An arithmetic reformulation of the 3x + 1 problem using signed Jacobsthal numbers

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Abstract

We establish a structural correspondence between the Collatz map and the signed Jacobsthal numbers, providing an arithmetic reformulation of the $3x+1$ problem. By representing Collatz iterations through powers of signed Jacobsthal numbers, we derive necessary and sufficient conditions for the existence of cycles and for the validity of the coefficient stopping time conjecture. This formulation translates the combinatorial dynamics of the Collatz map into explicit number-theoretic identities, revealing an underlying algebraic framework that connects iteration, recurrence, and integrality. The results suggest a pathway toward analyzing the conjecture through the intrinsic arithmetic structure of the signed Jacobsthal numbers.

1 Introduction

The Collatz conjecture, also known as the $3x+1$ problem, remains one of the most tantalizing unsolved questions in mathematics. Despite its deceptively simple definition, the conjecture exhibits unexpectedly complex combinatorial behavior, and no general proof of convergence has yet been found. In this work, we uncover a previously unexplored arithmetic connection between Collatz trajectories and the signed Jacobsthal numbers, leading to a reformulation of the problem in purely number-theoretic terms.

By expressing Collatz iterations through powers of signed Jacobsthal numbers, we derive necessary and sufficient conditions for the existence of cycles and show that the coefficient stopping time conjecture admits an explicit arithmetic formulation. This viewpoint transforms the parity-driven dynamics of the Collatz map into verifiable algebraic relations, providing a new avenue for analysis and suggesting a deeper arithmetic structure underlying this classic problem.

Lagarias [4, p. 18] succinctly articulated the core difficulty of the conjecture:

> "Empirical evidence seems to indicate that the $3x+1$ function on the domain $\mathbb{Z}$ retains the pseudorandomness property on its initial iterates until the iterates enter a periodic orbit. This supports the $3x+1$ conjecture and at the same time deprives us of any obvious mechanism to prove it, since mathematical arguments exploit the existence of structure, rather than its absence."

The present work is motivated directly by this observation. We aim to identify an underlying *arithmetic structure* within the Collatz dynamics—one that persists beneath its apparent pseudorandomness. By establishing a direct correspondence between Collatz iterations and the signed Jacobsthal numbers, we make this hidden structure explicit and reveal its role in governing the integrality conditions that characterize Collatz cycles.

To formalize these ideas, we begin by defining the fundamental objects of study: the Collatz map and its trajectories. These provide the foundation for expressing iterations through structured arithmetic vectors, which will then be linked to the signed Jacobsthal numbers.

1.1 Background and Scope

Definition 1. The **Collatz map** $C : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$C(x) = \begin{cases} x/2, & \text{if } x \text{ is even,} \\ 3x+1, & \text{if } x \text{ is odd.} \end{cases}$$

Definition 2. The **Collatz trajectory** of an integer $x \in \mathbb{Z}$ is the sequence

$$(x,\, C(x),\, C^2(x),\, C^3(x),\, \ldots),$$

where $C^k(x)$ denotes the k^{th} iterate of x under C.

The **Collatz conjecture** asserts that for every $x \in \mathbb{Z}^+$, there exists a $k \in \mathbb{Z}^+$ such that $C^k(x) = 1$.

For example, starting with $x = 3$, the Collatz trajectory is

$$(3,\, 10,\, 5,\, 16,\, 8,\, 4,\, 2,\, 1),$$

so $C^7(3) = 1$. Continuing the iteration yields the trivial cycle

$$(4,\, 2,\, 1,\, 4,\, 2,\, 1, \ldots).$$

Although the conjecture is traditionally formulated over the positive integers, the map C extends naturally to all of $\mathbb{Z}$. Among the negative iterates, three distinct cycles are known. Additionally there is a trivial fixed point $(0, 0)$ since $C(0) = 0$. In this paper, we show that the *signed Jacobsthal numbers* form the underlying arithmetic structure of all four known nonzero cycles—that is, excluding the trivial $(0, 0)$ case—and that both the existence of such cycles and the Collatz conjecture itself can be expressed entirely in terms of their arithmetic properties.

Definition 3. For $n \geq 0$, the n^{th} **Jacobsthal number** A001045 is defined recursively by

$$J(0) = 0, \quad J(1) = 1, \quad J(n) = 2J(n-2) + J(n-1) \quad \text{for } n \geq 2,$$

with closed form

$$J(n) = \frac{2^n - (-1)^n}{3}.$$

The sequence begins:

$$0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,\ 171,\ 341,\ 683,\ 1365,\ 2731,\ 5461,\ 10923, \ldots$$

Remark 4. All even-indexed nonzero Jacobsthal numbers, e.g.

$$1, 5, 21, 85, \ldots$$

are the only odd positive integers (besides the recurring 1) that appear in their own Collatz trajectories. For example, the trajectory of 5 is

$$(5,\ 16,\ 8,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1, \ldots),$$

which contains only 5 as an odd element apart from the repeating 1.

To extend this connection, we introduce the notion of *signed Jacobsthal numbers*.

Definition 5. For $n \geq 0$, the n^{th} **signed Jacobsthal number** is defined recursively as

$$J_s(0) = 0, \quad J_s(1) = 1, \quad J_s(n) = 2J_s(n-2) - J_s(n-1) \quad \text{for } n \geq 2,$$

with closed form

$$J_s(n) = (-1)^{n+1} J(n) = \frac{1 - (-2)^n}{3}. \tag{1}$$

The sequence begins:

$$0,\ 1,\ -1,\ 3,\ -5,\ 11,\ -21,\ 43,\ -85,\ 171,\ -341,\ 683,\ -1365,\ 2731,\ -5461, \ldots.$$

For all $n > 0$, this sequence is identical to A077925.

1.2 Related Prior Work

The Collatz map has been extensively investigated from arithmetic, probabilistic, and dynamical perspectives. Terras [6] introduced the concept of *stopping time* and showed that the Collatz conjecture is equivalent to the statement that every integer has finite stopping time. Garner [2] and Eliahou [1] established that any nontrivial cycle, if it exists, must contain thousands of terms. More recently, Tao [5] proved that almost all integer orbits eventually decrease, providing a measure-theoretic viewpoint on the problem.

In contrast, the present work develops an explicit arithmetic framework for analyzing Collatz iterations through *signed Jacobsthal numbers* and their powers. This formulation yields necessary and sufficient conditions for the existence of cycles (Theorems (38) and (47)) and introduces a new combinatorial interpretation of stopping times (Theorem (49)). The resulting framework complements prior probabilistic and analytic approaches by expressing Collatz dynamics in closed arithmetic form.

1.3 Motivation and Key Results

The $3x+1$ problem is remarkably simple to state yet remains unsolved. Any new structure underlying its iterations may yield fresh analytical insight. In this work, we show that the problem can be reformulated entirely in terms of the signed Jacobsthal numbers.

Although the Collatz problem has been widely studied (see, e.g., Lagarias [3, 4]), no previous work has established a systematic arithmetic correspondence between the Collatz dynamics and the Jacobsthal numbers beyond the observation noted in Remark (4). Here we develop a deeper and more general relationship between these two structures.

We begin by proving that for any odd integer x_o and any $m \in \mathbb{Z}^+$, there exists a vector $S_v = (s_1, s_2, \ldots, s_m)$ with $s_i \in \mathbb{Z}_{\geq 0}$ and $s_1 > s_2 > \cdots > s_m$ such that

$$\left(x_o + 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} = 0.$$

Here, the vector S_v represents the Collatz trajectory associated with the starting value x_o. This establishes a fundamental connection between the signed Jacobsthal numbers and the $3x+1$ dynamics.

Moreover, for every nonzero Collatz cycle—including those involving negative iterates—with m odd elements, the corresponding vector $S_v = (s_1, \ldots, s_m)$ must satisfy

$$\sum_{i=1}^{m} \left(\frac{(J_s(s_i))^i}{2^{s_i}} + \frac{3^m}{2^{s_1} - 3^m} \frac{1}{3^i 2^{s_i}} \right) \in \mathbb{Z}.$$

Conversely, every admissible vector satisfying this integrality condition corresponds to a unique Collatz cycle. Thus, for the Collatz conjecture to hold, this condition must fail for all admissible S_v with $2^{s_1} > 3^m$, except for the vector associated with the trivial cycle.

We further derive a criterion under which the *coefficient stopping time conjecture* holds. For all m and admissible suffix-sum vectors S_v satisfying

$$2^{s_1} > 3^m \quad \text{and} \quad 2^{s_1 - s_{i+1}} < 3^i \quad \text{for } 1 \leq i < m,$$

one of the following conditions must be satisfied:

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} = 1, \quad \text{or}$$

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} > \frac{3^m 2^{s_1}}{2^{s_1} - 3^m} \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}}.$$

The *coefficient stopping time conjecture* implies the validity of the classical Collatz conjecture.

Notably, these conditions depend only on the signed Jacobsthal numbers and their powers, and not on the particular integer being iterated. This transformation isolates the arithmetic essence of the $3x + 1$ problem, suggesting that its resolution may ultimately hinge solely on the number-theoretic properties of the signed Jacobsthal numbers.

The analysis presented in this paper also shows a relationship between the sequence A177789 and the signed Jacobsthal sequence A077925.

Together, these results show that the dynamics of the Collatz map can be viewed as manifestations of a single arithmetic structure governed by the signed Jacobsthal numbers.

2 Stopping Time

Definition 6. Stopping time (Terras [6, Definition 0.1]). The *stopping time* of a positive integer $x \in \mathbb{Z}^+$ is the smallest $k \in \mathbb{Z}^+$ such that

$$C^k(x) < x.$$

Following standard notation, the stopping time of n is denoted by $\sigma(n)$.

Garner [2, p. 1] observed that it is straightforward to verify the following values:

$$\begin{aligned}
\sigma(n) &= 1 && \text{if } n \text{ is even,}\\
\sigma(n) &= 3 && \text{if } n \equiv 1 \pmod{4},\\
\sigma(n) &= 6 && \text{if } n \equiv 3 \pmod{16},\\
\sigma(n) &= 8 && \text{if } n \equiv 11 \text{ or } 23 \pmod{32},\\
\sigma(n) &= 11 && \text{if } n \equiv 7, 15, \text{ or } 59 \pmod{128},\\
\sigma(n) &= 13 && \text{if } n \equiv 39, 79, 95, 123, 175, 199, \text{ or } 219 \pmod{256},
\end{aligned}$$

and so on.

The corresponding sequence of residues

$$1,\ 3,\ 11,\ 23,\ 7,\ 15,\ \ldots$$

is recorded in the OEIS as sequence A177789. As will be shown later, these residue classes are directly related to the signed Jacobsthal numbers and their powers.

Corollary 7. *As shown by Terras [6], the Collatz conjecture holds if and only if the stopping time is finite for all $x > 1$.*

Since every even integer eventually yields an odd integer under iteration, it suffices to study trajectories beginning with odd integers. Hence, throughout this paper, we restrict attention to iterations that start from odd integers only.

Definition 8. Even-run vector. Let $x_o \in \mathbb{Z}_{\text{odd}}$. The *even-run vector* associated with the first k terms of the Collatz trajectory of x_o is a finite sequence of nonnegative integers

$$(e_1, e_2, \ldots, e_m),$$

where each e_i denotes the number of consecutive even iterates immediately preceding the next odd iterate (or preceding the final term). Here $m \in \mathbb{Z}^+$ is the number of odd elements in the trajectory (excluding the last one, if any).

Example 9. Consider the Collatz trajectory of 11 up to nine terms:

$$(\mathbf{11}, 34, \mathbf{17}, 52, 26, \mathbf{13}, 40, 20, 10).$$

The odd terms are shown in bold. The corresponding even-run vector is

$$(1, 2, 2).$$

Corollary 10. *Only the final entry of an even-run vector can be zero—this occurs when the trajectory terminates at an even element immediately following an odd one. In Example (9), if the trajectory is truncated at* 40*, then the even-run vector becomes* $(1, 2, 0)$*.*

Definition 11. Suffix-sum even-run vector. Let $(e_1, e_2, \ldots, e_m)$ denote the even-run vector corresponding to the trajectory of $x_o \in \mathbb{Z}_{\text{odd}}$ for $k \in \mathbb{Z}^+$ iterations. The *suffix-sum even-run vector* is defined as

$$S(x_o; k) = (s_1, s_2, \ldots, s_m),$$

where

$$s_i = \sum_{j=i}^{m} e_j, \qquad 1 \leq i \leq m. \tag{2}$$

We will use the term *suffix-sum vector* to refer to the *suffix-sum even-run vector.*

Example 12. For the even-run vector in Example (9), the suffix-sum vector is

$$S(11; 8) = (5, 4, 2).$$

Corollary 13. *By definition,*

$$s_1 > s_2 > s_3 > \cdots > s_m \geq 0, \qquad \textit{and} \qquad s_m = e_m.$$

Corollary 14. *All members of a suffix-sum vector representing the complete cycle, starting from an odd member of a cycle, are positive integers.*

Definition 15. Admissible suffix-sum vector. A generic vector $S_v = (s_1, s_2, \ldots, s_m)$ is called *admissible* if it satisfies

$$s_1 > s_2 > \cdots > s_m \geq 0, \qquad s_i \in \mathbb{Z}_{\geq 0}.$$

Like Garner [2, p. 2] and many others, we will now write the term formula for the k^{th} iterate, but only for odd number as starting point and in the language of *even-run* and *suffix-sum* vectors.

Lemma 16. ***Term formula.*** *Let $x_o \in \mathbb{Z}_{odd}$, and let its trajectory of $k \in \mathbb{Z}^+$ iterations have even-run vector $(e_1, e_2, \ldots, e_m)$ and corresponding suffix-sum vector $(s_1, s_2, \ldots, s_m)$. Then the last element of this trajectory is given by*

$$C^k(x_o) = \frac{3^m}{2^{e_1+\cdots+e_m}} x_o + \sum_{i=1}^{m} \frac{3^{m-i}}{2^{e_i+\cdots+e_m}} \tag{3}$$

$$= \frac{3^m}{2^{s_1}} x_o + 3^m \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}}, \tag{4}$$

where $k = m + s_1$.

Corollary 17. *The suffix-sum vector $S_v = (s_1, s_2, \ldots, s_m)$ encodes key information about the corresponding Collatz trajectory. Specifically, the total number of iterations is given by the sum of its length and its first component, that is,*

$$\textit{Total iterations} = m + s_1.$$

Moreover, the number of odd elements in the trajectory satisfies:

- the minimum number of odd terms equals the length m of the vector, and
- the maximum is $m + 1$, attained when the trajectory terminates with an odd element.

Lemma 18. *For all $n \in \mathbb{Z}$,*

$$S(x_o + n2^{s_1}; k) = S(x_o; k).$$

Proof. The first k terms in the trajectories of x_o and $x_o + n2^{s_1}$ share the same parity sequence. Hence, their even-run vectors—and consequently their suffix-sum vectors—are identical. □

From the preceding lemma and Equation (4) we obtain an important consequence presented below.

Corollary 19. *If for some $x_o \in \mathbb{Z}_{odd}$,*

$$C^k(x_o) = \frac{3^m}{2^{s_1}} x_o + \sum_{i=1}^{m} \frac{3^{m-i}}{2^{s_i}},$$

then for any integer n,

$$C^k(x_o + n2^{s_1}) = C^k(x_o) + n3^m. \tag{5}$$

Definition 20. Coefficient stopping time (Garner [2, p. 2]). In Equation (4), the coefficient of x_o is

$$\frac{3^m}{2^{s_1}}.$$

The *coefficient stopping time* of x_o is the least $k \in \mathbb{Z}^+$ such that this coefficient is less than 1; equivalently, the least k satisfying

$$3^m < 2^{s_1}.$$

It follows directly from Equation (4) that the coefficient stopping time of a positive odd integer cannot exceed its ordinary stopping time. Garner [2, p. 4] proposed the following conjecture.

Conjecture 21. ***Coefficient stopping time conjecture (Garner [2, p. 4]).*** *For every positive odd integer $x_o > 1$, the coefficient stopping time of x_o equals its ordinary stopping time.*

Lemma 22. *If the coefficient stopping time conjecture holds, then the Collatz conjecture follows (Garner [2, Application to cycles. p. 4]).*

Proof. Suppose a Collatz trajectory enters a cycle that does not contain 1. Let x_o be the least term of this cycle. Then $C^k(x_o) = x_o$ for some $k \in \mathbb{Z}^+$. Hence the coefficient stopping time is less than k, implying that if the coefficient stopping time equals the stopping time, then x_o cannot be the least term of the cycle. Therefore, if stopping time and coefficient stopping time are always equal, the only possible cycle is the trivial $(1, 4, 2, 1)$ cycle. □

One of the main results of this paper provides a necessary and sufficient condition—expressed in terms of signed Jacobsthal numbers and their powers—for this conjecture to hold. We now proceed to rewrite Equation (4) using the signed Jacobsthal numbers.

3 Connection with signed Jacobsthal Numbers

In this section we establish a connection between Collatz trajectories and powers of signed Jacobsthal numbers. We begin with a simple observation, stated in the following corollary.

Corollary 23. *In view of Equation 1, it is easy to see that the modular inverse of 3^i modulo 2^{s_i} is given by*

$$(J_s(s_i))^i.$$

One application of this corollary is a reformulation of the sequence A381707. The sequence A381707 is related to unimodal Collatz glide sequence. The original formula for A381707 is

$$a(n) = 2^n(3^{-n} \bmod 2^{s_n}) - 1,$$

where

$$s_n = \max\big(2, \lceil \log_2(3^n) - n \rceil\big).$$

Using Corollary 23, the same expression can be rewritten as

$$a(n) = 2^n\big((J_s(s_n))^n \bmod 2^{s_n}\big) - 1.$$

We now direct our attention to a more detailed exploration of how Collatz trajectories connect to signed Jacobsthal numbers.

Lemma 24. ***Term formula using Jacobsthal numbers.*** *Equation (4) can be expressed in terms of the signed Jacobsthal numbers $J_s(n)$ as*

$$C^k(x_o) = \frac{3^m}{2^{s_1}}\left(x_o + 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}\right) + 3^m\sum_{i=1}^{m}\left(\frac{1}{3^i 2^{s_i}} - \frac{(J_s(s_i))^i}{2^{s_i}}\right). \tag{6}$$

Equation (6) differs from Equation (4) only by the addition and subtraction of the same compensating term,

$$3^m\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}},$$

introduced purely for algebraic convenience. The motivation for this adjustment becomes evident in the following lemma and theorem.

Lemma 25. *The term*

$$3^m\left(\frac{1}{3^i 2^{s_i}} - \frac{(J_s(s_i))^i}{2^{s_i}}\right)$$

is an integer for all i such that $1 \le i \le m$.

Remark 26. Lemma 25 is a direct consequence of Corollary 23.

Theorem 27. *Let $S(x_o; k) = (s_1, s_2, \ldots, s_m)$ be the suffix-sum vector of x_o. Then*

$$\left(x_o + 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 0.$$

Proof. From Lemma (25), the term

$$3^m\sum_{i=1}^{m}\left(\frac{1}{3^i 2^{s_i}} - \frac{(J_s(s_i))^i}{2^{s_i}}\right)$$

in Equation (6) is an integer. Since the right-hand side of Equation (6) must also yield an integer value, it follows that

$$\left(x_o + 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 0,$$

establishing the stated congruence. □

Theorem (27) exposes a structural congruence linking Collatz trajectories with powers of signed Jacobsthal numbers.

Corollary 28. *For every $x_o \in \mathbb{Z}_{odd}$ and $m \in \mathbb{Z}^+$, there exists a vector*

$$(s_1, s_2, \ldots, s_m)$$

such that

$$\left(x_o + 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} = 0.$$

Corollary 29. *Let there exist a vector*

$$(s_1, s_2, \ldots, s_m)$$

such that

$$\left(x_o + 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} = 0$$

for some $x_o \in \mathbb{Z}_{odd}$ and $m \in \mathbb{Z}^+$. Then the suffix-sum vector associated with x_o is

$$S(x_o; k) = (s_1, s_2, \ldots, s_m),$$

where $k = m + s_1$.

Thus, every odd integer is connected to its Collatz trajectory through powers of signed Jacobsthal numbers. Notably, key properties of Collatz dynamics, such as the existence of a cycle or absence of a nontrivial cycles in $\mathbb{Z}^+$, can all be expressed entirely in terms of signed Jacobsthal numbers and their powers. This observation motivates the next part of our discussion. Before proceeding, we establish several preparatory results.

Lemma 30. *If there exist x_o and k such that $S(x_o; k) = (s_1, s_2, \ldots, s_m)$ and*

$$x_o - C^k\big(x_o\big) = n\big(2^{s_1} - 3^m\big)$$

for some $n \in \mathbb{Z}$, then there exists a unique Collatz cycle and the cycle contains the element $x_o - n2^{s_1}$.

Proof. From Equation (5),

$$\begin{aligned}
\big(x_o - n2^{s_1}\big) - C^k\big(x_o - n2^{s_1}\big) &= \big(x_o - n2^{s_1}\big) - \big(C^k\big(x_o\big) - n3^m\big) \\
&= \big(x_o - C^k\big(x_o\big)\big) - n\big(2^{s_1} - 3^m\big) \\
&= 0.
\end{aligned}$$

Hence, $x_o - n2^{s_1}$ must be an element of a unique Collatz cycle. □

Lemma 31. *If*

$$x_o = n2^{s_1} - 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}},$$

then

$$x_o - C^k(x_o) = (2^{s_1} - 3^m)\left(n - \sum_{i=1}^{m}\left(\frac{(J_s(s_i))^i}{2^{s_i}} + \frac{3^m}{2^{s_1} - 3^m}\frac{1}{3^i 2^{s_i}}\right)\right). \tag{7}$$

Proof. Substituting $x_o = n2^{s_1} - 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}$ into Equation (6) gives

$$C^k(x_o) = 3^m n + 3^m\sum_{i=1}^{m}\left(\frac{1}{3^i 2^{s_i}} - \frac{(J_s(s_i))^i}{2^{s_i}}\right).$$

So, we have

$$\begin{aligned} x_o - C^k(x_o) &= n2^{s_1} - 2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}} - \left(3^m n + 3^m\sum_{i=1}^{m}\left(\frac{1}{3^i 2^{s_i}} - \frac{(J_s(s_i))^i}{2^{s_i}}\right)\right) \\ &= (2^{s_1} - 3^m)\left(n - \sum_{i=1}^{m}\left(\frac{(J_s(s_i))^i}{2^{s_i}} + \frac{3^m}{2^{s_1} - 3^m}\frac{1}{3^i 2^{s_i}}\right)\right), \end{aligned}$$

which is precisely Equation (7). □

3.1 Collatz Cycles and signed Jacobsthal Numbers

With the foundational results established, we are now prepared to examine the connection between Collatz cycles and signed Jacobsthal numbers.

Definition 32. Jacobsthal-Collatz Function. For an admissible suffix-sum vector

$$S_v = (s_1, s_2, \dots, s_m),$$

the *Jacobsthal–Collatz function* is defined by

$$J_C(S_v) = \sum_{i=1}^{m}\left(\frac{(J_s(s_i))^i}{2^{s_i}} + \frac{3^m}{2^{s_1} - 3^m}\frac{1}{3^i 2^{s_i}}\right). \tag{8}$$

Definition 33. Jacobsthal–Collatz Condition. An admissible suffix-sum vector S_v is said to satisfy the *Jacobsthal–Collatz condition* if and only if

$$J_C(S_v) \in \mathbb{Z}.$$

Using this definition, we can restate the condition from Lemma (31) in a more compact form.

Corollary 34. *Using Definition (33), the condition from Lemma (31) can be equivalently expressed as: If*

$$x_o = n2^{s_1} - 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}},$$

then

$$x_o - C^k(x_o) = (2^{s_1} - 3^m)\big(n - J_C(S_v)\big). \tag{9}$$

Lemma 35. *Let $S_v = (s_1, s_2, \ldots, s_m)$ be an admissible suffix-sum vector. If it satisfies the* Jacobsthal–Collatz condition*, namely*

$$J_C(S_v) \in \mathbb{Z},$$

then there exists a unique Collatz cycle associated with S_v.

Proof. Assume $J_C(S_v) = n'$ for some integer $n' \in \mathbb{Z}$. By the preceding corollary, we have

$$x_o - C^k(x_o) = (2^{s_1} - 3^m)(n - n'),$$

for suitable integers x_o and n. Invoking Lemma (30), it follows that there exists a unique Collatz cycle containing the element $x_o - (n - n')2^{s_1}$. Hence the existence of an admissible S_v satisfying $J_C(S_v) \in \mathbb{Z}$ implies existence of a corresponding Collatz cycle. □

Corollary 36. *If $2^{s_1} > 3^m$ and the* Jacobsthal–Collatz condition *holds, then the corresponding cycle must lie entirely in $\mathbb{Z}^+$. Indeed, for iterations over negative integers, if $\frac{2^{s_1}}{3^m} < 1$, then by Equation (4), the iterate has a smaller absolute value than the starting term, preventing the formation of a cycle.*

Lemma 37. *If a Collatz cycle of length k with m odd elements exists, then the full-run suffix-sum vector of the cycle must satisfy the* Jacobsthal–Collatz Condition*.*

Proof. Suppose a Collatz cycle of length k with m odd elements exists, and let x_o be one of its odd elements with suffix-sum vector $S(x_o; k) = (s_1, s_2, \ldots, s_m)$. By Theorem (27),

$$x_o = n2^{s_1} - 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \quad \text{for some } n \in \mathbb{Z}.$$

Since x_o is part of a cycle, we have $x_o - C^k(x_o) = 0$. Substituting into Equation (9) gives

$$J_C(S_v) = n \in \mathbb{Z}.$$

Therefore, the suffix-sum vector associated with any Collatz cycle necessarily satisfies the Jacobsthal–Collatz Condition. □

The following result establishes the central arithmetic equivalence of this paper. It identifies a precise condition—expressed entirely in terms of signed Jacobsthal numbers—under which a Collatz cycle can exist. This correspondence converts the iterative dynamics of the map into an explicit integrality criterion.

Theorem 38. ***Cycles and the signed Jacobsthal Connection.*** *A Collatz cycle exists* if and only if *there exists a corresponding admissible suffix-sum vector* $S_v = (s_1, s_2, \ldots, s_m)$ *satisfying the* Jacobsthal–Collatz Condition

$$J_C(S_v) \in \mathbb{Z}.$$

Proof. Assume that there exists an admissible suffix-sum vector S_v satisfying the Jacobsthal–Collatz condition, that is, $J_C(S_v) \in \mathbb{Z}$. Then, by Lemma (35), there exists a unique Collatz cycle corresponding to S_v.

Conversely, if a Collatz cycle exists, then by Lemma (37), its full-run suffix-sum vector must satisfy the *Jacobsthal–Collatz Condition*. Hence, the theorem is proved. □

Definition 39. Multifold suffix-sum vector. Let $S_v = (s_1, s_2, \ldots, s_m)$ be an admissible suffix-sum vector. For any $p \in \mathbb{Z}^+$, the *multifold suffix-sum vector* is defined as

$$S'_v = (s'_1, s'_2, \ldots, s'_{p \cdot m}),$$

where

$$s'_{i \cdot m+j} = (p - i - 1)s_1 + s_j, \qquad 0 \le i < p,\ 1 \le j \le m.$$

Corollary 40. *For any $p \in \mathbb{Z}^+$, a multifold suffix-sum vector may equivalently be referred to as a p*-fold suffix-sum vector*.*

Lemma 41. *Let $S_v = (s_1, s_2, \ldots, s_m)$ be an admissible suffix-sum vector representing a Collatz cycle. Then, for any $p \in \mathbb{Z}^+$, the p-fold repetition of this cycle is represented by the multifold suffix-sum vector S'_v defined in Definition (39).*

Proof. Let the even-run vector associated with S_v be

$$(e_1, e_2, \ldots, e_m).$$

Since S_v corresponds to a Collatz cycle, the p-fold repetition of this cycle produces the even-run vector

$$(e'_1, e'_2, \ldots, e'_{p \cdot m}) = \underbrace{\underbrace{e_1, e_2, \ldots, e_m}, \underbrace{e_1, e_2, \ldots, e_m}, \ldots, \underbrace{e_1, e_2, \ldots, e_m}}_{p \text{ times}}.$$

Hence, for all $0 \le i < p$ and $1 \le j \le m$, we have

$$e'_{i \cdot m+j} = e_j,$$

where $e'_{i\cdot m+j}$ denotes the j^{th} entry in the i^{th} repeated block.

By Definition (11), the corresponding suffix-sum entries of

$$S'_v = (s'_1, s'_2, \dots, s'_{p\cdot m})$$

satisfy

$$\begin{aligned} s'_{i\cdot m+j} &= \sum_{a=i\cdot m+j}^{p\cdot m} e'_a \\ &= \sum_{a=j}^{m} e_a + (p-i-1)\sum_{a=1}^{m} e_a \\ &= (p-i-1)s_1 + s_j, \quad \text{(by Equation 2)}. \end{aligned}$$

This completes the proof. □

Theorem 42. *Let $S_v = (s_1, s_2, \dots, s_m)$ be an admissible suffix-sum vector that satisfies the Jacobsthal–Collatz condition, i.e.,*

$$J_C(S_v) \in \mathbb{Z}.$$

Then, for every $p \in \mathbb{Z}^+$, the p-fold suffix-sum vector S'_v, as defined in Definition (39), also satisfies

$$J_C(S'_v) \in \mathbb{Z}.$$

Proof. By Lemma (35), if $J_C(S_v) \in \mathbb{Z}$, the vector S_v corresponds to a unique Collatz cycle. A p-fold repetition of this cycle remains a valid Collatz cycle. Hence, by Lemma (37), the corresponding multifold suffix-sum vector S'_v must also satisfy the Jacobsthal–Collatz condition, i.e. $J_C(S'_v) \in \mathbb{Z}$. □

Corollary 43. *The trivial Collatz cycle* (1, 4, 2, 1) *has period* 3*, and its corresponding suffix-sum vector is*

$$S(1;3) = (2).$$

By Lemma (41), the suffix-sum vector for the p-fold repetition of this cycle is given by

$$\begin{aligned} S\big(1;3p\big) &= \big(2p, 2\big(p-1\big), 2\big(p-2\big), \dots, 6, 4, 2\big) \\ &= \big(\underbrace{\dots, 10, 8, 6, 4, 2}_{p\text{-terms}}\big). \end{aligned}$$

Example 44. Let us compute $J_C\big(S(1;3p)\big)$, using the Equation (8), for several values of p. The trivial case occurs when $p = 1$, yielding

$$\begin{aligned} J_C\big(S(1;3)\big) &= J_C\big((2)\big) \\ &= \frac{J_s(2)^1}{2^2} + \frac{3^1}{2^2 - 3^1} \cdot \frac{1}{3^1 2^2} \\ &= \big(\frac{-1}{4} + \frac{1}{4}\big) = 0. \end{aligned}$$

Next, consider a larger case with $p = 5$. Then

$$J_C\big(S(1; 3 \times 5)\big) = J_C\big((10, 8, 6, 4, 2)\big).$$

We first evaluate the primary summation:

$$\begin{aligned}\sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} &= \frac{J_s(10)^1}{2^{10}} + \frac{J_s(8)^2}{2^8} + \frac{J_s(6)^3}{2^6} + \frac{J_s(4)^4}{2^4} + \frac{J_s(2)^5}{2^2} \\ &= \frac{(-341)^1}{2^{10}} + \frac{(-85)^2}{2^8} + \frac{(-21)^3}{2^6} + \frac{(-5)^4}{2^4} + \frac{(-1)^5}{2^2} \\ &= -\frac{79873}{2^{10}}.\end{aligned}$$

The correction term is computed as

$$\begin{aligned}\frac{3^m}{2^{s_1} - 3^m} \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}} &= \frac{3^5}{2^{10} - 3^5}\Big(\frac{1}{3^1 2^{10}} + \frac{1}{3^2 2^8} + \frac{1}{3^3 2^6} + \frac{1}{3^4 2^4} + \frac{1}{3^5 2^2}\Big) \\ &= \frac{3^5}{781} \cdot \frac{781}{3^5 2^{10}} = \frac{1}{2^{10}}.\end{aligned}$$

Combining both parts gives

$$J_C\big((10, 8, 6, 4, 2)\big) = -\frac{79873}{2^{10}} + \frac{1}{2^{10}} = -78,$$

which is indeed an integer.

Thus, as p ranges over $\mathbb{Z}^+$, the values of the Jacobsthal–Collatz function $J_C\big(S(1; 3p)\big)$ form a discrete integer sequence:

$$\begin{aligned}&0, 0, 1, -1, -78, 558, 102803, -3411763, -2219938716, \\ &307986970396, 773279732056805, -434776959996932005, \ldots\end{aligned}$$

This sequence corresponds to entry A389565 in the OEIS.

A similar construction for the cycle $(-1, -2, -1)$ yields

$$\begin{aligned}&0, 0, 1, 1, 6, 6, 119, 223, 9972, 11852, 3338237, \\ &12126925, 4561052818, -1079233822, 24799374299315, \ldots\end{aligned}$$

This sequence corresponds to entry A390493 in the OEIS.

Because the Jacobsthal–Collatz condition involves exponentiated ratios of rapidly growing terms, the magnitude of $J_C(S_v)$ escalates extraordinarily fast with p. For instance:

$$\begin{aligned}&J_C\big(S(-17; 1 \times 18)\big) = 200{,}453, \\ &J_C\big(S(-17; 2 \times 18)\big) = 567{,}822{,}615{,}635{,}892{,}838{,}467{,}858, \\ &J_C\big(S(-17; 3 \times 18)\big) \text{ is a 52-digit integer,} \\ &J_C\big(S(-17; 4 \times 18)\big) \text{ is a 95-digit integer.}\end{aligned}$$

These values correspond to the cycle defined by $C^{18}(-17) = -17$.

Altogether, these computations demonstrates the exponential scaling inherent in the Jacobsthal–Collatz formulation—an interplay of algebraic self-consistency and explosive numerical growth that highlights both the coherence of the structure and its formidable computational complexity.

Remark 45. Theorem (42) uncovers an intrinsic algebraic symmetry in the signed Jacobsthal numbers. While its proof follows directly from the Jacobsthal–Collatz framework, the theorem encapsulates a deeper structural phenomenon: the signed Jacobsthal numbers are recursively interconnected through their own powers, independently of any particular iterative map or trajectory. In other words, the Jacobsthal structure exhibits self-replication under systematic extension.

3.2 Nontrivial cycles of positive iterates

Having discussed Collatz cycles in general, we now focus specifically on the conditions governing the existence—or the nonexistence—of nontrivial cycles composed of positive iterates. This section establishes the necessary criteria distinguishing the trivial cycle from any potential nontrivial ones.

Lemma 46. *Let $S_v = (s_1, s_2, \dots, s_m)$ be an admissible suffix-sum vector such that*

$$J_C(S_v) \in \mathbb{Z}, \quad \textit{and}$$

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1.$$

Then S_v necessarily represents one or more complete runs of the trivial cycle $(1, 4, 2, 1)$.

Proof. By Corollary (29), the congruence

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1$$

implies that iterating the integer 1 yields precisely the same suffix-sum vector. The additional condition $J_C(S_v) \in \mathbb{Z}$ ensures that the corresponding iteration is cyclic. By Lemma (35), such a vector uniquely identifies a specific Collatz cycle. Since the iteration beginning with 1 produces the trivial cycle $(1, 4, 2, 1)$, the given suffix-sum vector must therefore correspond to one or more concatenated runs of this trivial cycle. This completes the proof. □

Theorem 47. *The Collatz conjecture holds* if and only if *for every admissible suffix-sum vector $S_v = (s_1, s_2, \dots, s_m)$ satisfying $2^{s_1} > 3^m$, one of the following conditions is true:*

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1, \quad \textit{or}$$

$$J_C(S_v) \notin \mathbb{Z}.$$

Proof. First, consider the case where

$$\left(-2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1.$$

By Lemma (46), this condition corresponds precisely to the trivial Collatz cycle.

Now suppose, on the contrary, that the above congruence does not hold and that $J_C(S_v) \in \mathbb{Z}$. By Corollary (36), the vector S_v then determines a unique Collatz cycle consisting entirely of positive integers whenever $2^{s_1} > 3^m$. The existence of such a nontrivial positive cycle would contradict the Collatz conjecture. Hence, for the conjecture to hold, it is necessary that $J_C(S_v) \notin \mathbb{Z}$ whenever the congruence condition fails.

Conversely, if the stated condition holds for all admissible S_v, no nontrivial positive cycle can exist, and thus the Collatz conjecture is true. □

We have now established a strong algebraic link between Collatz cycles and signed Jacobsthal numbers. In the next section, we will investigate the *coefficient stopping time* and its relationship with signed Jacobsthal numbers.

3.3 Stopping time and signed Jacobsthal numbers

Theorem (47) applies to all suffix-sum vectors satisfying the inequality $2^{s_1} > 3^m$. We now turn to the critical case where this condition first becomes true—that is, when the *coefficient stopping time* occurs.

We begin with a lemma that reformulates the coefficient stopping time conjecture in terms of Collatz trajectories, using the framework of suffix-sum vectors.

Lemma 48. *Let $x_o \in \mathbb{Z}_{odd}$ have the associated suffix-sum vector*

$$(s_1, s_2, \ldots, s_m),$$

and suppose that $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \le i < m$. If the coefficient stopping time conjecture *holds, then for all $x_o > 1$ satisfying $2^{s_1} > 3^m$, we have*

$$x_o - C^k(x_o) > 0.$$

Proof. Let the corresponding even-run vector be

$$(e_1, e_2, \ldots, e_m).$$

For each odd term in the Collatz trajectory (except possibly the final one), the coefficient of x_o in Equation (3) is given by

$$\frac{3^i}{2^{e_1+\cdots+e_i}} = \frac{3^i}{2^{e_1+\cdots+e_m-(e_{i+1}+\cdots+e_m)}} = \frac{3^i}{2^{s_1-s_{i+1}}}, \qquad 1 \le i < m.$$

Because $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \leq i < m$ implying that every odd iterate before the last is larger than x_o. When $2^{s_1} > 3^m$, the coefficient stopping time must therefore occur after the odd term immediately preceding the final one. By the coefficient stopping time conjecture, there must exist a subsequent iterate strictly smaller than x_o.

Furthermore, the odd iterate preceding the final term is followed by a larger even term, which then decreases monotonically toward the end of the trajectory. Consequently, the stopping time occurs at or before the final term, guaranteeing that

$$x_o - C^k(x_o) > 0.$$

□

Theorem 49. ***Criterion for the Coefficient Stopping Time Conjecture.*** *The Coefficient Stopping Time Conjecture* (21) *holds* if and only if, *for every $m \in \mathbb{Z}^+$ and admissible suffix-sum vector $S_v = (s_1, s_2, \ldots, s_m)$ satisfying $2^{s_1} > 3^m$ and $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \leq i < m$, one of the following conditions is satisfied:*

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1, \quad \textit{or}$$
$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} > \frac{3^m 2^{s_1}}{2^{s_1} - 3^m} \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}}.$$

Proof. First, consider the case where

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} = 1.$$

By Lemma (46), this corresponds to the trivial cycle. Hence, it suffices to examine the complementary case where this congruence does not hold.

Let

$$x_o = n2^{s_1} - 2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}},$$

where n is chosen such that $0 < x_o < 2^{s_1}$. Then we have

$$x_o = \left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1}, \tag{10}$$

$$n = \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} + \frac{1}{2^{s_1}}\left(\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1}\right). \tag{11}$$

By Lemma 48, the conditions $2^{s_1} > 3^m$ and $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \leq i < m$ imply that

$$x_o - C^k(x_o) > 0$$

is necessary for the Coefficient Stopping Time Conjecture (21) to hold. From Equation (7), this is equivalent to

$$n > \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} + \frac{3^m}{2^{s_1} - 3^m} \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}}.$$

Substituting n from Equation (11) then gives

$$\left(-2^{s_1} \sum_{i=1}^{m} \frac{(J_s(s_i))^i}{2^{s_i}} \right) \bmod 2^{s_1} > \frac{3^m 2^{s_1}}{2^{s_1} - 3^m} \sum_{i=1}^{m} \frac{1}{3^i 2^{s_i}},$$

which ensures that the coefficient stopping time of x_o coincides with its ordinary stopping time. □

Theorem (49) establishes a condition on the signed Jacobsthal numbers whose validity implies the *coefficient stopping time conjecture*. Although the condition is expressed purely in terms of signed Jacobsthal numbers, its consequence directly concerns the behavior of the Collatz trajectory.

Definition 50. Stopping time vector. An admissible suffix-sum vector $(s_1, s_2, \ldots, s_m)$ is called a *stopping time vector* if it satisfies the following conditions:

i $2^{s_1} > 3^m$,

ii $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \leq i < m$, and

iii s_1 is the smallest possible integer satisfying (i).

Corollary 51. *For a given value of m, the smallest integer s_1 satisfying the inequality $2^{s_1} > 3^m$ is given by*

$$s_1 = \lceil m \log_2 3 \rceil .$$

Example 52. Representative examples of *stopping time vectors* are:

$$\begin{aligned}
&(2),\\
&(4,3),\\
&(5,4,2),\ (5,4,3),\\
&(7,6,5,3),\ (7,6,5,4),\ (7,6,4,3),
\end{aligned}$$

and so on.

For each of these vectors, the corresponding smallest positive integer can be obtained using Equation (10), yielding:

$$
\begin{aligned}
x_o &= 1 \quad \text{for } (2),\\
x_o &= 3 \quad \text{for } (4,3),\\
x_o &= 11 \quad \text{for } (5,4,2),\\
x_o &= 23 \quad \text{for } (5,4,3),\\
x_o &= 7 \quad \text{for } (7,6,5,3).
\end{aligned}
$$

All elements of the integer sequence A177789 can likewise be generated using Equation (10). Because Equation (10) incorporates powers of the signed Jacobsthal numbers (sequence A077925), it reveals a previously unnoticed structural link between the two sequences.

4 Conclusion

This paper establishes a structural link between the Collatz map and the signed Jacobsthal numbers, providing a reformulation of the $3x + 1$ problem in purely arithmetic terms. By representing Collatz iterations through the suffix-sum even-run vector and connecting its parameters to signed Jacobsthal numbers (Theorem (27)), we show that every odd integer is related to its Collatz trajectory via powers of signed Jacobsthal numbers. Leveraging this fundamental connection, we derive necessary and sufficient conditions for the existence of cycles. In particular, Theorems (38), (47), and (49) show that both the existence of Collatz cycles and the validity of the coefficient stopping time conjecture can be characterized by explicit arithmetic conditions involving powers of signed Jacobsthal numbers. Theorem (42) demonstrates that the Jacobsthal structure reproduces itself under systematic repetition, manifesting naturally in Collatz cycles. Collectively, these results establish that the Collatz conjecture is equivalent to a concrete arithmetic condition expressed entirely in terms of signed Jacobsthal numbers. It is noteworthy that the analysis presented in this paper reveals a mathematically-interesting connection linking sequence A177789, and sequence A381707, to the signed Jacobsthal numbers (sequence A077925).

Recasting the Collatz problem in the language of Jacobsthal arithmetic offers a new structural perspective. Rather than studying the parity-driven dynamics of the map directly, the problem is transformed into verifying explicit number-theoretic constraints. This reformulation opens the door for the application of established techniques from recurrence theory, modular arithmetic, and Diophantine analysis to further explore the conjecture.

While this work establishes a clear and promising connection between the Collatz dynamics and signed Jacobsthal numbers, much depth remains to be explored. Notably, at no point in this paper was it necessary to expand the powers of the signed Jacobsthal numbers themselves, despite their central role in the presented framework. Thus, the current results only begin to reveal the underlying arithmetic structure that links Collatz trajectories to the signed Jacobsthal numbers. Future research directions may include:

- Investigating whether the Jacobsthal–Collatz condition (Definition (33)) admits nontrivial integer solutions when $2^{s_1} > 3^m$;
- Determining whether a positive lower bound exists for

$$\left(-2^{s_1}\sum_{i=1}^{m}\frac{(J_s(s_i))^i}{2^{s_i}}\right) \bmod 2^{s_1} - \frac{3^m 2^{s_1}}{2^{s_1}-3^m}\sum_{i=1}^{m}\frac{1}{3^i 2^{s_i}},$$

  under the conditions $2^{s_1} > 3^m$ and $2^{s_1 - s_{i+1}} < 3^i$ for all $1 \leq i < m$. A strictly positive bound would imply the Collatz conjecture;
- Extending the framework to generalized $px+q$ maps by developing analogous relationships with suitable linear recurrence sequences.

Ultimately, the results presented here suggest that the intricate combinatorial behavior of the Collatz map can be encoded within the arithmetic structure of a classical integer sequence—the Jacobsthal numbers—pointing toward a potentially fruitful direction for future analytical investigation.

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