Energy Conservation in General Relativity

Stefan Bernhard Rüster[1,*] and Antonino Del Popolo[2,3,†]

Abstract
This article shows that the vanishing covariant divergence of the energy-momentum tensor of the matter is a conservation law. Furthermore, it is explained why energy-momentum pseudotensors of the gravitational field cannot represent its energy density, but this is described up to a factor by the Einstein tensor. The necessarily existing conservation law of total energy, momentum, and stress in general relativity is derived, thereby explaining the phenomena of dark energy and dark matter and solving the cosmological constant problem and the cuspy halo problem. In Newton's theory of gravity, it is the modified Poisson equation that fulfills the requirement of conservation of total energy. Using a model that solves the modified Poisson equation, it turns out that dark matter in modified Newtonian cosmology is nothing other than a central point-like mass, probably a supermassive primordial black hole, thus refuting the cosmological principle and explaining both the Hubble and S_8 tensions. A simple but fairly accurate model is presented that solves the modified Poisson equation to fit the calculated rotation curves to the observed speeds in spiral galaxies, which consist of several components: the central black hole, the bulge, the disk, and the dark matter.

[1] *Am Wiebelsberg 12, 63579 Freigericht, Germany.*
[2] *Dipartimento di Fisica e Astronomia, University of Catania, Viale Andrea Doria 6, 95125 Catania, Italy.*
[3] *INFN Sezione di Catania, Via Santa Sofia 64, 95123 Catania, Italy.*
*__Corresponding author__: dr.ruester@t-online.de
†E-mail address: antonino.delpopolo@unict.it

1. Introduction

Instead of exhibiting the expected Keplerian fall, the rotation curves of spiral galaxies at great distances from the galactic center turned out to be flat; a phenomenon that fueled the hypothesis that large amounts of dark matter particles must be present in their halos. The mutual gravitational attraction between dark matter particles and ordinary matter would result in large accumulations of dark matter particles being located in regions with a higher density of ordinary matter. In the case of a galaxy, this would mean that there would have to be a high density of dark matter particles in its center, which decreases with increasing distance, so that dark matter particles would be present in smaller quantities precisely where they are most needed, namely in its halo. However, especially in dwarf galaxies, a homogeneous rather than a very steep distribution of dark matter is observed in their centers, which leads to the so-called cuspy halo problem [1]. Additionally, the assumption of dark matter particles raises further problems: the missing satellites problem and the too-big-to-fail problem. Although there are proposed solutions to these problems, the assumption that dark matter particles could resolve the discrepancies between theory and observations seems highly speculative, as there is currently no evidence for this.

It is observed that the Universe is expanding at an accelerating rate. Einstein's field equations describe this with a positive value for the cosmological constant Λ. Until now, the cosmological constant has been associated with the energy density of the vacuum, known as dark energy. However, this leads to the so-called cosmological constant problem, as there is a large discrepancy between its predicted and observed value.

In this article, tensor indices, regardless of whether Greek or Latin letters, take on the values $0, 1, 2, 3$.

2. Matter tensor

By definition, the energy-momentum tensor of the matter,

$$T^{\mu\nu} = T^{\mu\nu}_{(\mathrm{pf})} + T^{\mu\nu}_{(\mathrm{em})} + \dots , \tag{1}$$

also called the energy-momentum tensor or simply the matter tensor, consists of all types of matter-energy, but does *not* include the energy of the gravitational field [2, 3]. In Eq. (1),

$$T^{\mu\nu}_{(\mathrm{pf})} = \left(\varrho + \frac{P}{c^2}\right) u^\mu u^\nu + P g^{\mu\nu} \tag{2}$$

is the energy-momentum tensor of a perfect fluid and

$$T^{\mu\nu}_{(\mathrm{em})} = \frac{1}{\mu_0}\left(F^{\mu\alpha} F^{\nu}{}_{\alpha} - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right) \tag{3}$$

is the energy-momentum tensor of the electromagnetic field [4].

In special relativity, i.e. in an inertial frame as well as in a local inertial frame, the vanishing partial divergence of the matter tensor,

$$\partial_\nu T^{\mu\nu} = 0 , \tag{4}$$

demonstrates that the sum of all types of matter-energy are conserved [2, 5]. But Landau and Lifshitz state in § 96 of Ref. [3] regarding the vanishing covariant divergence of the matter tensor, i.e. Eq. (96.1),

$$T^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial (T^k_i \sqrt{-g})}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0 , \tag{5}$$

that "in this form, however, this equation does not generally express any conservation law whatever. Because the integral

$$\int T^k_i \sqrt{-g}\, dS_k \tag{6}$$

is conserved only if the condition

$$\frac{\partial (\sqrt{-g}\, T^k_i)}{\partial x^k} = 0 \tag{7}$$

is fulfilled, and not (96.1)." Corresponding arguments can also be found on page 27 of Ref. [4] and in § 126 of Ref. [6]. In contrast, Eq. (7) is not covariant [7] and therefore does not generally represent a valid condition for the conservation of the integral (6), which is why the latter cannot in general represent a conserved integral.

Fließbach demonstrates in Eq. (20.31) of Ref. [2], that the vanishing covariant divergence of the matter tensor represents a conservation law by applying the principle of covariance on the vanishing partial divergence of the matter tensor in a local inertial frame,

$$\partial_\nu T^{\mu\nu} = 0 \quad \longrightarrow \quad \nabla_\nu T^{\mu\nu} = 0 . \tag{8}$$

Weinberg proves in Sec. 12.3 of Ref. [8] that the vanishing covariant divergence of the matter tensor (5) represents a conservation law by performing an infinitesimal transformation of the dynamical variables in the change of the matter action given by Eq. (12.2.2),

$$\delta I_M = \frac{1}{2} \int d^4x \sqrt{g(x)}\, T^{\mu\nu}(x)\, \delta g_{\mu\nu}(x) . \tag{9}$$

He emphasizes that "*the energy-momentum tensor defined by Eq.* (12.2.2) *is conserved* (*in the covariant sense*) *if and only if the matter action is a scalar*. Also, with I_M a scalar, (12.2.2) shows immediately that $T^{\mu\nu}$ is a symmetric *tensor*, so this definition of the energy-momentum tensor has all the properties for which one could wish. This proof, that general covariance implies energy-momentum conservation, has an exact analog in the proof that gauge invariance implies charge conservation."

Therefore, one concludes that the vanishing covariant divergence of a tensor represents a conservation law in general relativity [9, 10]. Hence, the matter tensor is conserved in general relativity contrary to the wording of Landau and Lifshitz in § 96 of Ref. [3] that "in this form, however, this equation does not generally express any conservation law whatever." But they actually relate this statement to conservation of *total* energy by pointing out that "this is related to the fact that in a gravitational field the four-momentum of the matter alone must not be conserved, but rather the four-momentum of matter plus gravitational field; the latter is not included in the expression for T^k_i." This is also why total energy equals matter-energy plus energy of the gravitational field.

3. Completed field equations

In Sec. 3.4 of Ref. [4], Straumann states that "a general conservation law for energy and momentum does *not* exist in GR. This has been disturbing to many people, but one simply has to get used to this fact. There is no 'energy-momentum tensor for the gravitational field'. Independently of any formal arguments, Einstein's equivalence principle tells us directly that there is no way to localize the energy of the gravitational field: The 'gravitational field' (the connection $\Gamma^\mu_{\alpha\beta}$) can be locally transformed away. But if there is no field, there is locally no energy and no momentum." Corresponding arguments can also be found in §20.4 of Ref. [5].

In Newton's theory of gravity, the energy density of the gravitational field is given by

$$\varepsilon_{\mathrm{gf}}(\mathbf{r}) = -\frac{[\boldsymbol{\nabla}\Phi(\mathbf{r})]^2}{8\pi G} . \tag{10}$$

In the Poisson equation,

$$\Delta\Phi(\mathbf{r}) = 4\pi G \varrho(\mathbf{r}) , \tag{11}$$

however, there does not appear the energy density of the gravitational field, but only the mass distribution on the right-hand side as a source of gravitation. The Poisson equation (11) shows the zero-zero component of Einstein's field equations in the Newtonian limit under the assumption of a perfect fluid with negligible pressure. However, to meet the requirements of a precise theory of gravity, all types of energy must be taken into account [9, 10]. The energy-momentum tensor of the gravitational field must be present in Einstein's field equations not only for this reason, but also to satisfy the correspondence principle due to the existence and localizability of the energy density of the gravitational field in Newton's theory of gravity, see Eq. (10). Furthermore, without the conservation of energy, our Universe would be in chaos or, in the worst case, would dissolve into nothing.

By introducing the energy-momentum tensor of the gravitational field,

$$A_{im} = -\kappa^{-1} G_{im}\,, \tag{12}$$

where $\kappa = 8\pi G/c^4$ and G_{im} is the Einstein tensor, Einstein's field equations

$$G_{im} = \kappa T_{im} \tag{13}$$

can be rearranged to obtain the Levi-Civita field equations,

$$T_{im} + A_{im} = 0\,. \tag{14}$$

The latter are criticized by Einstein in §6 of Ref. [11], where he argues that "in (14) the components of the *total energy* vanish everywhere. The equations (14), for example, do not exclude the possibility ... that a material system dissolves into just nothing without leaving any trace. Because the total energy in (14) ... is zero from the beginning: the conservation of this energy value does not demand the continued existence of the system in any form." However, a simple modification makes Einstein's objection superfluous: One simply needs to introduce the *non-zero* total energy-momentum tensor L_{im} on the right-hand side of Eq. (14), so that the completed Levi-Civita field equations read [9, 10]

$$T_{im} + A_{im} = L_{im}\,. \tag{15}$$

This modification highlights that Einstein's field equations (13) must be incomplete and violate the conservation of total energy. Nevertheless, they can be used as a good approximation, provided that the total energy-momentum tensor L_{im} does not play a significant role.

According to Lovelock's theorem – see e.g. Sec. 3.2.2 in Ref. [4], especially Theorem 3.1 and Eq. (3.51) – Einstein's field equations in their maximally possible modified form are Einstein's field equations with the cosmological constant Λ,

$$G_{im} = \kappa T_{im} - \Lambda g_{im}\,, \tag{16}$$

which can be rearranged to obtain the completed Levi-Civita field equations (15), where

$$L_{im} = \kappa^{-1} \Lambda g_{im}\,. \tag{17}$$

3.1 Cosmological constant

If one replaces the cosmological constant Λ by the universal constant λ in Eq. (16) and then takes the trace, one obtains

$$\kappa T + R = 4\lambda\,, \tag{18}$$

which can be differentiated,

$$\partial_\mu(\kappa T + R) = 0\,. \tag{19}$$

The general solution to these differential equations is

$$\kappa T + R = 4\Lambda\,, \tag{20}$$

which corresponds to the trace of Eq. (16), where Λ, unlike λ, is a constant of integration and thus a *parameter* and not a universal constant [12]. This reflects the fact that different gravitational systems have different total energy densities [9, 10]. Therefore, for each gravitational system, a different metric ds^2 exists with a different, *non-zero* Λ. This finding solves the cosmological constant problem and does not contradict Lovelock's theorem.

The value of the cosmological constant Λ with respect to the metric ds^2 of each gravitational system must be determined by observations, as it is initially unknown. The cosmological constant Λ with respect to the metric ds^2 of our solar system must be tiny, otherwise the calculated angle of Mercury's perihelion shift would not match the observed one.

With these new insights, Einstein's condition for emptiness, $G_{im} = 0$, is obsolete and must now be $G_{im} = -\Lambda g_{im}$ to satisfy the requirement of conservation of total energy. In fact, "empty" spacetime is not truly empty, since it consists of the energy of the gravitational field. Therefore, it is more appropriate to refer to it as *matter-free* rather than "empty" spacetime. In matter-free spacetime, $T_{im} = 0$, and consequently, in this case, the total energy density equals the energy density of the gravitational field.

The connection of the cosmological constant with the energy density of the vacuum leads to a significant discrepancy between its theoretical and observed value, which raises the cosmological constant problem. It is important to classify general relativity as a classical rather than a quantum theory. Therefore, the cosmological constant cannot be related to the energy density of the vacuum, since the latter only occurs in quantum theory. In matter-free spacetime, the cosmological constant corresponds, apart from a factor, to a scalar curvature of spacetime,

$$\Lambda = \frac{R}{4}\,, \tag{21}$$

which can be easily recognized from Eq. (20).

Assuming a perfect fluid, the *modified* Poisson equation,

$$\Delta\Phi(\mathbf{r}) = 4\pi G\left(\varrho(\mathbf{r}) + \frac{3P(\mathbf{r})}{c^2}\right) - \Lambda c^2\,, \tag{22}$$

which is obtained in the Newtonian limit from the zero-zero component of the Einstein field equations (16) with the cosmological constant Λ as a *parameter*, in contrast to the Poisson equation (11), fulfills the requirement of conservation of total energy in Newton's theory of gravity. Nevertheless, Eq. (11) can be used as a good approximation if the cosmological constant Λ and the pressure P do not play a significant role.

3.2 Energy-momentum tensor of the gravitational field

The field equations (15) are tensor equations that must contain the energy, momentum and stress of the gravitational field in the form of a tensor, which is true for Eq. (12). In addition, this tensor has the unit of measurement of an energy density, which is required to represent a tensor of any type of energy.

The component

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \tag{23}$$

of the metric tensor of the Schwarzschild metric contains the Newtonian gravitational potential

$$\Phi(r) = -\frac{GM}{r}\,, \tag{24}$$

which is why the metric tensor g_{im} is a quantity that must be related to the gravitational field and hence to its energy density. It is of great importance that in the mixed tensor representation of the field equations (15),

$$T_i^k + A_i^k = L_i^k\,, \tag{25a}$$

$$A_i^k = -\kappa^{-1} G_i^k\,, \tag{25b}$$

$$L_i^k = \kappa^{-1} \Lambda \delta_i^k\,, \tag{25c}$$

all metric tensors and their first two derivatives occur in the Einstein tensor G_i^k, which is why the energy density of the gravitational field is represented by A_0^0 [9, 10].

The energy-momentum tensor of the gravitational field A_i^k, given by Eq. (25b), consists of terms with squared Christoffel symbols. This is consistent with Newton's theory of gravity, in which the analogous expression $[\nabla\Phi(\mathbf{r})]^2$ appears in the energy density of the gravitational field (10).

The vanishing of the covariant divergence of the matter tensor, $T^k_{i;k} = 0$, means that the matter tensor is conserved in general relativity. Likewise, the vanishing of the covariant divergence of the energy-momentum tensor of the gravitational field, $A^k_{i;k} = 0$, means that the energy-momentum tensor of the gravitational field is conserved. Consequently, matter-energy is *not* converted into energy of the gravitational field and vice versa.

From the Eq. (25a) in matter-free spacetime, it is easy to see that due to a required *non-zero* total energy-momentum tensor (25c), the energy-momentum tensor of the gravitational field (25b) does *not* vanish, which then also holds in a local inertial frame. This finding refutes Straumann's statement in Sec. 3.4 of Ref. [4] that "if there is no field, there is locally no energy and no momentum" and that of Misner, Thorne and Wheeler in §20.4 of Ref. [5] that "no local gravitational field means no 'local gravitational energy-momentum'."

Energy-momentum pseudotensors of the gravitational field cannot describe its energy density because they are not tensors and vanish in a local inertial frame. Nevertheless, they are used to represent the energy density of the gravitational field and to form a "conservation law" of energy, momentum, and stress that is not covariant and thus violated [7]. Misner, Thorne and Wheeler correctly state in §20.4 of Ref. [5] regarding energy-momentum pseudotensors of the gravitational field that "there is no unique formula for it, but a multitude of quite distinct formulas. The two cited are only two among an infinity." However, contrary to the statements in §20.4 of Ref. [5], Eq. (25b) demonstrates the necessarily existing unique formula for the local gravitational energy-momentum. It is thus localizable and neither contradicts nor is forbidden by the equivalence principle.

4. Conservation law of total energy, momentum and stress

The *total* energy-momentum tensor (25c) is *conserved*, which is why its covariant divergence vanishes,

$$L^k_{i;k} = \nabla_k (T_i^k + A_i^k) = 0\,. \tag{26}$$

By exploiting the special property of the Kronecker tensor,

$$\delta^k_{i;k} = \delta^k_{i,k} = 0\,, \tag{27}$$

this conservation law of total energy and momentum can be simplified,

$$L^k_{i;k} = L^k_{i,k} = \partial_k (T_i^k + A_i^k) = 0\,. \tag{28}$$

One can even go further and take the derivative instead of the divergence, because

$$\delta^k_{i;j} = \delta^k_{i,j} = 0\,. \tag{29}$$

This means that not only the divergences vanish, but also the derivatives, so that

$$L^k_{i;j} = L^k_{i,j} = \partial_j (T_i^k + A_i^k) = 0\,, \tag{30}$$

which is the conservation law of total energy, momentum and stress in general relativity in its differential form [9, 10].

One can consider a closed region with volume V. Volume integration over $L^k_{i,0}$ in Eq. (30) yields

$$\frac{\partial}{\partial t}\int_V dV\, L_i^k = \frac{\partial}{\partial t}\int_V dV\,(T_i^k + A_i^k) = 0\,, \tag{31}$$

whereby the conserved total energy, momentum and stress within the closed region with volume V are given by

$$E_i^k = \kappa^{-1}\Lambda V \delta_i^k = \int_V dV\, L_i^k = \int_V dV\,(T_i^k + A_i^k) = \text{constant}, \tag{32}$$

which can only be achieved in the mixed tensor representation [9, 10].

4.1 Example: Non-rotating star

As a simple example, consider a non-rotating star with mass M and radius R without electromagnetic fields, which occupies a closed region with a volume of $V = \frac{4}{3}\pi R^3$. The total energy contained in V is

$$E_{\text{tot}} = E_0^0 = E_{\text{M}} + E_{\text{gf}} = \frac{4\pi\Lambda R^3}{3\kappa} = \text{constant}, \tag{33}$$

where

$$E_{\text{M}} = \int_V T_0^0\, dV = -4\pi c^2 \int_0^R dr\, r^2 \varrho(r) = -Mc^2 \tag{34}$$

is the mass-energy of the star and

$$E_{\text{gf}} = \int_V A_0^0\, dV = -4\pi\kappa^{-1} \int_0^R dr\, r^2 G_0^0(r) \tag{35}$$

is the energy of the gravitational field. The mass

$$M = M_{\text{c}} + \frac{E_{\text{pot}}}{c^2} \tag{36}$$

is the gravitational mass and thus the physical mass of the star, while in the metric of the star

$$M_{\text{c}} = 4\pi \int_0^R dr\, r^2 \varrho(r) \sqrt{g_{11}(r)} \tag{37}$$

is its constituent mass, which represents the unbound mass of the star. The metric coefficient

$$g_{11} = \left(1 - \frac{2Gm(r)}{c^2 r} - \frac{\Lambda r^2}{3}\right)^{-1}, \tag{38}$$

where

$$m(r) = 4\pi \int_0^r dr'\, r'^2 \varrho(r') \tag{39}$$

is the mass within the radius r of the star.

In contrast to Newton's theory of gravity, in general relativity the energy of the gravitational field is stored as the curvature of spacetime, which is why it is *not* synonymous with the gravitational potential energy E_{pot}, which, apart from the sign, corresponds to the gravitational binding energy [9].

5. Dark matter and dark energy

Either the "mass" density of dark matter, which is characterized by a *negative* value of the cosmological constant Λ,

$$\varrho_{\text{dm}} = \frac{\Lambda}{\kappa c^2} \qquad (\Lambda < 0), \tag{40}$$

or that of dark energy, which is characterized by a *positive* value of the cosmological constant Λ,

$$\varrho_{\text{de}} = \frac{\Lambda}{\kappa c^2} \qquad (\Lambda > 0), \tag{41}$$

is homogeneously distributed in spacetime with respect to the metric ds^2 of the respective gravitational systems. Since Eq. (21) holds in matter-free spacetime, the "mass" density of dark matter, apart from a factor, is in fact nothing other than the constant, negative scalar curvature of matter-free spacetime, while that of dark energy is nothing other than the constant, positive scalar curvature of matter-free spacetime. It results from the conservation of total energy with respect to the metric ds^2 of the respective gravitational systems [9, 10].

Clearly, the "mass" density of dark matter (40), as already mentioned, is homogeneously distributed in spacetime, which immediately solves the cuspy halo problem. Its negative value should not pose a problem, since a negative value of the cosmological constant in the modified Poisson equation (22) acts like a fictitious positive mass distribution, causing gravitational attraction and thus explaining the phenomenon of dark matter. A positive value of the cosmological constant, on the other hand, acts like a fictitious negative mass distribution, causing gravitational repulsion and thus explaining the phenomenon of dark energy.

6. Modified Newtonian cosmology

The fact that dark matter is described by a *negative* value of the cosmological constant Λ cannot explain the discrepancy in cosmology between the total gravitational mass M and the gravitational mass of baryonic matter plus radiation $M_{\text{b+r}}$, which is called dark matter M_{d},

$$M = M_{\text{d}} + M_{\text{b+r}}\,. \tag{42}$$

However, as shown below, this problem can be solved within the framework of modified Newtonian cosmology using a model that solves the modified Poisson equation (22). In this model, the mass density of the baryonic matter (b) plus radiation (r),

$$\varrho(t) = \varrho_{\text{b}}(t) + \varrho_{\text{r}}(t)\,, \tag{43}$$

as well as the pressure $P(t)$ depend on time t and are considered spatially homogeneous, and the dark matter is replaced by a point-like mass M_{d} at the center of the Universe, probably a supermassive primordial black hole. The Newtonian cosmological model was adopted from Ref. [13] and modified to account for both the central point-like mass and the accelerated expansion of the Universe, the latter of which is described by a *positive* value of the cosmological constant Λ.

Due to spherical symmetry, only the radial coordinates of a test particle need to be considered,

$$r(L,t) = a(L,t)\, L\,, \tag{44}$$

where $r(L,t)$ is its physical coordinate and L is its fixed coordinate. Due to the inhomogeneity caused by the central point-like mass, the scale factor $a(L,t)$ depends not only on the time t but also on the coordinate L.

The modified Hubble law reads

$$v(L,t) = \dot{r} = \dot{a}L = \dot{a}(L,t)\,\frac{r(L,t)}{a(L,t)} = H(L,t)\,r(L,t)\,, \quad (45)$$

where

$$H(L,t) = \frac{\dot{a}(L,t)}{a(L,t)} \quad (46)$$

is the modified Hubble parameter and

$$H_0(L) = H(L,t_0) \quad (47)$$

is the modified Hubble constant, which corresponds to the modified Hubble parameter at the current time t_0. Therefore, the modified reduced Hubble constant is

$$h(L) = \frac{H_0(L)}{100\,\frac{\mathrm{km}}{\mathrm{s}}/\mathrm{Mpc}}\,. \quad (48)$$

The modified Poisson equation (22) yields the modified field equation of Newtonian gravity in spherical coordinates,

$$\boldsymbol{\nabla}\cdot\ddot{\mathbf{r}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\ddot{r}\right) = -4\pi G\left(\varrho + \frac{3P}{c^2}\right) + \Lambda c^2\,. \quad (49)$$

This results in the gravitational acceleration in modified Newtonian cosmology,

$$\ddot{r} = -\frac{\partial\Phi}{\partial r} = -\frac{GM}{r^2} + \frac{\Lambda c^2 r}{3} = -\frac{GM}{a^2L^2} + \frac{\Lambda c^2 aL}{3} = \ddot{a}L\,, \quad (50)$$

where Φ is the modified gravitational potential. Inside the sphere with radius $r(L,t)$ and volume

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi a^3 L^3 \quad (51)$$

is the total gravitational mass M, given by Eq. (42), consisting of the constant central point-like mass M_d and the gravitational mass of the baryonic matter plus radiation,

$$M_\mathrm{b+r} = 4\pi\int_0^r dr'\,r'^2\varrho' = \varrho' V\,, \quad (52)$$

where

$$\varrho'(t) = \varrho(t) + \frac{3P(t)}{c^2} \quad (53)$$

is its mass density at time t.

By dividing Eq. (50) by aL one obtains the modified second Friedmann equation,

$$\frac{\ddot{a}}{a} = -\frac{GM_\mathrm{d}}{a^3L^3} - \frac{4\pi G}{3}\left(\varrho + \frac{3P}{c^2}\right) + \frac{\Lambda c^2}{3}\,. \quad (54)$$

By using the internal energy of baryonic matter plus radiation,

$$U = \varrho c^2 V\,, \quad (55)$$

in the differential,

$$\mathrm{d}U = -P\,\mathrm{d}V\,, \quad (56)$$

one obtains

$$c^2\,\mathrm{d}(\varrho V) = -P\,\mathrm{d}V\,, \quad (57)$$

which can be rearranged to

$$\mathrm{d}\varrho = -\left(\varrho + \frac{P}{c^2}\right)\frac{\mathrm{d}V}{V}\,. \quad (58)$$

From Eq. (51) one yields

$$\frac{\mathrm{d}V}{V} = 3\,\frac{\mathrm{d}a}{a}\,, \quad (59)$$

which can be used in Eq. (58) to obtain the conservation laws for baryonic matter and radiation,

$$\dot{\varrho} = -3\,\frac{\dot{a}}{a}\left(\varrho + \frac{P}{c^2}\right), \quad (60a)$$

$$\varrho_\mathrm{b}a^3 = \text{constant} \qquad (\varrho = \varrho_\mathrm{b},\ P = 0)\,, \quad (60b)$$

$$\varrho_\mathrm{r}a^4 = \text{constant} \qquad (\varrho = \varrho_\mathrm{r},\ P = \varrho_\mathrm{r}c^2/3)\,. \quad (60c)$$

Solving Eq. (60a) for the pressure, inserting it in Eq. (54) and then by multiplying the latter with $2a\dot{a}$ yields

$$2\dot{a}\ddot{a} = -\frac{2GM_\mathrm{d}}{L^3}\frac{\dot{a}}{a^2} + \frac{8\pi G}{3}\left(\dot{\varrho}a^2 + 2\varrho a\dot{a}\right) + \frac{2\Lambda c^2 a\dot{a}}{3}\,. \quad (61)$$

Integration over t gives

$$\dot{a}^2 = \frac{2GM_\mathrm{d}}{aL^3} + \frac{8\pi G\varrho a^2}{3} + \frac{\Lambda c^2 a^2}{3} - kc^2\,. \quad (62)$$

The last term in Eq. (62) represents a constant of integration, where $k = k(L)$. By dividing by a^2 one obtains the modified first Friedmann equation,

$$H^2 = \frac{2GM_\mathrm{d}}{a^3L^3} + \frac{8\pi G\varrho}{3} + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}\,. \quad (63)$$

With the current scale factor $a_0(L) = a(L,t_0)$, the current mass density of the baryonic matter $\varrho_{\mathrm{b},0} = \varrho_\mathrm{b}(t_0)$ and that of the radiation $\varrho_{\mathrm{r},0} = \varrho_\mathrm{r}(t_0)$ and the dimensionless quantities

$$x(L,\tau) = \frac{a(L,t)}{a_0(L)}\,, \quad (64a)$$

$$\tau(L) = H_0(L)\,t\,, \quad (64b)$$

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{\dot{a}}{a_0}H_0^{-1} = \frac{\dot{a}}{a}\frac{a}{a_0}H_0^{-1} = \frac{H}{H_0}x\,, \quad (64c)$$

the modified cosmic equation of motion

$$\left(\frac{\mathrm{d}x}{\mathrm{d}\tau}\right)^2 = \frac{\Omega_\mathrm{r}}{x^2} + \frac{\Omega_\mathrm{m}}{x} + \Omega_\mathrm{k} + \Omega_\Lambda x^2 \quad (65)$$

results from multiplying Eq. (63) with x^2/H_0^2, where

$$\Omega_{\rm r}(L) = \frac{8\pi G \varrho_{\rm r} a^4}{3H_0^2 a_0^4} = \frac{\varrho_{\rm r,0}}{\varrho_{\rm c}} \approx 0\,, \tag{66a}$$

$$\Omega_{\rm m}(L) = \Omega_{\rm d}(L) + \Omega_{\rm b}(L) \approx 0.3\,, \tag{66b}$$

$$\Omega_{\rm d}(L) = \frac{2GM_{\rm d}}{H_0^2 a_0^3 L^3} \approx 0.25\,, \tag{66c}$$

$$\Omega_{\rm b}(L) = \frac{8\pi G \varrho_{\rm b} a^3}{3H_0^2 a_0^3} = \frac{\varrho_{\rm b,0}}{\varrho_{\rm c}} \approx 0.05\,, \tag{66d}$$

$$\Omega_{\rm k}(L) = -\frac{kc^2}{H_0^2 a_0^2} \approx 0\,, \tag{66e}$$

$$\Omega_\Lambda(L) = \frac{\Lambda c^2}{3H_0^2} \approx 0.7 \tag{66f}$$

are the density parameters and

$$\varrho_{\rm c}(L) = \frac{3H_0^2}{8\pi G} \tag{67}$$

is the critical density.

It should be noted here that McCrea and Milne rediscovered the Friedmann equations within the framework of Newton's theory of gravity [14]. Therefore, it would be correct to attribute the equations obtained above to McCrea and Milne. However, the authors of this article refrain from doing so, as the term "Friedmann equations" has become established.

Eq. (66c) confirms that dark matter in cosmology is indeed nothing other than a central point-like mass $M_{\rm d}$. The cosmological principle implies that the entire Universe is isotropic and homogeneous. This principle is violated and thus refuted by the existence of the central point-like mass. Because of this finding, the FLRW metric is not really suitable to adequately describe the entire Universe.

The inhomogeneity caused by the central point-like mass leads not only to the Hubble tension, which manifests itself in the dependence of the modified Hubble constant (47) on the coordinate L, but also to the S_8 tension,

$$S_8(L) = \sigma_8 \sqrt{\frac{\Omega_{\rm m}(L)}{0.3}}\,, \tag{68}$$

where σ_8 represents the root mean square of the mass fluctuations within a sphere of radius $8h^{-1}$ Mpc. Therefore σ_8 quantifies the clustering of matter in the Universe, with a higher value of σ_8 indicating a more clustered Universe, while a lower value suggests a more uniform distribution of matter. However, σ_8 is not a suitable quantity to adequately describe the extent of clustering in cosmology, since this is described by σ_8 within spheres with different radii due to the Hubble tension [15, 16].

7. Model of spiral galaxies

This section presents a simple but fairly accurate model that is consistent with observations. It solves the modified Poisson equation (22) with vanishing pressure to fit the calculated rotation curves to the observed speeds in spiral galaxies, which consist of several components [10, 17, 18]: the central black hole (bh), the bulge (b), the disk (d), and the dark matter (dm).

The squared circular speed of a test particle located in the galactic disk at a distance r from the galactic center and caused by the gravitational potential Φ of a spiral galaxy is given by

$$v^2(r) = r\frac{\partial \Phi}{\partial r} = \sum_m v_m^2(r)\,, \tag{69}$$

where

$$v_m^2(r) = r\frac{\partial \Phi_m}{\partial r} \tag{70}$$

is the squared circular speed caused by the gravitational potential Φ_m of the m-th component of the spiral galaxy [17]. The square root of Eq. (69) is the computed rotation curve, which can be fitted to the n observed speeds $v_{\rm obs}(r_i)$ and their respective uncertainties $\sigma_{v_{\rm obs}}(r_i)$ using the weighted least squares method [19],

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{v_{\rm obs}(r_i) - v(r_i)}{\sigma_{v_{\rm obs}}(r_i)} \right)^2\,, \tag{71a}$$

$$\frac{\partial \chi^2}{\partial M_{\rm bh}} = \frac{\partial \chi^2}{\partial M_{\rm b}} = \frac{\partial \chi^2}{\partial M_{\rm d}} = \frac{\partial \chi^2}{\partial R_{\rm b}} = \frac{\partial \chi^2}{\partial R_{\rm d}} = \frac{\partial \chi^2}{\partial \Lambda} = 0\,, \tag{71b}$$

where the quantities by which differentiation is performed are the model parameters given below.

It is assumed that at the center of the spiral galaxy there is a black hole with mass $M_{\rm bh}$, which is considered to be point-like [17]. The squared circular speed caused by the central black hole is

$$v_{\rm bh}^2(r) = r\frac{\partial \Phi_{\rm bh}}{\partial r} = \frac{GM_{\rm bh}}{r}\,. \tag{72}$$

The matter in the bulge is assumed to be exponentially and isotropically distributed [17],

$$\varrho_{\rm b}(r) = \varrho_0 \exp\left(-\frac{r}{R_{\rm b}}\right)\,, \tag{73}$$

where ϱ_0 is the central mass density of the bulge and $R_{\rm b}$ is the bulge scale length. The mass of the exponential bulge within the sphere of radius r is given by

$$\begin{aligned} M_\varrho(r) &= 4\pi \int_0^r dr'\, r'^2 \varrho_{\rm b}(r') \\ &= M_{\rm b}\left[1 - \exp\left(-\frac{r}{R_{\rm b}}\right)\left(1 + \frac{r}{R_{\rm b}} + \frac{r^2}{2R_{\rm b}^2}\right)\right]\,, \end{aligned} \tag{74}$$

where

$$M_{\rm b} = 8\pi \varrho_0 R_{\rm b}^3 \tag{75}$$

is the total mass of the bulge. The squared circular speed caused by the bulge is given by

$$v_{\mathrm{b}}^2(r) = r\frac{\partial \Phi_{\mathrm{b}}}{\partial r} = \frac{GM_{\varrho}(r)}{r}\,. \tag{76}$$

The disk of the spiral galaxy can be considered infinitesimally thin and has an exponentially decreasing surface mass distribution [20, 21], which is given by

$$\Sigma(r) = \Sigma_0 \exp\left(-\frac{r}{R_{\mathrm{d}}}\right), \tag{77}$$

where Σ_0 is the central surface mass density of the disk and R_{d} is the disk scale length. The mass of the disk within radius r is [21]

$$\begin{aligned} M_{\Sigma}(r) &= 2\pi \int_0^r dr'\, r' \Sigma(r') \\ &= M_{\mathrm{d}}\left[1 - \exp\left(-\frac{r}{R_{\mathrm{d}}}\right)\left(1 + \frac{r}{R_{\mathrm{d}}}\right)\right], \end{aligned} \tag{78}$$

where

$$M_{\mathrm{d}} = 2\pi\Sigma_0 R_{\mathrm{d}}^2 \tag{79}$$

is the total mass of the disk. The squared circular speed caused by the disk is [20, 21]

$$v_{\mathrm{d}}^2(r) = r\frac{\partial \Phi_{\mathrm{d}}}{\partial r} = 4\pi G\Sigma_0 R_{\mathrm{d}} y^2 [I_0(y)K_0(y) - I_1(y)K_1(y)]\,, \tag{80}$$

where

$$y = \frac{r}{2R_{\mathrm{d}}}\,. \tag{81}$$

The contribution of dark matter in the modified Poisson equation (22) is

$$\Delta\Phi_{\mathrm{dm}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi_{\mathrm{dm}}}{\partial r}\right) = -\Lambda c^2\,, \tag{82}$$

where the *parameter* Λ is the cosmological constant with respect to the spiral galaxy under consideration [18]. The squared circular speed caused by dark matter is given by

$$v_{\mathrm{dm}}^2(r) = r\frac{\partial \Phi_{\mathrm{dm}}}{\partial r} = -\frac{\Lambda c^2 r^2}{3}\,. \tag{83}$$

8. Results and discussion

Fig. 1 shows the calculated rotation curves fitted to the H I/Hα observational data of some spiral galaxies from SPARC [22]. The model shown in Sec. 7 was used, omitting the mass models from Ref. [22]. This is because the latter is based on SPARC surface photometry data measured at only one wavelength, 3.6 micrometers in the infrared region of the electromagnetic spectrum. However, spiral galaxies are known to become bluer with increasing distance from the galactic center [23]. This means that matter in the outer regions, which tends to emit at shorter wavelengths, may not be detected to the same extent at 3.6 micrometers as in the center of a spiral galaxy. Not only for this reason, the authors use the model given in Sec. 7, but also because it demonstrates the exact formulas of the dark matter component, which is not the case with the mass models of Ref. [22].

For the model calculations in Sec. 7, the spreadsheet program `LibreOffice Calc` and its included `Solver` tool were used. To solve the Eqs. (71), suitable search intervals were defined for the six model parameters. The calculated rotation curves fit the observed speeds remarkably well, see Fig. 1. The values of the fitted model parameters of the considered spiral galaxies are given in Tab. 1. These results differ from those in Tab. 1 of Ref. [18], because, in contrast to the present work, the unweighted least squares method was used there.

However, a problem can arise when fitting a calculated rotation curve to the observed speeds in a spiral galaxy using the least squares method, since more than one minimum can exist with respect to χ^2. One can even obtain solutions that differ greatly from each other, yet still reproduce the observed speeds very well. Even the lowest minimum, i.e., the correct mathematical solution, is not necessarily the correct physical solution if it produces a nonsensical result. The reason for this is that the model presented in Sec. 7 offers enough freedom that even different parameter combinations can produce rotation curves that fit the observed data very well. Therefore, additional information, i.e., additional observational data, is necessary to decide which of the solutions is physically the correct one. It may also be the case that two reasonable solutions are so close to each other that choosing the right solution can be very difficult. However, these shortcomings do not detract from the main point, namely that the simple model given in Sec. 7 describes the observed rotation curves remarkably well.

9. Conclusions and outlook

The vanishing covariant divergence of the matter tensor, $T^k_{i;k} = 0$, shows that matter-energy is conserved in general relativity. Similarly, the vanishing covariant divergence of the energy-momentum tensor of the gravitational field, $A^k_{i;k} = 0$, shows that the energy of the gravitational field is conserved in general relativity. Einstein's field equations, with the cosmological constant Λ as a *parameter*, fulfill the requirement of conservation of total energy, momentum, and stress. They thus explain the phenomena of dark energy and dark matter and solve the cosmological constant problem as well as the cuspy halo problem. Using the model in Sec. 6, which solves the modified Poisson equation (22), it turns out that dark matter in modified Newtonian cosmology is nothing other than a central point-like mass, probably a supermassive primordial black hole, thus refuting the cosmological principle and explaining both the Hubble and S_8 tensions.

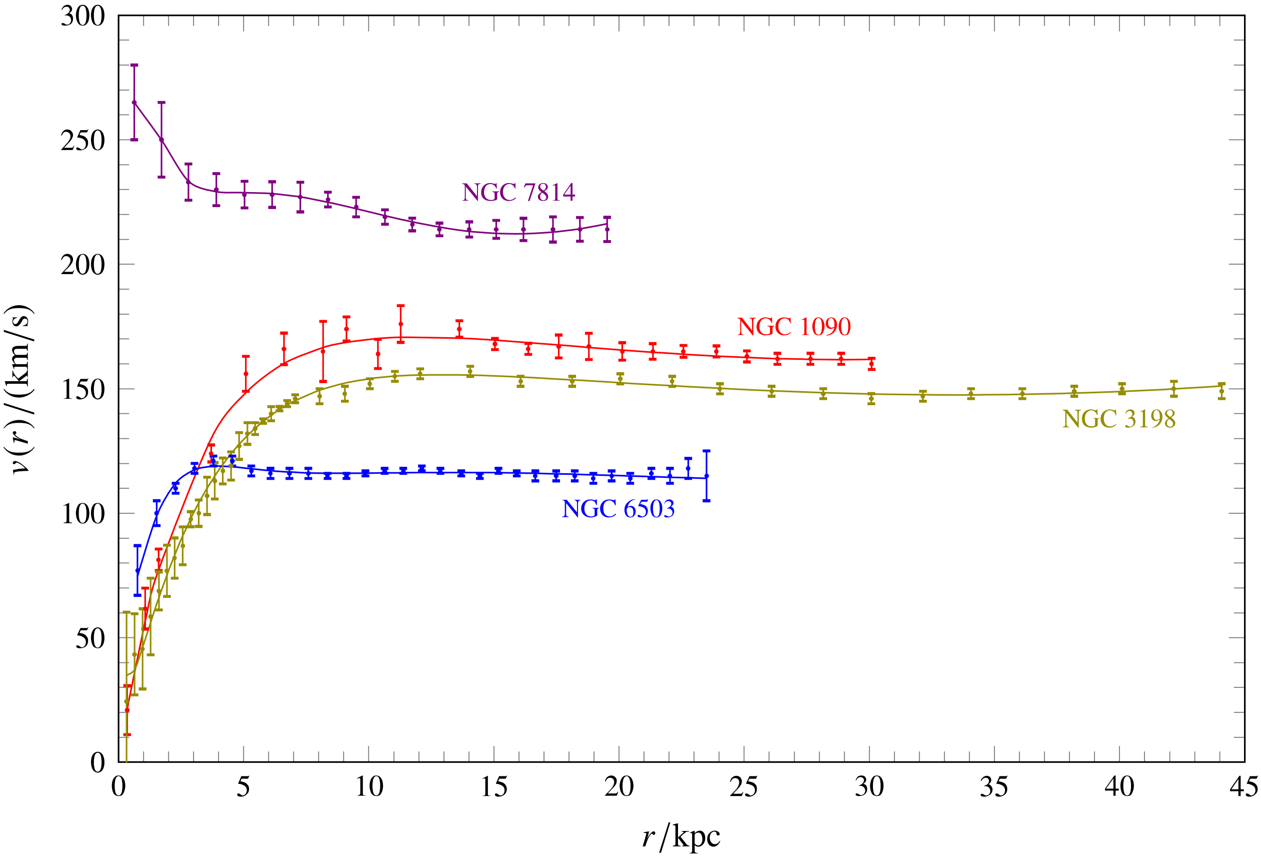

Figure 1. The markers with error bars indicate the speeds in some spiral galaxies taken from SPARC [22]. The fitted rotation curves are shown by the corresponding solid lines [10]. The fitted model parameters of the considered spiral galaxies are listed in Tab. 1.

Table 1. This table lists the fitted model parameters of the spiral galaxies considered in Fig. 1 [10].

spiral galaxy	$M_{\rm bh}$ (10^9 $M_\odot$)	$M_{\rm b}$ (10^9 $M_\odot$)	$M_{\rm d}$ (10^9 $M_\odot$)	$R_{\rm b}$ (kpc)	$R_{\rm d}$ (kpc)	Λ (10^{-48} m^{-2})
NGC 1090	0.0	37.3107	72.2153	2.1227	7.3978	−0.3301
NGC 3198	0.0726	44.5548	67.5261	2.5049	9.6868	−0.1905
NGC 6503	0.2867	11.6626	42.2537	0.8919	7.663	−0.1511
NGC 7814	4.0504	17.2821	74.4207	0.3088	3.4627	−2.0907

The rotation curves of spiral galaxies can be described remarkably well by using the model in Sec. 7, which contains the exact formulas for the dark matter component, and solving the *modified* Poisson equation (22). The latter satisfies the requirement of conservation of total energy in Newton's gravitational theory. Since there are many spiral galaxies and each spiral galaxy has different model parameters, many rotation curves still need to be calculated by fitting them to the observed speeds in the respective spiral galaxies. Other types of gravitational systems that can be modeled with the *modified* Poisson equation (22) are outside the scope of this work and are the subject of future research.

An extension of general relativity is the Einstein–Cartan theory, which allows for torsion related to spin [24]. This leads to a *repulsive* gravitational interaction within matter, which prevents the formation of singularities and explains the inflation of the early Universe. However, the Einstein–Cartan theory only plays a significant role at enormous mass densities, while in matter-free spacetime, there is no difference between the Einstein–Cartan theory and general relativity. In order to fulfill the requirement of conservation of total energy, momentum and stress in the Einstein–Cartan theory, the cosmological constant must be included as a *parameter*.

Acknowledgments

The authors would like to thank Ricardo Gobato, Bernard Howard Lavenda, Dirk–Hermann Rischke, Jürgen Schaffner-

Bielich and Scott Duncan Tremaine for their support.

References

[1] A. Del Popolo, *The Invisible Universe, Dark Matter, Dark Energy, and the Origin and End of the Universe*, World Scientific, Singapore (2021).

[2] T. Fließbach, *Allgemeine Relativitätstheorie*, 5. Auflage, Elsevier GmbH, München (2006).

[3] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields, Course of Theoretical Physics, Volume 2*, Fourth Revised English Edition, Reed Educational and Professional Publishing Ltd, Oxford (1975).

[4] N. Straumann, *General Relativity, Graduate Texts in Physics*, Second Edition, Springer, Dordrecht (2013).

[5] C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, W. H. Freeman and Company, San Francisco (1973).

[6] C. Møller, *The Theory of Relativity*, Clarendon Press, Oxford (1955).

[7] B. Lavenda, *Where Physics Went Wrong*, World Scientific Publishing, Singapore (2015).

[8] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, John Wiley & Sons, Inc., New York (1972).

[9] S. B. Rüster, Parana J. Sci. Educ. **8**, 6 (2022).

[10] S. B. Rüster, Parana J. Sci. Educ. **9**, 6 (2023).

[11] A. Einstein, *On Gravitational Waves, The Collected Papers of Albert Einstein, Volume 7, The Berlin Years: Writings, 1918–1921*, pp. 9–27, Princeton University Press, Princeton (2002).

[12] S. B. Rüster and V. B. Morozov, Parana J. Sci. Educ. **7**, 10 (2021).

[13] J. Binney and S. Tremaine, *Galactic Dynamics, Princeton Series in Astrophysics*, Second Edition, Princeton University Press, Princeton (2008).

[14] W. H. McCrea and E. A. Milne, Q. J. Math. **os-5**, 1 (1934).

[15] A. G. Sánchez, Phys. Rev. D **102**, 123511 (2020).

[16] M. Forconi, A. Favale and A. Gómez-Valent, Phys. Rev. D **112**, 023517 (2025).

[17] Y. Sofue, Publ. Astron. Soc. Japan **69**, 1, R1 (2017).

[18] S. B. Rüster, Parana J. Sci. Educ. **8**, 8 (2022).

[19] P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences*, Third Edition, McGraw-Hill, New York (2003).

[20] K. C. Freeman, ApJ **160**, pp. 811–830 (1970).

[21] J. Binney and S. Tremaine, *Galactic Dynamics, Princeton Series in Astrophysics*, Princeton University Press, Princeton (1987).

[22] F. Lelli, S. S. McGaugh and J. M. Schombert, AJ **152**, 6, 157 (2016).

[23] A. Unsöld and B. Baschek, *Der neue Kosmos, Einführung in die Astronomie und Astrophysik*, p. 429, 7. Auflage, Springer, Berlin (2002).

[24] A. Trautman, *Einstein–Cartan Theory, Encyclopedia of Mathematical Physics, Volume 2*, J.-P. Françoise, G. L. Naber and S. T. Tsou, pp. 189–195, Elsevier, Oxford (2006).