Prime Number Distribution as an Iterative Spiral

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Abstract

A deterministic arithmetic reformulation of the mod 6 lattice revealing geometric symmetry in the distribution of primes. Rotational symmetry when applied to these residuals visits all of the non-prime numbers stepwise, which we refer to as “hops”, that generates a deterministic arithmetic framework that reproduces the sequence of primes and their gaps.

We present a framework for understanding the distribution of primes using modular arithmetic and iterative hop sequences. Visual patterns such as the Ulam spiral are shown to arise naturally from rotational symmetries within this framework. We provide both an intuitive explanation and a formal arithmetic treatment that reproduces the sequence of primes and their gaps.

As first noted by Ulam [2] and popularized by Gardner [1], the arrangement of integers in a spiral lattice reveals that prime numbers tend to cluster along diagonal lines. However, this observation alone does not explain the *mechanism* of the clustering.

The hop-based interpretation offers a natural explanation: primes occupy loci defined by arithmetic propagation rather than arbitrary geometric coincidence.

1 Modular Structure of Prime Distribution and Hop-Based Composite Generation

1.1 Definitions

Definition:[Prime Candidate Sequences] Define two sequences based on congruence classes modulo 6:

$$A = \{a_k \mid a_k = 6k+1,\ k \in \mathbb{N}_0\}, \quad B = \{b_k \mid b_k = 6k+5,\ k \in \mathbb{N}_0\}.$$

These sequences contain all integers congruent to 1 and 5 modulo 6, respectively. □

Definition:[Hop Sequence] For any $x \in A \cup B$, define its *hop sequence* as:

$$H(x) = \{x + m \cdot x \mid m \in \mathbb{N}_{>0}\} = \{x(1+m) \mid m \in \mathbb{N}_{>0}\}.$$

This sequence enumerates composite numbers generated by repeated addition of x to itself, without explicit multiplication or division—only index counting. □

Definition:[Rotational Symmetry Mapping] Define a helper sequence:

$$C = B^{\text{rev}} \cup \{1\} \cup A,$$

where B^{rev} is the reverse of B. Define a mapping:

$$\phi : A \to B,\ \phi(a_k) = b_k,$$

illustrating that backward hops along A project into B, forming a symmetric modular structure. □

1.2 Theorems

Theorem 1 (Prime Containment) *For every prime $p > 3$,*

$$p \in A \cup B.$$

Proof: Every integer n satisfies $n \equiv r \pmod 6$ for some $r \in \{0, 1, 2, 3, 4, 5\}$.

- If $r = 0, 2, 4$, then n is divisible by 2.
- If $r = 3$, then n is divisible by 3.

Thus, primes greater than 3 cannot lie in these classes. The remaining classes are $r = 1$ and $r = 5$, corresponding to sequences A and B □

Theorem 2 (Composite Generation Without Division) *Every composite number $c > 3$ in $A \cup B$ can be expressed as:*

$$c \in H(p) \text{ for some } p \in A \cup B.$$

Thus, composites emerge from hop sequences of primes, and primality can be tested by verifying absence from all hop sequences except its own origin.

Theorem 3 (Hop Closure Property — index-based form) *Let $x, y \in A \cup B$ and suppose $x \mid y$, so $y = kx$ for some integer k. If the hop families are taken with multipliers drawn from M_1 and M_5 as above, then*

$$H^{\pm}(y) \subseteq H^{\pm}(x),$$

and, in particular,

$$\bigcup_{p \text{ prime}} \big(H^{+}(p) \cup H^{-}(p)\big) = \bigcup_{x \in A \cup B} \big(H^{+}(x) \cup H^{-}(x)\big).$$

Proof: Write $y = kx$ with $k \in \mathbb{N}$. Take an arbitrary element of $H^{+}(y)$:

$$h = y(1 + 6m) = kx(1 + 6m).$$

Since $k(1 + 6m)$ is an integer, we can write

$$h = x \cdot \big(k(1 + 6m)\big),$$

so h is an element of $H^{+}(x)$ whenever $k(1 + 6m) \in M_1$ or more generally whenever we consider the closure with respect to all integer multipliers; but crucially h is an integer multiple of x and therefore lies on a spoke generated by x. The same argument applies to $H^{-}(y)$ using multipliers of the form $5 + 6m$.

Consequently every composite hop sequence originating from a composite y is contained in the union of hop sequences originating from its prime divisors. Hence the union of prime-generated hop sequences covers all composite hops. □

1.3 Index-based Hop Definition and Example

Definition:[Index-multiplier Hop Sequences] Let $x \in A \cup B$. Define two families of index-based multipliers, those congruent to $1 \pmod 6$ and those congruent to $5 \pmod 6$:

$$M_1 = \{\, 1 + 6m \mid m \in \mathbb{N},\, m \geq 1 \,\}, \qquad M_5 = \{\, 5 + 6m \mid m \in \mathbb{N}_0 \,\}.$$

Then define the *forward* and *backward* hop sequences of x by

$$H^+(x) = \{\, x \cdot t \mid t \in M_1 \,\}, \qquad H^-(x) = \{\, x \cdot t \mid t \in M_5 \,\}.$$

Elements of M_1 and M_5 are exactly the integers congruent to 1 and 5 modulo 6, so these hops remain in the residue classes $A \cup B$. □

Worked numeric example. Using the above definition for $x = 5$ and $x = 7$:

$$H^+(5) = \{\, 5(1 + 6m) \mid m \geq 1 \,\} = \{35, 65, 95, 125, 155, 185, 215, 245, \dots\},$$
$$H^+(7) = \{\, 7(1 + 6m) \mid m \geq 1 \,\} = \{49, 91, 133, 175, 217, 259, 301, \dots\},$$

$$H^-(5) = \{\, 5(5 + 6m) \mid m \geq 0 \,\} = \{25, 55, 85, 115, 145, 175, \dots\},$$
$$H^-(7) = \{\, 7(5 + 6m) \mid m \geq 0 \,\} = \{35, 77, 119, 161, 203, 245, \dots\}.$$

A short numeric snapshot (sorted) showing initial overlaps:

value	25	35	49	55	65	77	85	91	95
$H^-(5)$	✓			✓			✓		
$H^-(7)$		✓				✓			
$H^+(5)$		✓			✓				✓
$H^+(7)$			✓					✓	

Note how e.g. 35 appears in both $H^+(5)$ and $H^-(7)$, and 175 appears in several lists: these are the overlap points (intersections of spokes).

1.4 Consequences of Hop Closure

The hop lattice is therefore *closed* under multiplication: composite hops do not introduce new elements beyond those already reached by primes. Geometrically, this manifests as a rotationally symmetric structure in which each prime defines a fundamental "spoke" or orbit on the modular spiral, and all composite-derived hops lie along previously visited trajectories. This closure explains why additional hopping from composites causes no harm—these operations simply revisit already marked positions, reinforcing the deterministic coverage of the composite landscape.

1.5 The Ulam Spiral and the Hop-Based Interpretation

When integers are arranged in a spiral lattice, as first demonstrated by Ulam [2] and popularized by Gardner [1], the prime numbers tend to align along distinct diagonal bands (Figure 1). This simple visualization—now known as the *Ulam spiral*—revealed an unexpected order in the distribution of primes, where certain diagonals correspond to quadratic polynomials that yield long runs of primes (e.g., $n^2 + n + 41$).

Although striking, this geometric arrangement is descriptive rather than explanatory. It shows that primes cluster along structured trajectories but does not clarify *why* these alignments occur. In particular, the Ulam spiral illustrates symmetry without accounting for the arithmetic mechanism behind it.

The hop-based interpretation provides a complementary and causal explanation. In this model, integers are represented as *hop loci* arising from multiplicative and reciprocal projections $H(p)$ and $H'(p)$. These hops propagate radially, forming structured intersections whose geometry mirrors the diagonal bands of the Ulam spiral. Thus, the visual regularities observed by Ulam emerge naturally as a consequence of arithmetic propagation rather than arbitrary geometric coincidence.

In essence, while the Ulam spiral reveals the *symmetry* of the primes, the hop-based model describes the *mechanism* that generates it. The diagonal structures are not merely an artifact of plotting in polar coordinates but correspond to the loci where forward and backward hops resonate. This interpretation transforms the Ulam pattern from an aesthetic curiosity into a predictable geometric manifestation of multiplicative structure.

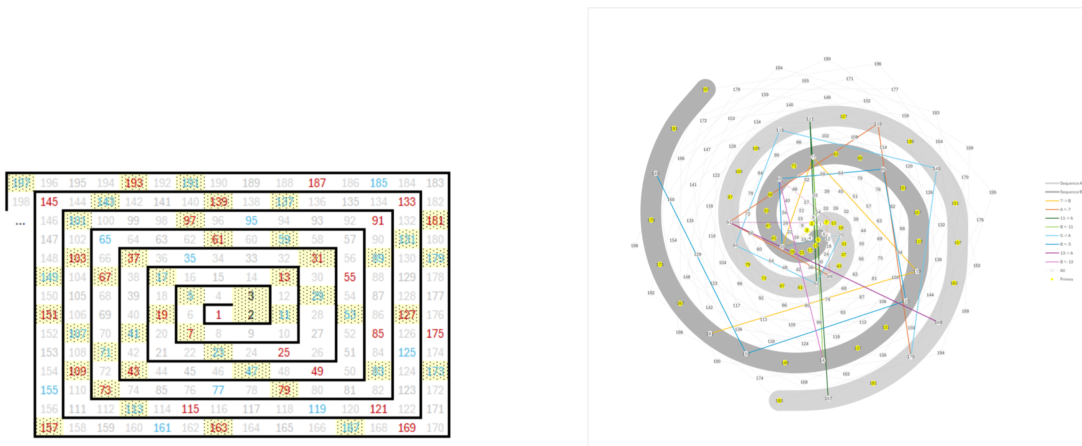

Figure 1: **Left:** Traditional Ulam spiral of integers showing prime diagonals. **Right:** Equivalent hop-based mapping revealing similar diagonal structure through arithmetic propagation.

Formally, the loci of alignment in the Ulam spiral correspond to quadratic residues mod p, while in the hop-based framework, these same loci arise where integer multiples and their reciprocals overlap in the polar plane. This correspondence suggests a deeper equivalence between visual diagonals and algebraic modular structure.

1.6 Interpretation

This framework explains prime gaps deterministically: gaps occur where hop sequences overlap, leaving no unmarked candidates. The approach uses only modular indexing and additive hops—no multiplication or division—yet fully accounts for the emergence of composites and the apparent irregularity/asymmetry of prime gaps.

1.7 Visualising Hop Sequences Intuitively

In addition to the formally defined proof above, we now demonstrate the hop mechanism in a simple intuitive way.

STEP ONE-IDENTIFY FORWARD PRODUCTS IN SEQUENCE A (Figure 2)
Starting with the second element of Sequence A, the number 7. Noting that whilst the number 1 is essential in the sequence for the symmetry to work, it touches each and every other number

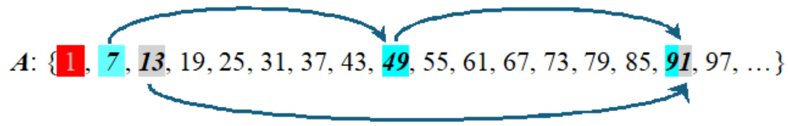

Figure 2: Forward Hops into Sequence A

and is therefore skipped.

- Every subsequent 7th "hop" along the line is a composite factor of 7, and hence not prime. {49,91,133,...}

- Taking the next element of Sequence A; the number 13, every subsequent 13th "hop" along the line is a composite factor of 13, and hence not prime. {91,169,247,...}

- Continue the process with each subsequent element of the set to infinity.

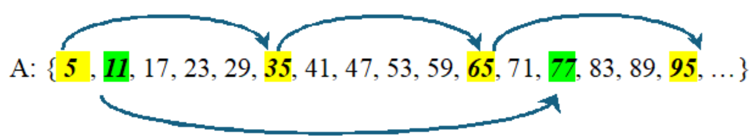

Figure 3: Forward Hops into Sequence B

STEP TWO-IDENTIFY FORWARD PRODUCTS IN SEQUENCE B (Figure 3) Repeat with the first element of Sequence B-the number 5.

- Every subsequent 5th "hop" along the line is a composite factor of 5, and hence not prime. {35,65,95,...}

- Taking the next element of Sequence B; the number 11, every subsequent 11th "hop" along the line is a composite factor of 11, and hence not prime. {77,143,209,...}

- Continue the process with each subsequent element of the set to infinity.

STEP THREE-CONSTRUCT HELPER SEQUENCE C (Figure 4) With the first sets of Products so-identified by forward counting, construct a new sequence by reversing Sequence B, simply by multiplying Sequence B indices by -1 and appending Sequence A to the end.

Note there is no strict necessity to actually "create" Sequence C, Sequences A and B are still the same, but it helps the intuition to visualise it in this way.

Figure 4: Helper Sequence C

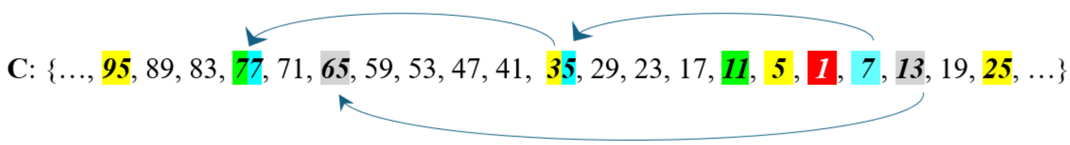

Figure 5: Reverse Hops from Sequence A into Sequence B.

STEP FOUR-IDENTIFY FURTHER PRODUCTS BY HOPPING INTO SEQUENCE B FROM SEQUENCE A (Figure 5) The remaining Products come from repeating the procedure in reverse.

- Starting with the second element of Sequence A, count backwards that number into Sequence B, i.e. backwards from Sequence A into the Sequence B elements of Sequence C, so ordered.

- Every 7th "hop" backwards is a composite of 7 and hence not prime. {35,77,119,...}
- Every 13th "hop" backwards is a composite of 13 and hence not prime. {65,143,221,...}
- Continue the process with each subsequent element of the set to infinity.

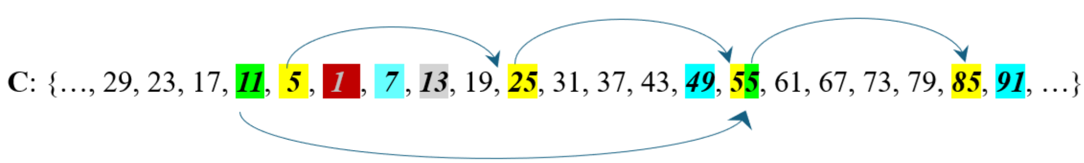

Figure 6: Reverse Hops from Sequence B into Sequence A.

STEP FIVE-IDENTIFY REMAINING PRODUCTS BY HOPPING INTO SEQUENCE A FROM SEQUENCE B (Figure 6) Starting with the first element of Sequence B, count forwards that number into Sequence A, i.e. forwards from Sequence B into the Sequence A elements of Sequence C, so ordered.

- Every 5th hop along the line is a composite factor of 5, and hence not prime. {25,55,85,...}
- Every 11th hop along the line is a composite factor of 11 and hence not prime. {55,121,187,...}
- Continue the process with each subsequent element of the set to infinity.

SUMMARY-PRIMES HAVE BEEN IDENTIFIED BY THE SPIRAL SIEVE Every composite number >3 lies at a hop position determined by an element of A or B-hence every composite number >3 is marked by this iteration, leaving precisely the primes unmarked. Each of these hops themselves are the multiplication of the prime numberss themselves, arising in perferctly predictable order with simple hop-counting.

1.8 The Hop Spiral

Definition:[Hop Spiral] For each $n \in A \cup B$, define:

$$x_n = \cos(n) \cdot \big(\lfloor n/6 \rfloor + 1\big), \quad y_n = \sin(n) \cdot \big(\lfloor n/6 \rfloor + 1\big)$$

This produces polar coordinates that produce the spiral patterns corresponding to each sequence. □

1.9 Example: The Residuals of the Five-Hop Sequence

To illustrate the hop-based construction intuitively (Figure 7), we consider the case of the five-hop, where the base sequence $S = \{1, 2, 3, 4, 5, 6, 7, 8, \dots\}$ is projected by hop length $t = 5$.

For each index i, the hop relation selects those terms of S whose index satisfies $i \bmod t = 0$. That is,

$$X_{(5)} = \{\, s_i \in S \mid i \bmod 5 = 0 \,\} = \{5, 10, 15, 20, 25, 30, \dots\}.$$

However, rather than following the arithmetic sequence of multiples directly, we interpret these as *hops* along the spiral—indexed by their order in the sequence, rather than by their arithmetic value, it can be observed that these are in prime order, so it's

$$5 \cdot 7, 5 \cdot 13$$

and so on. The forward and backward projections of the five-hop thus form two intertwined subsequences, denoted A_5 and B_5:

$$A_5 = \{35, 65, 95, 125, 155, 185, 215, 245, \dots\},$$
$$B_5 = \{25, 55, 85, 115, 145, 175, 205, 235, \dots\}.$$

When plotted according to

$$x_n = \cos(n) \cdot (\lfloor n/6 \rfloor + 1), \qquad y_n = \sin(n) \cdot (\lfloor n/6 \rfloor + 1),$$

these residuals trace two mirrored spiral arms—corresponding precisely to the distinct trajectories followed by the forward and backward hops. The structure that emerges under this projection reveals the same rotational symmetry observed in the Ulam spiral, but now with a clear generative rule based on discrete hop interactions.

1.10 The Residuals of the Seven-Hop Sequence

The residuals of the Hop 7 sequence (Figure 8), and all others, follow the same logic as hop 5, so we won't clutter by repeating, rather just show the figure that is generated as a further uncluttered worked example.

1.11 Close: The Residuals of the first 4 primes demonstrating the aritmetic sequence of prime numbers and the gaps

We end with a zoomed in view of the residuals of 5,11 and 7,13 (Figure 9), which already creates an image slightly tricky to read, but with the intuition from the hops and the residuals of the 5, you can follow from the source point observing the rotation.

References

[1] Martin Gardner. Mathematical games: The remarkable patterns of the ulam spiral. *Scientific American*, pages 150–154, March 1964.

[2] Stanislaw M. Ulam. Patterns of integers. *The American Mathematical Monthly*, 71(10):1058–1066, 1964.

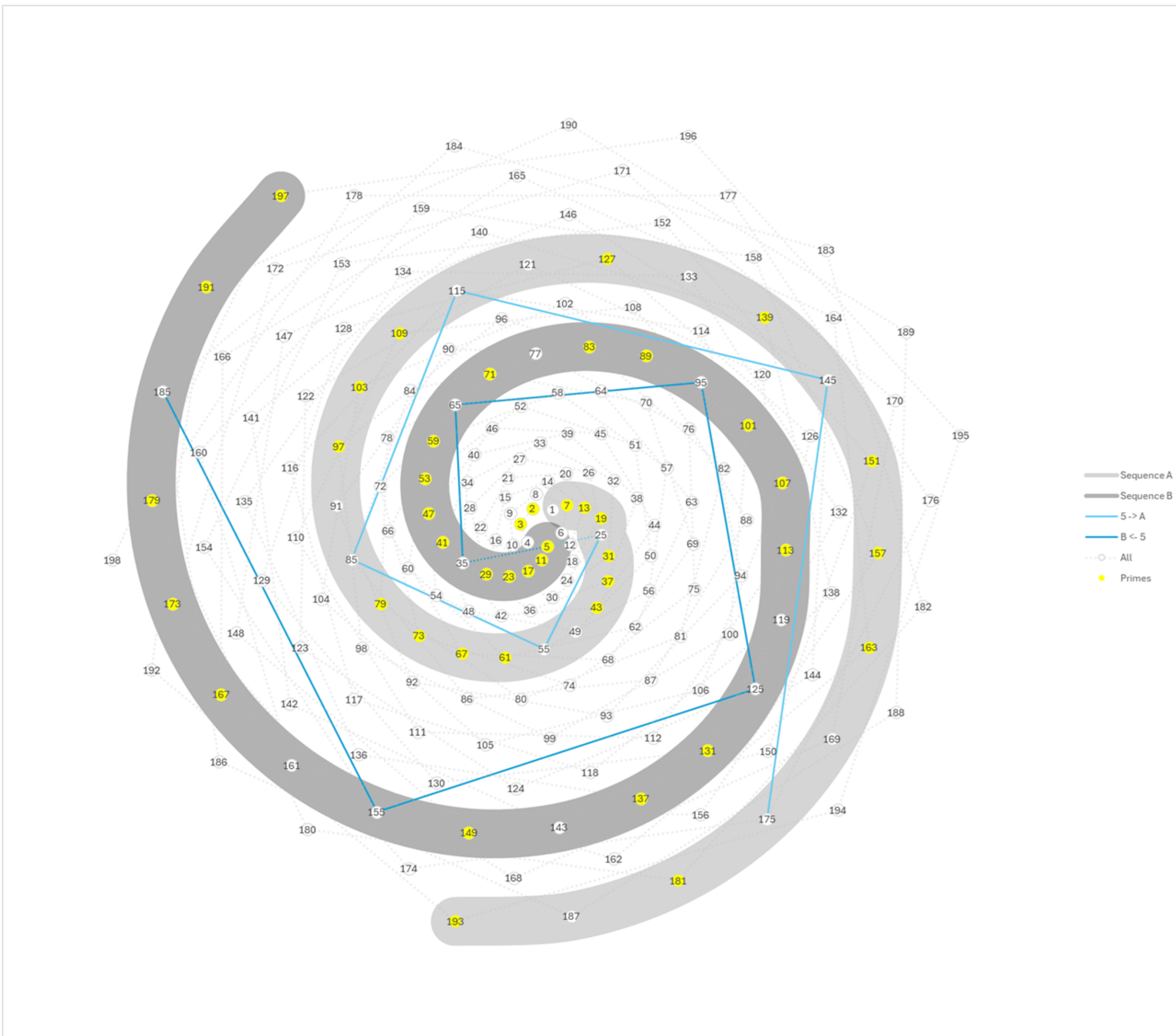

Figure 7: The residuals of the five-hop projected onto the spiral, showing the emergence of sequences A_5 and B_5.

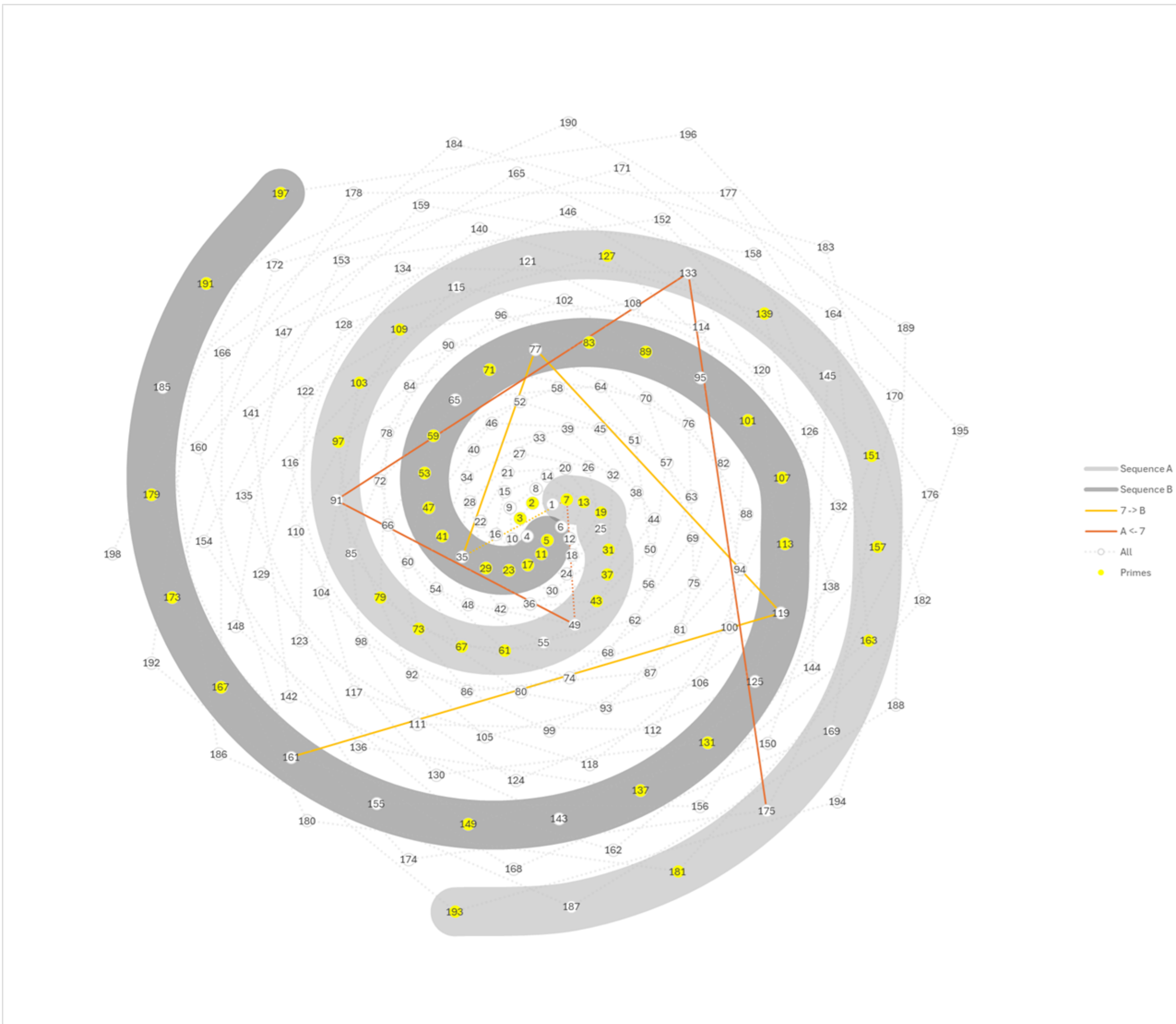

Figure 8: The residuals of the seven-hop projected onto the spiral, showing the emergence of sequences A_7 and B_7.

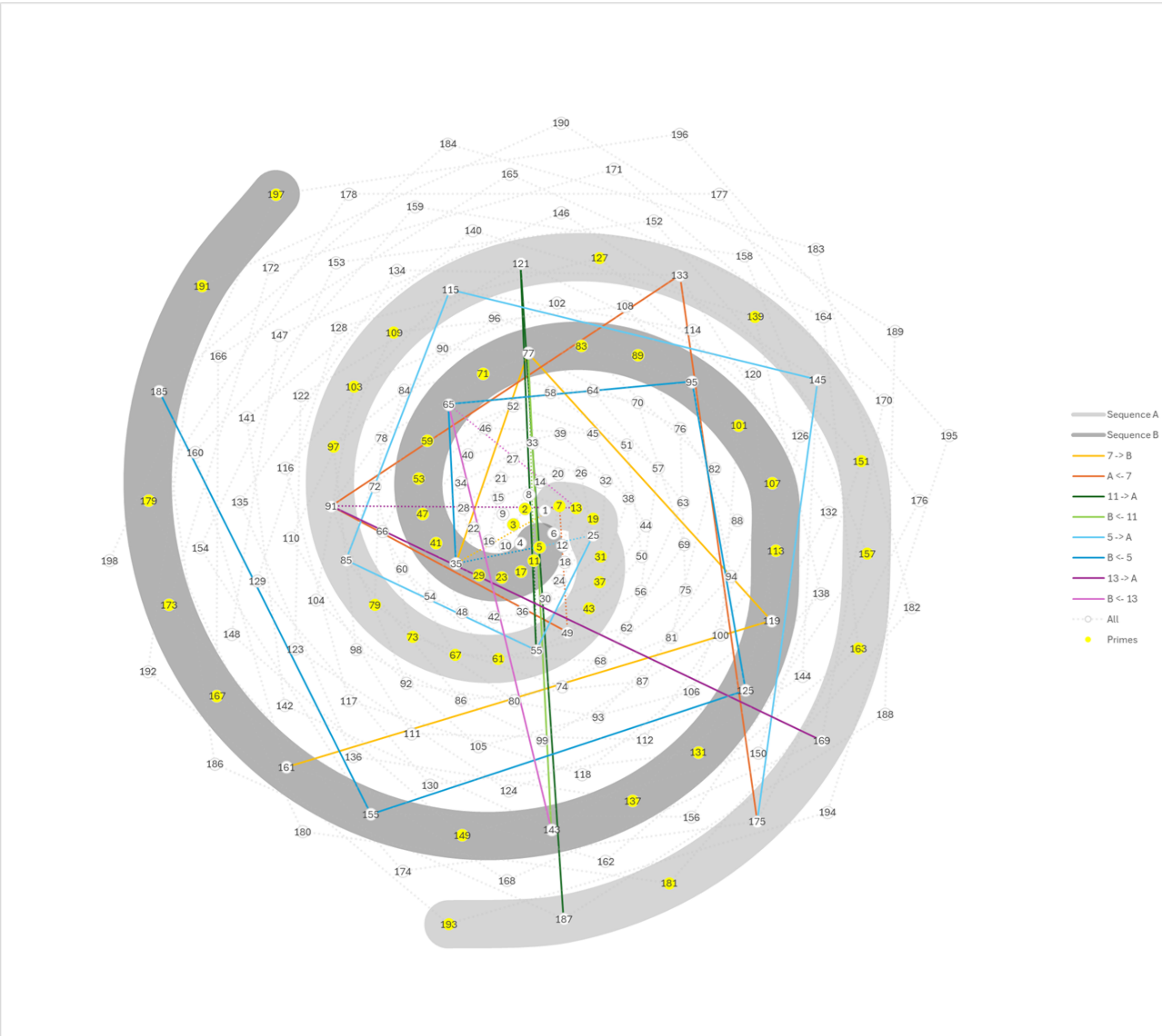

Figure 9: The residuals of the 5,11 and 7,13 hops projected onto the spiral, showing the structure of the prime gaps up to 198, the sequence continues to infinity.