Ω_{132}^* and $\overline{B_1B_2}$ in Ω_4^*) can match, i.e, their lengths are same and directions are opposite to each other. Now, we use combination theorem to obtain a Fuchsian group $\Omega_{1325} = \langle \Omega_{132}^*, \Omega_5 \rangle$ with signature $(1, 0, 2, 0, 3)$. After that, we will glue the groups Ω_{1325} and Ω_4^* on the harmony of the technique that we have mentioned just before (i.e., the direction of the curve lines $\overline{B_3B_4}$ in Ω_{1325} and $\overline{B_5B_6}$ in Ω_4^*). Again, one can utilize combination theorem to get a Fuchsian group $\Omega_{13254} = \langle \Omega_{1325}, \Omega_4^* \rangle$ with signature $(1, 0, 2, 1, 4)$. Therefore, the fundamental domain of the arising group Ω_{13254} is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B_7', b_9, b_{18}, B_8', b_{17}, B^*, b_{16}, B_{10}, B_9, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$, and a_1 .

Step 3. Furthermore, we take two Fuchsian groups, one is our constructed group, i.e., Ω_{13254} and other is Ω_6 with signatures $(1, 0, 2, 1, 4)$ and $(0, 0, 1, 0, 3)$ respectively. To glue these two groups, we simply apply direct procedure keeping in mind that the attaching sides can match. In fact, we take the group Ω_6 in such a way that this operation can happen. Clearly, from figure, the region joining the points $c_1, c_{12}, C_3, C_5', c_{11}, c_{10}, C_6', C_8, c_9, C_8, C_6, c_7, c_6, C_5, C_4, c_5, c_4, C_2, C_1$, and c_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_6 = \langle h_{13}, h_{14}, h_{15} \rangle$) with signature $(0, 0, 1, 0, 3)$. Now, to glue two Fuchsian groups Ω_{13254} and Ω_6 with signatures $(1, 0, 2, 1, 4)$ and $(0, 0, 1, 0, 3)$, we need to take care of the lengths of the closed geodesic boundaries of the corresponding conformal holes, i.e., the curve lines $\overline{C_1C_2}$ in Ω_6 and $\overline{B_9B_{10}}$ in Ω_{13254} can match, i.e, their lengths are same and directions are opposite to each other. Now, after utilizing the combination theorem, we get a Fuchsian group $\Omega_{132546} = \langle \Omega_{13254}, \Omega_6 \rangle$ with signature $(1, 0, 1, 1, 7)$. So, the fundamental domain of the arising group Ω_{132546} is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B_7', b_9, b_{18}, B_8', b_{17}, B^*, b_{16}, C_1, c_1, c_{12}, C_3, C_5', c_{11}, c_{10}, C_6', C_8, c_9, c_8, C_6, c_7, c_6, C_5, C_4, c_5, c_4, B_9, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$, and a_1 .

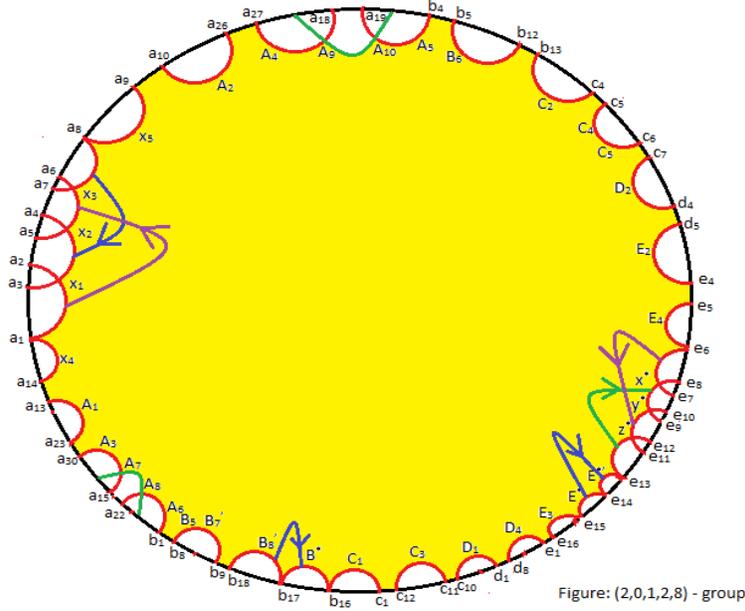


Figure : 2.2.1

Step 4. Finally, we take two Fuchsian groups Ω_{132546} and Ω_7 with signatures $(1, 0, 1, 1, 7)$ and $(1, 0, 0, 1, 2)$ respectively. To glue these two groups, we apply same technique as we have performed in the overhead construction, i.e., cutting one funnel end each for the groups Ω_{132546} and Ω_7 along the neck of their corresponding funnels and adding a new Fuchsian group Ω_8 with signature $(0, 0, 2, 0, 1)$. We take the region joining the points $D_1, d_1, d_8, D_4, D_3, d_5, d_4, D_2$, and D_1 represents the fundamental domain of a Fuchsian group (denoted by $\Omega_8 = \langle h_{16}, h_{17} \rangle$) with signature $(0, 0, 2, 0, 1)$. Now, the domain joining the points $e_2, E_1, e_1, e_{16}, E_3, e_{15}, e_{14}, e_{13}, z^*, y^*, x^*, e_6, E_4, e_5, e_4, E_2, e_3$, and e_2 represents a fundamental domain of a Fuchsian group (denoted by $\Omega_7 = \langle h_{18}, h_{19}, h_{20}, h_{21} \rangle$) with signature $(1, 0, 0, 1, 2)$. Now, to glue two Fuchsian groups Ω_{132546} and Ω_7 with signatures $(1, 0, 1, 1, 7)$ and $(1, 0, 0, 1, 2)$, we cut the two Fuchsian surface groups one funnel end each along the neck of their funnels. After this operation, it reduces two groups, one is Ω_{132546}^* and other one Ω_7^* . Then adding a Fuchsian group Ω_8 in such a way that the neck length can match with the groups Ω_{132546}^* and Ω_7^* . Note that, when we paste the group Ω_{132546}^* with Ω_8 the curve lines (i.e., $\overline{C_7C_8}$ in Ω_{132546}^* and $\overline{D_1D_2}$ in Ω_8^*) can match, i.e, their lengths are same and directions are opposite to each other. Now, we use combination theorem to obtain a Fuchsian group $\Omega_{1325468} = \langle \Omega_{132546}^*, \Omega_8^* \rangle$ with signature $(1, 0, 1, 1, 8)$. After that, we will attach the groups $\Omega_{1325468}$ and Ω_7^* on the harmony of the technique that we have mentioned in the above construction (i.e., the direction of the curve lines $\overline{D_3D_4}$ in $\Omega_{1325468}$

and $\overline{E_1 E_2}$ in Ω_7^*). Again, by utilizing the combination theorem, we get a Fuchsian group $\Omega_{13254687} = \langle \Omega_{1325468}, \Omega_7^* \rangle$ (say, Ω^*) with signature $(2, 0, 1, 2, 8)$. Hence, the fundamental domain of the arising group $\Omega_{13254687}$ is the region bounded by curve lines joining the points $a_1, x_4, a_{14}, a_{13}, A_1, a_{23}, a_{30}, A_3, A_7, A_8, A_6, b_1, b_8, B_5, B_7', b_9, b_{18}, B_8', b_{17}, B^*, b_{16}, C_1, c_1, c_{12}, C_3, c_{11}, c_{10}, D_1, d_1, d_8, E_1, e_1, e_{16}, E_3, E_{15}, E^*, e_{14}, E^{*'}, e_{13}, z^*, y^*, x^*, e_6, E_4, e_5, e_4, E_2, d_5, d_4, D_2, c_7, c_6, C_5, C_4, c_5, c_4, C_2, b_{13}, b_{12}, B_6, b_5, b_4, A_5, A_{10}, A_9, A_4, a_{27}, a_{26}, A_2, a_{10}, a_9, x_5, a_8, x_3, x_2, x_1$, and a_1 .

So, Theorem 1.1 is proven. This establishes the first main goal of this paper.

8. PROOF OF THE SECOND MAIN THEOREM

In this section, our goal is to construct four kinds of infinite type generalized non-tight flute surfaces and four types of infinite type generalized tight flute surfaces via fundamental groups. At first, we are looked into non-tight surfaces. After that, we will investigate the tight ones.

Proof of Theorem 1.2(a). (Constructing infinite type generalized non-tight flute surface: Type 1).

Let, $\Gamma_1 (= \langle h_1, h_2, h_3 \rangle)$ and $\Gamma_2 (= \langle h_4, h_5, h_6 \rangle)$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 4)$ consisting of neck lengths $\kappa_1, \kappa_2, \kappa_3, \kappa_4$, and $\kappa_5, \kappa_6, \kappa_7, \kappa_8$ respectively, where h_i 's are hyperbolic elements. Now, we wish to attach these two groups. For this purpose, we cut the two corresponding surfaces of the groups Γ_1 and Γ_2 (reduce to Γ_1^* and Γ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are κ_4 and κ_5 respectively. After that, we add a Fuchsian group $\Gamma_3 = \langle h_7, h_8, h_9 \rangle$ of signature $(0, 0, 2, 0, 2)$ consisting two funnel ends of neck lengths κ_9, κ_{10} and two conformal holes with lengths of their closed geodesic boundaries are κ_4 and κ_5 . In particular, we take Γ_3 in such a way that it can match with the groups Γ_1^* and Γ_2^* . So, taking the notion of gluing Fuchsian groups from above sections, we use the combination theorem to get a Fuchsian group $\Gamma_{132} = \langle \Gamma_1^*, \Gamma_3, \Gamma_2^* \rangle$ of signature $(0, 0, 0, 0, 8)$. In the second step, we consider two groups one is Γ_{132} and other is $\Gamma_4 = \langle h_{10}, h_{11}, h_{12} \rangle$ of signature $(0, 0, 0, 0, 4)$ consisting four funnel ends of neck lengths $\delta_{11}, \delta_{12}, \delta_{13}$, and δ_{14} . Then, we proceed in a similar way as we have carried out in the preceding construction, i.e., cut the two corresponding surfaces of the groups Γ_{132} and Γ_4 (transform to Γ_{132}^* and Γ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_7 and δ_{11} respectively. Furthermore, we add a Fuchsian group $\Gamma_5 = \langle h_{13}, h_{14}, h_{15} \rangle$ of signature $(0, 0, 2, 0, 2)$ with neck lengths δ_{15}, δ_{16} , and closed geodesic boundaries lengths are δ_7 and δ_{11} . Now, we again apply the combination theorem to obtain a

Fuchsian group $\Gamma_{13254} = \langle \Gamma_{132}^*, \Gamma_5, \Gamma_4^* \rangle$ of signature $(0, 0, 0, 0, 12)$. So, in this way, one can continue this process infinitum number of times to get an infinitely generated Fuchsian group $\Gamma_{13254\dots} = \langle \Gamma_1^*, \Gamma_3, \Gamma_2^{**}, \Gamma_5, \Gamma_4^{**}, \Gamma_7, \Gamma_6^{**}, \dots \rangle = \langle h_1, h_2, h_3, h_4, \dots \rangle$. This infinitely generated group of type $(0, 0, 0, 0, \infty)$ provides infinite type generalized non-tight flute surface where each block is of infinite hyperbolic area.

Proof of Theorem 1.2(b). (Constructing infinite type generalized non-tight flute surface: Type 2).

Let, $\Delta_1 = \langle h_1, h_2 \rangle$ and $\Delta_2 = \langle h_3, h_4 \rangle$ be two Fuchsian groups with signature $(0, 0, 0, 0, 3)$ consisting neck lengths $\alpha_1, \alpha_2, \alpha_3$, and $\alpha_4, \alpha_5, \alpha_6$ respectively, where h_i 's are hyperbolic elements. Now, we want to glue these two groups. So, we cut the two corresponding surfaces of the groups Δ_1 and Δ_2 (changing to Δ_1^* and Δ_2^*) one funnel end each, along the neck of their funnels, where the lengths of their necks are α_3 and α_4 respectively. After that, we add a Fuchsian group $\Delta_3 = \langle h_5, h_6 \rangle$ of signature $(0, 0, 2, 0, 1)$ consisting one funnel end of neck length α_7 , and two conformal holes with lengths of their closed geodesic boundaries are α_3 and α_4 . More precisely, we consider Δ_3 in such a way that it can match with the groups Δ_1^* and Δ_2^* . Now, we use the combination theorem to obtain a Fuchsian group $\Delta_{132} = \langle \Delta_1^*, \Delta_3, \Delta_2^* \rangle$ of signature $(0, 0, 0, 0, 5)$. In the second stage, we take two groups, one is Δ_{132} and the other is Δ_4 of signature $(0, 0, 0, 0, 3) = \langle h_7, h_8 \rangle$. After that, we move forward in an analogous way as we have carried out earlier, i.e., cut the two corresponding surfaces of the groups Δ_{132} and Δ_4 (transforming to Δ_{132}^* and Δ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are α_6 and α_8 respectively. Further, we add a Fuchsian group $\Delta_5 = \langle h_9, h_{10} \rangle$ of signature $(0, 0, 2, 0, 1)$ with neck length α_{11} , and closed geodesic boundaries lengths are α_6 and α_8 . Now, we again utilize the combination theorem to obtain a Fuchsian group $\Delta_{13254} = \langle \Delta_{132}^*, \Delta_5, \Delta_4^* \rangle$ of signature $(0, 0, 0, 0, 7)$. Therefore, in this way, one can continue this process infinitum number of times to get an infinitely generated Fuchsian group $\Delta_{13254\dots} = \langle \Delta_1^*, \Delta_3, \Delta_2^{**}, \Delta_5, \Delta_4^{**}, \Delta_7, \Delta_6^{**}, \dots \rangle = \langle h_1, h_2, h_3, h_4, \dots \rangle$. This infinitely generated group $(0, 0, 0, 0, \infty)$ provide infinite type generalized non-tight flute surfaces where each block is of infinite hyperbolic area, but, each block of the Fuchsian surface group $\Delta_{132547698\dots}$ contains one funnel end (apart from the first one), whereas for the group $\Gamma_{13254\dots}$, two funnels appear in each block.

Proof of Theorem 1.2(c). (Constructing infinite type generalized non-tight flute surface: Type 3).

Let, $\Omega_1 = \langle h_1, h_2 \rangle$ and $\Omega_2 = \langle h_3, h_4 \rangle$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 3)$ consisting neck lengths $\beta_1, \beta_2, \beta_3$, and $\beta_4, \beta_5, \beta_6$ respectively. Now, to glue these two groups Ω_1 and Ω_2 , we cut the two corresponding surfaces (reduce to Ω_1^* and Ω_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are β_3 and β_4 respectively. After this operation, we add a Fuchsian group $\Omega_3 = \langle h_5, h_6 \rangle$ of signature $(0, 0, 3, 0, 0)$ consisting three conformal holes with lengths of their closed geodesic boundaries are β_3, β_4 , and β_7 . In fact, we take Ω_3 in such a way that it can match with the groups Ω_1^* and Ω_2^* . After that, we use the combination theorem to obtain a Fuchsian group $\Omega_{132} = \langle \Omega_1^*, \Omega_3, \Omega_2^* \rangle$ of signature $(0, 0, 1, 0, 4)$. In the next stage, we take two groups one is Ω_{132} and other is $\Omega_4 = \langle h_7, h_8 \rangle$ of signature $(0, 0, 0, 0, 3)$. Then, we proceed a similar path as we have done in the earlier construction, i.e., cut the two corresponding surfaces of the groups Ω_{132} and Ω_4 (transform to Ω_{132}^* and Ω_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are β_6 and β_8 respectively. After that, we add a Fuchsian group $\Omega_5 = \langle h_9, h_{10} \rangle$ of signature $(0, 0, 3, 0, 0)$ with closed geodesic boundaries lengths are β_6, β_8 and β_{11} . So, again utilizing the combination theorem, we obtain a Fuchsian group $\Omega_{13254} = \langle \Omega_{132}^*, \Omega_5, \Omega_4^* \rangle$ of signature $(0, 0, 2, 0, 5)$. Consequently, in this fashion, one can continue this process infinitum number of times to get an infinitely generated Fuchsian group $\Omega_{13254\dots} = \langle \Omega_1^*, \Omega_3, \Omega_2^*, \Omega_5, \Omega_4^*, \Omega_7, \Omega_6^*, \dots \rangle = \langle h_1, h_2, h_3, h_4, \dots \rangle$. This infinitely generated group of type $(0, 0, \infty, 0, \infty)$ supplies infinite type generalized non-tight flute surface, where 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area, but, 2nd, 4th, 6th, 8th, ... blocks are of finite area (i.e., π), h_i 's are hyperbolic elements, $i \in \mathbb{N}$.

Proof of Theorem 1.2(d). (Constructing infinite type generalized non-tight flute surface: Type 4).

Let, $\Phi_1 (= \langle h_1, h_2, h_3 \rangle)$ and $\Phi_2 (= \langle h_4, h_5, h_6 \rangle)$ be two Fuchsian surface groups with signature $(0, 0, 1, 0, 3)$ consisting one conformal hole each with lengths of their closed geodesics being γ_4 and γ_8 respectively and three funnel ends (each) of neck lengths $\gamma_1, \gamma_2, \gamma_3$, and $\gamma_5, \gamma_6, \gamma_7$ respectively. Now, we wish to glue these two groups, but assume that $\gamma_4 \neq \gamma_8$. So, we cut the two corresponding surfaces of the groups Φ_1 and Φ_2 (changing to Φ_1^* and Φ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are γ_3 and γ_5 respectively. After that, we add a Fuchsian group $\Phi_3 (= \langle h_7, h_8, h_9 \rangle)$ of signature $(0, 0, 3, 0, 1)$ consisting

one funnel end of neck length γ_9 , and three conformal holes with lengths of their closed geodesic boundaries are γ_3 , γ_5 , and γ_{10} . In particular, we take Φ_3 in such a way that it can match with the groups Φ_1^* and Φ_2^* . So, using the combination theorem, we get a Fuchsian group $\Pi_{132} = \langle \Phi_1^*, \Phi_3, \Phi_2^* \rangle$ of signature $(0, 0, 3, 0, 5)$. In the second stage, we consider two groups one is Π_{132} and other is $\Phi_4 (= \langle h_{10}, h_{11}, h_{12} \rangle)$ of signature $(0, 0, 1, 0, 3)$ with neck length γ_{12} , and lengths of the closed geodesic boundaries are γ_{11} , γ_{13} , and γ_{14} . Then, we proceed in a similar way as we have operated in the earlier construction, i.e., cut the two corresponding surfaces of the groups Π_{132} and Φ_4 (reduce to Π_{132}^* and Φ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_7 and γ_{11} respectively. After that, we add a Fuchsian group $\Phi_5 (= \langle h_{13}, h_{14}, h_{15} \rangle)$ of signature $(0, 0, 3, 0, 1)$ with neck length γ_{15} , and closed geodesic boundaries lengths are γ_7 , γ_{11} , and γ_{16} . Now, again utilizing the combination theorem, we obtain a Fuchsian group $\Pi_{13254} = \langle \Pi_{132}^*, \Phi_5, \Phi_4^* \rangle$ of signature $(0, 0, 5, 0, 7)$. Therefore, in this way, we can continue this process infinite number of times to get an infinitely generated Fuchsian group $\Pi_{13254\dots} = \langle \Phi_1^*, \Phi_3, \Phi_2^{**}, \Phi_5, \Phi_4^{**}, \Phi_7, \Phi_6^{**}, \dots \rangle = \langle h_1, h_2, h_3, h_4, \dots \rangle$. This infinitely generated group of type $(0, 0, \infty, 0, \infty)$ indicates infinite type generalized non-tight flute surface, where each block is of infinite hyperbolic area, h_i 's are hyperbolic elements, $i \in \mathbb{N}$.

In the following, we have constructed four new kinds of infinite type generalized tight flute surfaces via fundamental non-compact Fuchsian (surface) groups.

Proof of Theorem 1.2(e). (Constructing infinite type generalized tight flute surface: Type 1).

Let, $\Xi_1 (= \langle h_1, h_2, p_1 \rangle)$ and $\Xi_2 (= \langle h_3, h_4, p_2 \rangle)$ be two Fuchsian surface groups with signature $(0, 0, 0, 1, 3)$ consisting neck lengths α_1 , α_2 , α_3 , and α_4 , α_5 , α_6 , where h_i 's are hyperbolic elements and where p_i 's are parabolic elements respectively. Now, to obtain a new surface we will glue these two groups. For this reason, we cut the two corresponding surfaces of the groups Ξ_1 and Ξ_2 (reduce to Ξ_1^* and Ξ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are α_3 and α_4 respectively. After cutting, we add a Fuchsian group $\Xi_3 (= \langle h_5, h_6, p_3 \rangle)$ of signature $(0, 0, 2, 1, 1)$ consisting one funnel end of neck length α_7 , one cusp, and two conformal holes with lengths of their closed geodesic boundaries are α_3 and α_4 . In particular, we consider Ξ_3 in such a way that it helps to match with the groups Ξ_1^* and Ξ_2^* . Now, we can use the combination theorem to obtain a Fuchsian group $\Xi_{132} = \langle \Xi_1^*, \Xi_3, \Xi_2^* \rangle$ of signature $(0, 0, 0, 3, 5)$. In the next stage, we take two groups one is Ξ_{132} and other is $\Xi_4 (= \langle h_7, h_8, p_4 \rangle)$ of signature $(0, 0, 0, 1, 3)$ with neck lengths α_8 , α_9 , and α_{10} . After that, we

proceed a similar way as we have approached in the construction just above, i.e., cut the two corresponding surfaces of the groups Ξ_{132} and Ξ_4 (reduce to Ξ_{132}^* and Ξ_4^*), one funnel end each along the neck of their funnels where the lengths of their necks are α_6 and α_8 respectively. Now, we move forward in a similar path as we have operated in the preceding framework, i.e., cutting the corresponding surfaces arise from the groups Ξ_{132} and Ξ_4 (reduce to Ξ_{132}^* and Ξ_4^*), one funnel end each along the neck of their funnels, where the lengths of their necks are α_6 and α_8 respectively. After that, we add a Fuchsian group $\Xi_5 (= \langle h_9, h_{10}, p_5 \rangle)$ of signature $(0, 0, 2, 1, 1)$ with neck length α_{11} , and closed geodesic boundaries lengths are α_6 and α_8 . Now, again applying the combination theorem, we obtain a Fuchsian group $\Xi_{13254} = \langle \Xi_{132}^*, \Xi_5, \Xi_4^* \rangle$ of signature $(0, 0, 5, 0, 7)$. So, in this way, we can continue this process infinitum number of times to get an infinitely generated Fuchsian group $\Xi_{13254\dots} = \langle \Xi_1^*, \Xi_3, \Xi_2^{**}, \Xi_5, \Xi_4^{**}, \Xi_7, \Xi_6^{**}, \dots \rangle = \langle h_1, h_2, h_3, \dots; p_1, p_2, p_3, \dots \rangle$. This infinitely generated group of type $(0, 0, 0, \infty, \infty)$ gives rise to infinite type generalized tight flute surface, where each block is of infinite hyperbolic area, h_i 's and p_i 's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.2(f). (Constructing infinite type generalized tight flute surface: Type 2).

Let, $\Psi_1 (= \langle h_1, h_2 \rangle)$ and $\Psi_2 (= \langle h_3, h_4 \rangle)$ be two Fuchsian surface groups with signature $(0, 0, 0, 0, 3)$ consisting neck lengths $\beta_1, \beta_2, \beta_3$, and $\beta_4, \beta_5, \beta_6$ respectively. Now, we aim to glue these two groups to obtain a new surface. So, we cut the two corresponding surfaces of the groups Ψ_1 and Ψ_2 (reduce to Ψ_1^* and Ψ_2^*) one funnel end each along the neck of their funnels where the lengths of their necks are β_3 and β_4 respectively. After this operation, we add a Fuchsian group $\Pi_3 (= \langle h_5, p_1 \rangle)$ of signature $(0, 0, 2, 1, 0)$ consisting one cusp end with two conformal holes where lengths of their closed geodesic boundaries are β_3 and β_4 . In fact, we take Π_3 in such a way that it will fit for the groups Ψ_1^* and Ψ_2^* . So, applying the combination theorem, we get a Fuchsian group $\Psi_{132} = \langle \Psi_1^*, \Pi_3, \Psi_2^* \rangle$ of signature $(0, 0, 0, 1, 4)$. In the second stage, we take two groups one is Ψ_{132} and other is $\Psi_4 (= \langle h_6, h_7 \rangle)$ of signature $(0, 0, 0, 0, 3)$ of neck lengths β_7, β_8 , and β_9 . Now, we can continue in an analogous way as we have performed in the preceding formation, i.e., cutting the corresponding surfaces arise from the groups Ψ_{132} and Ψ_4 (reduce to Ψ_{132}^* and Ψ_4^*), one funnel end each along the neck of their funnels, where the lengths of their necks are β_6 and β_7 respectively. Then, we add a Fuchsian group $\Pi_5 (= \langle h_8, p_2 \rangle)$ of signature $(0, 0, 2, 1, 0)$, where the closed geodesic boundaries lengths are β_6 and β_7 . Now, again we use the combination theorem and we obtain a Fuchsian group

$\Psi_{13254} = \langle \Psi_{132}^*, \Pi_5, \Psi_4^* \rangle$ of signature $(0, 0, 0, 2, 5)$. Hence, in this way, one can continue this process infintium number of times to get an infinitely generated Fuchsian group $\Psi_{13254\dots} = \langle \Psi_1^*, \Pi_3, \Psi_2^{**}, \Pi_5, \Psi_4^{**}, \Pi_7, \Psi_6^{**}, \dots \rangle = \langle h_1, h_2, h_3, \dots; p_1, p_2, p_3, \dots \rangle$. This infinitely generated group of type $(0, 0, 0, \infty, \infty)$ produces an infinite type generalized tight flute surface, where 1st, 3rd, 5th, 7th, ... blocks are of infinite hyperbolic area, whereas 2nd, 4th, 6th, 8th, ... blocks are of finite area (more precisely, π), h_i 's and p_i 's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.2(g). (Constructing infinite type generalized tight flute surface: Type 3).

Let, $\Sigma_1 (= \langle h_1, h_2 \rangle)$ and $\Upsilon_1 (= \langle h_3, p_1 \rangle)$ be two Fuchsian surface groups with signatures $(0, 0, 0, 0, 3)$ and $(0, 0, 0, 1, 2)$ consisting neck lengths $\gamma_1, \gamma_2, \gamma_3$, and γ_4, γ_5 respectively. Now, our goal is to attach these two groups to obtain a new group. Now, we cut the two corresponding surfaces of the groups Σ_1 and Υ_1 (reduce to Σ_1^* and Υ_1^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_3 and γ_4 respectively. After cutting these surfaces, we add a Fuchsian group $\Pi_1 (= \langle h_4, h_5 \rangle)$ of signature $(0, 0, 3, 0, 0)$ consisting three conformal holes where lengths of their closed geodesic boundaries are γ_3, γ_4 , and γ_6 . In particular, we take Π_1 in such a way that it can match with the groups Σ_1^* and Υ_1^* . So, by utilizing the combination theorem, we get a Fuchsian group $\Omega_{111} = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^* \rangle$ of signature $(0, 0, 1, 1, 3)$. In the second step, we consider two groups one is Ω_{111} and other is $\Pi_2 (= \langle h_6, h_7 \rangle)$ of signature $(0, 0, 1, 0, 2)$ consisting one conformal hole with length of the closed geodesic is γ_9 , and onto two funnels end of neck length γ_7 and γ_8 . After that, we will continue in a similar fashion as we have performed in the overhead construction, i.e., cut the two corresponding surfaces of the groups Ω_{111} and Π_2 (transform to Ω_{111}^* and Π_2^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_5 and γ_7 respectively. Furthermore, we add a Fuchsian group $\Sigma_2 (= \langle h_8, h_9 \rangle)$ of signature $(0, 0, 2, 0, 1)$ with neck length γ_{10} , and the lengths of closed geodesic boundaries are γ_5 and γ_7 . Now, by applying the combination theorem, we get a Fuchsian group $\Omega_{11122} = \langle \Omega_{111}^*, \Sigma_2, \Pi_2^* \rangle$ of signature $(0, 0, 2, 1, 4)$. In the third step, we further take two groups one is Ω_{11122} and other is $\Sigma_3 (= \langle h_{10}, h_{11} \rangle)$ of signature $(0, 0, 0, 0, 3)$ consisting three funnel ends with neck lengths γ_{11}, γ_{12} , and γ_{13} . Furthermore, we proceed in a similar way as we have carried out in the preceding formation, i.e., cut the two corresponding surfaces of the groups Ω_{11122} and Σ_3 (convert to Ω_{11122}^* and Σ_3^*), one funnel end each along the neck of their funnels where the lengths of

their necks are γ_8 and γ_{11} respectively. In this situation, we again add a Fuchsian group Υ_2 ($=\langle h_{12}, p_2 \rangle$) of signature $(0, 0, 2, 1, 0)$, where length of the closed geodesic boundary is γ_8 and γ_{11} . So, we can again utilize the combination theorem to get a Fuchsian group $\Omega_{1112223} = \langle \Omega_{11122}^*, \Upsilon_2, \Sigma_3^* \rangle$ of signature $(0, 0, 2, 2, 5)$. In the fourth step, we will again consider two groups one is $\Omega_{1112223}$ and other is Υ_3 ($=\langle h_{13}, p_3 \rangle$) of signature $(0, 0, 0, 1, 2)$ consisting one cusp end and two funnel ends with neck lengths γ_{14} and γ_{15} . Furthermore, we cut the two corresponding surfaces of the groups $\Omega_{1112223}$ and Υ_3 (reduce to $\Omega_{1112223}^*$ and Υ_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are γ_{13} and γ_{14} respectively to glue them to each other. Finally, we again add a Fuchsian group Π_3 ($=\langle h_{14}, h_{15} \rangle$) of signature $(0, 0, 2, 0, 1)$ with neck length γ_{16} , and the lengths of their closed geodesic boundaries are γ_{13} , γ_{14} , and γ_{16} . Hence, by using the combination theorem, we get a Fuchsian group $\Omega_{1112223333} = \langle \Omega_{1112223}, \Pi_3, \Upsilon_3^* \rangle$ of signature $(0, 0, 3, 3, 5)$. Note that, this fourth step is basically a repetition of our previous first step. Therefore, in this way, one can continue this process for an infinitum numbers of times to get an infinitely generated Fuchsian group $\Upsilon = \Omega_{1112223333\dots} = \langle \Sigma_1^*, \Pi_1, \Upsilon_1^{**}, \Sigma_2, \Pi_2^{**}, \Upsilon_2, \Sigma_3^{**}, \Pi_3, \Upsilon_3^{**}, \dots \rangle = \langle h_1, h_2, h_3, \dots; p_1, p_2, p_3, \dots \rangle$ of type $(0, 0, \infty, \infty, \infty)$, where 2nd, 5th, 8th, ... blocks are of non-compact infinite hyperbolic areas, whereas 2nd, 4th, 6th, ... blocks are of compact and hence finite areas (more precisely, π), on the other hand, 3rd, 6th, 9th, ... blocks are non-compact but finite hyperbolic areas, h_i 's and h_i 's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

Proof of Theorem 1.2(h). (Constructing infinite type generalized tight flute surface: Type 4).

Let, Λ_1 ($=\langle h_1, h_2, h_3 \rangle$) and Ω_1 ($=\langle h_4, h_5, p_1 \rangle$) be two Fuchsian surface groups with signatures $(0, 0, 0, 0, 4)$ and $(0, 0, 0, 1, 3)$ consisting neck lengths $\delta_1, \delta_2, \delta_3, \delta_4$, and $\delta_5, \delta_6, \delta_7$ respectively. Now, we aim to glue these two groups to obtain a new group. So, we cut the two corresponding surfaces of the groups Λ_1 and Ω_1 (reduce to Λ_1^* and Ω_1^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_4 and δ_5 respectively. After cutting these surfaces, we add a Fuchsian group Θ_1 of signature $(0, 0, 3, 0, 1)$ consisting three conformal holes where lengths of their closed geodesic boundaries are δ_4, δ_5 , and δ_8 . In particular, we take Θ_1 ($=\langle h_5, h_6, h_7 \rangle$) in such a way that it can match with the groups Λ_1 and Ω_2 . So, by utilizing the combination theorem, we get a Fuchsian group $\Pi_{132} = \langle \Lambda_1^*, \Theta_1, \Omega_1^* \rangle$ of signature $(0, 0, 1, 1, 6)$. In the second step, we consider two groups one is Π_{111} and other is Θ_2 ($=\langle h_8, h_9, h_{10} \rangle$) of signature $(0, 0, 3, 0, 1)$ consisting three conformal holes with length of

their closed geodesics are δ_{10} , δ_{12} , δ_{13} and one funnel end of neck length δ_{11} . After that, we proceed in a similar way as we have carried out in the preceding construction, i.e., cut the two corresponding surfaces of the groups Π_{111} and Θ_2 (convert to Π_{111}^* and Θ_2^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_7 and δ_{10} respectively. Furthermore, we add a Fuchsian group Λ_5 ($=\langle h_{11}, h_{12}, h_{13} \rangle$) of signature $(0, 0, 2, 0, 2)$ with neck lengths δ_{14} , δ_{15} , and closed geodesic boundaries lengths are δ_7 and δ_{10} . Now, we again use the combination theorem to obtain a Fuchsian group $\Pi_{11122} = \langle \Pi_{111}^*, \Lambda_2, \Theta_2^* \rangle$ of signature $(0, 0, 2, 1, 9)$. In the third step, we further take two groups one is Π_{11122} and other is Λ_3 ($=\langle h_{14}, h_{15}, h_{16} \rangle$) of signature $(0, 0, 0, 0, 4)$ consisting four funnel ends with neck lengths δ_{16} , δ_{17} , δ_{18} , and δ_{19} . Furthermore, we proceed in a similar way as we have done in the overhead formation, i.e., cut the two corresponding surfaces of the groups Π_{11122} and Λ_3 (transform to Π_{11122}^* and Λ_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_{12} and δ_{16} respectively. Now, again, we add a Fuchsian group Ω_2 ($=\langle h_{17}, h_{18}, p_2 \rangle$) of signature $(0, 0, 2, 1, 1)$ with neck length δ_{20} , and closed geodesic boundaries lengths are δ_{12} and δ_{16} . So, one can again utilize the combination theorem to get a Fuchsian group $\Pi_{1112223} = \langle \Pi_{11122}^*, \Omega_2, \Lambda_3^* \rangle$ of signature $(0, 0, 2, 2, 12)$. In the fourth step, we again consider two groups one is $\Pi_{1112223}$ and other is Ω_3 ($=\langle h_{19}, h_{20}, p_3 \rangle$) of signature $(0, 0, 0, 1, 3)$ consisting one cusp end and three funnel ends with neck lengths δ_{21} , δ_{22} , and δ_{23} . Further, we cut the two corresponding surfaces of the groups $\Pi_{1112223}$ and Ω_3 (reduce to $\Pi_{1112223}^*$ and Ω_3^*), one funnel end each along the neck of their funnels where the lengths of their necks are δ_{18} and δ_{21} respectively to glue each other. Finally, again, we add a Fuchsian group Θ_3 ($=\langle h_{21}, h_{22}, h_{23} \rangle$) of signature $(0, 0, 3, 0, 1)$ with neck length δ_{24} , and closed geodesic boundaries lengths are δ_{18} , δ_{21} , and δ_{25} . Therefore, by the combination theorem, we obtain a Fuchsian group $\Pi_{111222333} = \langle \Pi_{1112223}^*, \Theta_3, \Omega_3^* \rangle$ of signature $(0, 0, 3, 3, 14)$. Note that, this fourth step is basically a repetition of our previous first step. Therefore, in this way, one can continue this process for an infinitum numbers of times to get an infinitely generated Fuchsian group $\Psi_{111222333\dots} = \langle \Lambda_1^*, \Theta_1, \Omega_1^{**}, \Lambda_2, \Theta_2^{**}, \Omega_2, \Lambda_3^{**}, \Theta_3, \Omega_3^{**}, \dots \rangle = \langle h_1, h_2, h_3, \dots; p_1, p_2, p_3, \dots \rangle$ of type $(0, 0, \infty, \infty, \infty)$, where each block is of infinite hyperbolic area, h_i 's and h_i 's are hyperbolic and parabolic elements respectively, $i \in \mathbb{N}$.

This completes the proof of Theorem 1.2. In particular, our second main goal of this article is established.

Notion of limit sets. Limit sets of our aforementioned constructed infinitely generated groups of types $(0, 0, 0, 0, \infty)$ and $(0, 0, \infty, 0, \infty)$ that give rise to infinite type generalized non-tight flute surfaces are Cantor sets, whereas for the types $(0, 0, 0, \infty, \infty)$ and $(0, 0, \infty, \infty, \infty)$ lead to infinite types generalized tight flute surfaces, are the whole circle at infinity. Indeed, this type of situation occurred due to the existence of parabolic elements in the infinitely generated Schottky groups $(0, 0, 0, \infty, \infty)$ and $(0, 0, \infty, \infty, \infty)$.

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9. DATA AVAILABILITY STATEMENT

Our manuscript has no associated data.

10. STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

REFERENCES

- [1] Alessandrini, D., Liu, L., Papadopoulos, A., Su, W. and Sun, Z., *On Fenchel-Nielsen coordinates on Teichmüller spaces of surfaces of infinite type*, Ann. Acad. Sci. Fenn. Math., **36(2)** (2011), 621-659.
- [2] Alessandrini, D., Liu, L., Papadopoulos, A. and Su, Z., *On various Teichmüller spaces of a surface of infinite topological type*, Proc. Amer. Math. Soc., **140(2)** (2012), 561-574.
- [3] Basmajian, A., *Hyperbolic invariants for infinitely generated Fuchsian groups*, Doctoral Thesis, State University of New York, Stony Brook, U.S.A., October **1987**, 89pp.
- [4] Basmajian, A., *Hyperbolic structures for surfaces of infinite type*, Trans. Amer. Math. Soc., **336(1)** (1993), 421-444.
- [5] Basmajian, A. and Sarić, D., *Geodesically complete hyperbolic structures*, Math. Proc. Cambridge Philos. Soc., **166(2)** (2019), 219-242.
- [6] Borthwick, D., *Spectral theory of infinite-area hyperbolic surfaces*, second ed., Progr. Math. **318**, Birkhäuser/Springer, Boston, Basel, Berlin, **2016**.
- [7] Ford, L., R., *Automorphic Functions*, (Second Edition) Chelsea, New York, **1951**.
- [8] Keen, L., *Canonical polygons for finitely generated Fuchsian groups*, Acta Math., **115** (1966), 1-16.
- [9] Keen, L., *Intrinsic moduli on Riemann surfaces*, Ann. of Math., **84(3)** (1966), 404-420.
- [10] Liu, L. and Papadopoulos, A., *Some metrics on Teichmüller spaces of surfaces of infinite type*, Trans. Amer. Math. Soc., **363(8)** (2011), 4109-4134.

- [11] Marden, A., *Hyperbolic manifolds: an introduction in 2 and 3 dimensions*, Cambridge University Press, **2016**.
- [12] Matsuzaki, K., and Taniguchi, M., *Hyperbolic manifolds and Kleinian groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford Univ. Press, New York, **1998**.
- [13] Shaikh, A. A. and Roy, U., *Construction of Fuchsian Schottky group with conformal boundary at infinity*, Palest. J. Math., **13(4)** (2024), 504-517.
- [14] Shaikh, A. A. and Roy, U., *Theory of semi-Fuchsian Schottky groups*, preprint, Archive ouverte HAL, (2025), 1-42.
- [15] Schwartz, R. E., *The limit sets of some infinitely generated Schottky groups*, Doctoral Thesis, Princeton University, Princeton, NJ, U. S. A., **1991**.

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