THEORY OF SEMI-FUCHSIAN SCHOTTKY GROUPS

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ABSTRACT. The purpose of this expository article is to introduce the notion of a new type of (classical and non-classical) Schottky structures in the discrete subgroups of the projective special linear group over the real numbers of degree 2. In particular, in this manuscript, we have investigated the classical and non-classical structures of a kind of Schottky group (which we named as semi-Fuchsian Schottky groups) in the hyperbolic plane. In general, a Schottky group is called classical if the Schottky curves used in the Schottky construction are Euclidean circles; on the other hand, it is said to be non-classical if the defining curves are Jordan curves, except the Euclidean circles. In fact, in this article, we initiated the concept of classical semi-Fuchsian Schottky groups of rank 2 (hence any finite rank) in the upper-half plane model. This study yields new and interesting surfaces in Riemann surface theory, specifically, various types of hyperbolic pairs of pants with infinite area (indeed, we proposed the notion of a pair of full pants) and a hyperbolic torus with one infinite end. After that, we constructed a structure of the rank 2 semi-Fuchsian Schottky group with non-classical generating sets in the hyperbolic plane by using two suitable Möbius transformations. More precisely, in this paper, we have produced a non-trivial example of the semi-Fuchsian Schottky group of rank 2 with non-classical generating sets within the hyperbolic plane.

1. INTRODUCTION

In 1877, Schottky [11] introduced the classical structure of Schottky groups. A long time later, in 1974, Marden ([7], [8]) first provided the concept of non-classical Schottky groups (though his proof was non-constructive). Indeed, Marden [7] established that the intersection of the closure of classical Schottky space with Schottky space is not the whole of Schottky space, and hence there are Schottky groups that are not classical. In 1975, Zarrow [16] claimed to have found an example of a rank 2 non-classical Schottky group; however, it was later proved to be classical by Sato [15]. In particular, Sato [10] proved that the non-classical Schottky group constructed by Zarrow [16] was, in fact, classical on a different set of generators. In 1991, Yamamoto [15] first constructed the rank 2 non-classical Schottky group in the Riemann sphere.

After that, in 2009, Williams [14] also supplied an example of the rank 2 non-classical Schottky group by utilizing the structure of the non-classical Schottky group proposed in the paper of Yamamoto [15]. So, most of the works surveyed in the literature of non-classical Schottky groups are in Kleinian flavor (i.e., in the hyperbolic space dimension 3). However, in the hyperbolic space dimension 2, almost nothing has been studied about the non-classical Schottky structure in the Fuchsian group theory. In particular, Button [4] established that all Fuchsian Schottky groups are classical Schottky groups, but not always on the same set of generators. Additionally, Button in [4] presented an example (along with a graphic) of a Fuchsian group that is Schottky on a specific generating set but non-classical on those generators. Marden ([7], [8]) also demonstrated that all Fuchsian Schottky groups are classical Schottky groups. So, after these works, although it is a very challenging task, it remains open to investigate a non-classical generating set for a Schottky group in a Fuchsian flavor. In [13], we have provided two non-trivial examples of the rank 2 Fuchsian Schottky groups with non-classical generating sets in the upper-half plane with the circle at infinity as the boundary. It is well known that a Schottky group is said to be classical if, for at least one set of generators, at least one set of curves (utilized in the Schottky construction) can be used as Euclidean circles in the complex plane. Moreover, if a Schottky group is classical on any specific generating set, the group is classical. In contrast, if a Schottky group is non-classical on particular generating sets, one can not claim that the group is a non-classical Schottky group. It signifies that if a Schottky group is not classical on a specific generating set, we can not conclude that the group is not classical. As we have mentioned earlier, this kind of phenomenon occurred in the work of Zarrow [16]. Indeed, if a Schottky group is non-classical in every generating set, the Schottky group turns out to be non-classical. In this article, to develop the literature on the Fuchsian group theory, we aim to investigate the classical and non-classical structures on a type of Schottky group (which we named as semi-Fuchsian Schottky group) in the two-dimensional hyperbolic space. More precisely, in this manuscript, after introducing the notion of any rank 2 (hence any finite rank) classical semi-Fuchsian Schottky groups (initiating the idea of infinite area hyperbolic pair of pants, i.e., a pair of pants attached with at least an infinite (hyperbolic) area funnel in the (closed) geodesic boundary of its compact core), we have presented a non-trivial example of the rank 2 semi-Fuchsian Schottky group with non-classical generating sets within the upper-half plane model without the boundary $\mathbb{R} \cup \{\infty\}$ by utilizing two appropriate Möbius transformations.

Question 1. Does there exist any group that originates all possible hyperbolic pairs of pants with infinite area?

In the literature, it is well known that a pair of pants is of finite area (more precisely, π). In this paper, we introduce a Schottky group (which we call semi-Fuchsian Schottky groups) equipped with a certain structure that provides an affirmative answer to the above question. It is evident that any Schottky group is of two types, one is classical and the other is non-classical. In particular, in this article, we have delivered a comprehensive notion of this group in classical (see, Section 2 for details) and non-classical (see, Sections 3, 4, and 5 for elaboration and more precisely, Theorem 1 in the following) aspects with most of the parts geometric (see, Figures $1(a)$, $1(b)$, $1(c)$, $1(d)$, $1(e)$, $1(g)$, $1(h)$, $1(i)$, $1(j)$, $1(k)$, $1(l)$, $1(m)$, $1(n)$, $1(o)$, and $1(p)$, in Section 2, for any rank 2 classical semi-Fuchsian Schottky groups in the upper-half plane model, on the other hand, in Section 3, Figure 4 supplies the representation of a rank 2 semi-Fuchsian Schottky group with non-classical generating sets within the hyperbolic plane). This exposition leads to various types of hyperbolic surfaces (viz., a 3-funnel surface, i.e., a pair of full pants (see, Figure $2(a)$), two 2-funnel surfaces (see, Figure $2(b)$, and $2(f)$), two 1-funnel surfaces (see, $2(c)$, and $2(d)$), and a torus with one infinite end (see, Figure $2(e)$)) arises from any rank 2 classical semi-Fuchsian Schottky groups. Furthermore, we initiate a non-classical Schottky structure (looks analogous to Yamamoto's construction [15] but it's not) in the upper-half plane without the boundary (i.e., the circle at infinity) in Fuchsian flavor. In the following diagrams $1(a)$, $1(b)$, $1(c)$, $1(d)$, $1(e)$, $1(f)$, $1(g)$, $1(h)$, $1(i)$, $1(j)$, $1(k)$, $1(l)$, $1(m)$, $1(n)$, $1(o)$, $1(p)$, and 4, we denote h_i, $i = 1, 2,...,26$, h_τ^*, and h_τ^{**} are hyperbolic Möbius transformations and p_i, $i = 1, 2, ...,9$ are parabolic Möbius transformations that pair the semi-circles in the upper-half plane. In these figures, SC_1, SC_2, SC_3, and SC_4 are arbitrary semi-circles (the defining curves that represent the various types of classical Schottky structures in the semi-Fuchsian Schottky groups) with centers lying in the real axis and r_i, $i = 1, 2,...,44$ are positive real numbers that indicate the radii of these semi-circles. In Figure 4, $SC_{\mu,\tau}$, and $SC_{-\mu,\tau}$ are semi-circles with diameters situated in the vertical height (from the real axis) of the value τ, where $\tau \to 0^+$. Apart from that, $R_{(\mu-1),\tau}$ and $R_{(\mu+1),\tau}$ are regarded as a small rectangle (at the vertical height τ, $\tau \to 0^+$ from the horizontal axis) and big rectangle (at the vertical height $\frac{\tau}{2}$, $\tau \to 0^+$ from the real axis) within the hyperbolic plane, respectively. In particular, $SC_{\mu,\tau}$, $SC_{-\mu,\tau}$, $R_{(\mu-1),\tau}$, and $R_{(\mu+1),\tau}$ exhibit the defining Jordan curves in the non-classical semi-Fuchsian Schottky structures within the hyperbolic plane. Figure $1(f)$ is not the structure of a semi-Fuchsian Schottky group in the upper-half plane. This is simply a justification for the non-existence of a

classical semi-Fuchsian Schottky group in the mentioned plane. In this paper, we have explored the torsion-free (surface) groups, so there is no elliptic element in the group.

We are now going to state the main theorem of this manuscript.

Theorem 1. The Schottky group $\Gamma_{S_\tau}^N$ generated by the Möbius transformations h_τ^* and h_τ^{**} is a non-classical semi-Fuchsian Schottky group in the hyperbolic plane without the boundary (i.e., circle at infinity) for $\tau < 10^{-17}$. $\qquad\qquad\square$

In Section 3, the Möbius transformations h_τ^* and h_τ^{**} are clarified in detail.

In particular, we have the following:

Corollary 1.0.1. *The Schottky group $\Gamma_{S_{\tau^*}}^N$ generated by the hyperbolic elements $h_{\tau^*}^1$ and $h_{\tau^*}^2$ is a non-classical semi-Fuchsian Schottky group within the upper-half plane model without the boundary (i.e., $\mathbb{R} \cup \{\infty\}$) for $\tau^* < 5 \times 10^{-18}$.*

In Section 6, we have furnished the hyperbolic elements $h_{\tau^*}^1$ and $h_{\tau^*}^2$ in detail.

Outline of this article. The rest of this manuscript is arranged as follows: In Section 2, firstly, we have initiated the notion of semi-Fuchsian Schottky groups of rank 2 (hence any finite rank) and proposed a classical Schottky structure in the hyperbolic plane. After that, we have discussed the nature of the associated hyperbolic surfaces related to that group. Sections 3 and 4 are the technical core of the paper. In Section 3, we have organized the setup for the non-classical structure (more precisely, non-classical generating set) in the semi-Fuchsian Schottky group of rank 2 (denoted by $\Gamma_{S_\tau}^N$) in the upper-half plane model. Section 4 deals with five necessary lemmas that will help to establish our main Theorem 1 later in Section 5. First, Lemma 4.1 is concerned with the existence of the fundamental domain for the rank 2 classical semi-Fuchsian Schottky group, which is encircled by four loops lying on $\mathbb{R} \cup \{\infty\}$. After that, in Lemma 4.2, under some suppositions, we have estimated the lengths of the components of the domain of discontinuity of the group $\Gamma_{S_\tau}^N$ in the hyperbolic plane. Then, in Lemma 4.3, we have further evaluated the lengths of components of the domain of discontinuity of the group $\Gamma_{S_\tau}^N$ that intersects the upper-imaginary axis. Lemma 4.4 is associated with the regions in Lemma 4.2 and Lemma 4.3 with the distance between the curves in the set SC_τ^*, where $SC_\tau^* = \{SC_\tau^1, SC_\tau^2, SC_\tau^3, ..., SC_\tau^{\mathbf{I}^+}\}$, ($\mathbf{I}^+$ represents the positive natural number) is the total list of images of the curves $SC_{\mu,\tau}, SC_{-\mu,\tau}, R_{(\mu+1),\tau}$, and $R_{(\mu-1),\tau}$, applying by the semi-Fuchsian Schottky group $\Gamma_{S_\tau}^N$. In Lemma 4.5, we have shown that the distance between the consecutive loops in the set SC_τ^* is less than $\frac{1}{50}$. Here, we have assessed the distance along the upper

imaginary axis or the real axis. After establishing all these lemmas, we have proved the main goal of this paper in Section 5 (see Theorem 1). Section 6 is equipped with the non-trivial example of the rank 2 semi-Fuchsian Schottky group with a non-classical generating set within the upper-half plane without the boundary, i.e., the extended real numbers (see Corollary 1.0.1). Finally, in this section, we have suggested some open problems in this regard.

2. STRUCTURE OF CLASSICAL SEMI-FUCHSIAN SCHOTTKY GROUPS WITH RELATED HYPERBOLIC SURFACES

In [12], we have fabricated a finitely generated Fuchsian group (equipped with a certain classical Schottky structure) which produces a conformally compact hyperbolic Riemann surface. We say a hyperbolic Riemann surface is conformally compact if it is an infinite area surface equipped with a metric of constant Gaussian curvature -1, with a finitely generated non-abelian fundamental group, and without cusp ends (i.e., only funnels end exist in the boundary of the compact core of the hyperbolic surface). Note that this type of group is not a usual convex cocompact group. It is rather an infinite area convex cocompact group. For an encounter of conformally compact hyperbolic Riemann surfaces, the readers may go through the reference, for instance, the book [3]. More precisely, in [12], we have constructed explicitly the structure of an arbitrary finite rank classical Fuchsian Schottky group in the hyperbolic plane with the extended real numbers as the boundary on the analogy of the real Schottky groups with two supplementary ensuing conditions: (see Figure 1(a) in the following)

(i) The semi-circles lying in the upper-half plane are situated by the reflection of the upper imaginary axis.

(ii) The positions of the semi-circles on the boundary of the hyperbolic plane are non-tangential.

In this paper, our main goal is to set up the construction of the rank 2 semi-Fuchsian Schottky group with non-classical generating sets within the hyperbolic plane without the boundary i.e., the circle at infinity. But what does the significance imply for a group to be semi-Fuchsian Schottky? The existence of such a group is an open problem. So, hereafter, we have to clarify the term "semi-Fuchsian" in the Fuchsian group theory. Note that the term "semi-Fuchsian" is not standard (or not even usual) in the literature of the Fuchsian group theory. For this purpose, firstly, we aim to define the rank 2 semi-Fuchsian Schottky group in the classical flavor. After that, we will look into the non-classical phenomena. In fact, for the classical one, the structure of the rank 2 semi-Fuchsian Schottky group inherits from the framework of the rank 2 classical Fuchsian Schottky group (see, Figure 1(a)) by rearranging the position of

the semi-circles in the hyperbolic plane by adding a parabolic Möbius transformation in the scenario. However, for the non-classical type, the construction of the rank 2 semi-Fuchsian Schottky group equipped with a non-classical generating set (see Figure 4 in the following) is a bit different from the framework of the rank 2 Fuchsian Schottky group with the non-classical generating sets (see Section 2, Figure 2 in [13]) in the upper-half plane model.

The succeeding Figure 1(a) supplies the structure of the rank 2 classical Fuchsian Schottky group in the upper-half plane model.
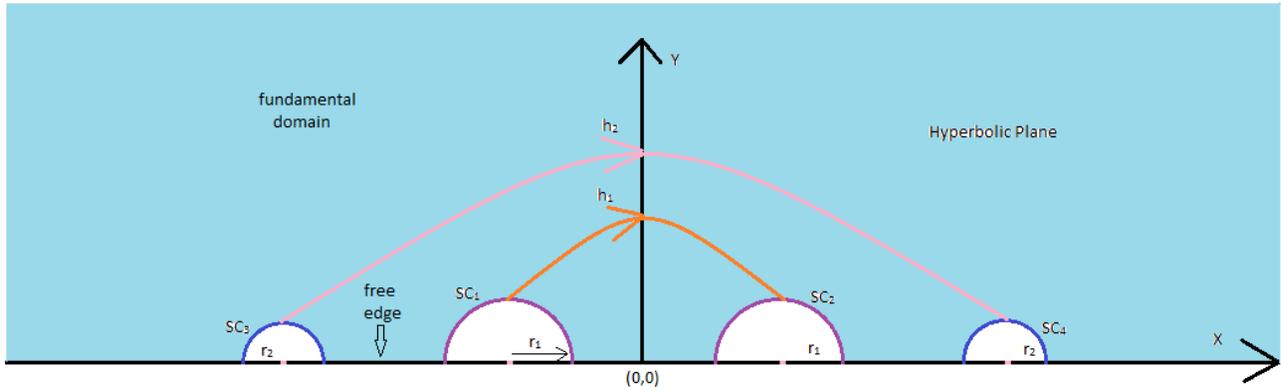
The foregoing framework represents the construction of the rank 2 Fuchsian Schottky group (a type of semi-Fuchsian Schottky groups) in the upper-half plane model.

The above structure of a rank 2 semi-Fuchsian Schottky group provides a (0,0,0,0,3) - group (see, Figure 2(a)).

$Figure: 1(a)$

Indeed, in the literature on Fuchsian group theory, the concept of a semi-Fuchsian Schottky group (classical or non-classical) is somewhat new. Therefore, at first, we aim to describe the group explicitly. In the following, from the group's theoretical point of view, we have initiated a structure in the hyperbolic plane. After that, we discuss the nature of that group from the point of view of the hyperbolic surface in further detail. We define a group as a (classical) semi-Fuchsian Schottky group if it is a hyperbolic Riemann surface containing at least one end in its compact core that is an infinite funnel. In this regard, we introduce the following definition in a precise manner.

We say a group is a classical semi-Fuchsian Schottky group of rank 2 if it is a discrete subgroup of $PSL(2, \mathbb{R})$ equipped with one of the following conditions (a), (b), (c), or (d) that represent the structure of the group in the upper-half plane:

(a) Two semi-circles out of four occupy the position in the hyperbolic plane by the reflection of the upper imaginary axis. The other two semi-circles that are not situated by their reflection of the imaginary axis are tangential to each other in the circle at infinity, but placed with various

circumstances (e.g., intersect, not intersect, tangential) with the former two semi-circles in the hyperbolic plane with $\mathbb{R} \cup \{\infty\}$; (e.g., Figure 1(b), 1(c), 1(i), 1(o), and 1(p)).

(b) All semi-circles are non-tangential at the boundary of the hyperbolic plane situated by the reflection of the upper vertical axis (e.g., Figure 1(a)).

(c) Two semi-circles (paired with a hyperbolic transformation) out of four are situated in the hyperbolic plane by the reflection of the vertical imaginary axis. The other two semi-circles (linked with the other hyperbolic transformation) may or may not be located by their reflection of the imaginary axis, but placed with various circumstances (e.g., intersect, tangential) with the previous first two semi-circles in the hyperbolic plane with $\mathbb{R} \cup \{\infty\}$; (e.g., Figure 1(d), 1(e), 1(h), 1(m), and 1(n)).

(d) If the preceding three cases (a), (b), and (c) do not happen, then only one of the semi-circles in a particular pair (linked with a hyperbolic element) is located within the other pair of semi-circles (twins with the other hyperbolic element) in the upper-half plane; (e.g., Figure 1(g), 1(j), 1(k), and 1(l)).

Remarks. It is possible that someone doesn't use the 'Reflection' procedure in the above framework. For that kind of situation, the existence of free edges (at least one) in the hyperbolic plane with a boundary must be clarified.

Some observations and examples: Note that we included the Fuchsian Schottky groups into the semi-Fuchsian Schottky groups by the preceding condition (b). In this manuscript, for a rank 2 semi-Fuchsian Schottky group, we allow the possibility to generate the group with one hyperbolic and one parabolic element. But we have exempted the case when the two generators are both parabolic (for example, see, Figure 1(f)). Parabolic elements can not be raised as a funnel on the hyperbolic surface. This sort of phenomenon is known in the literature (for instance, the congruence modulo 2 subgroup of the modular group $PSL(2, \mathbb{Z})$). In particular, for a rank 2 congruence modulo 2 subgroup, the corresponding associated hyperbolic surface is a pair of pants (look like) with three cusp ends (see [5]). Apart from that, when situation (d) occurs, then the corresponding hyperbolic surface is a torus with one infinite funnel end (see Figure 1(g)). However, when this type of case occurs for both generators are parabolic, then it becomes a torus with one cusp end (for example, the commutator subgroup of $PSL(2, \mathbb{Z})$; see [5]). Finally, for a rank 2 semi-Fuchsian Schottky group with a non-classical generating set, we have used both elements hyperbolic as generators (see Section 3 below).

The following diagrams represent the classical structures (with fundamental domains) of any rank 2 classical semi-Fuchsian Schottky groups in the upper-half plane model.
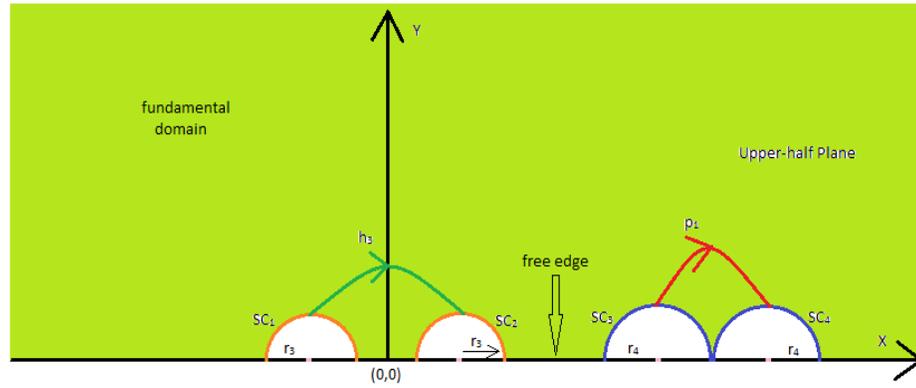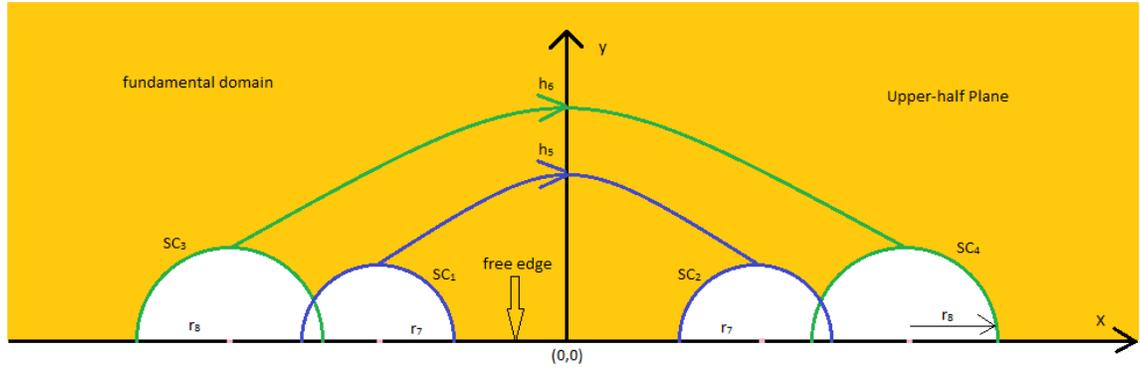

The preceding construction of a rank 2 semi-Fuchsian Schottky group indicates a (0,0,0,1,2) - group (see, Figure 2(b)).
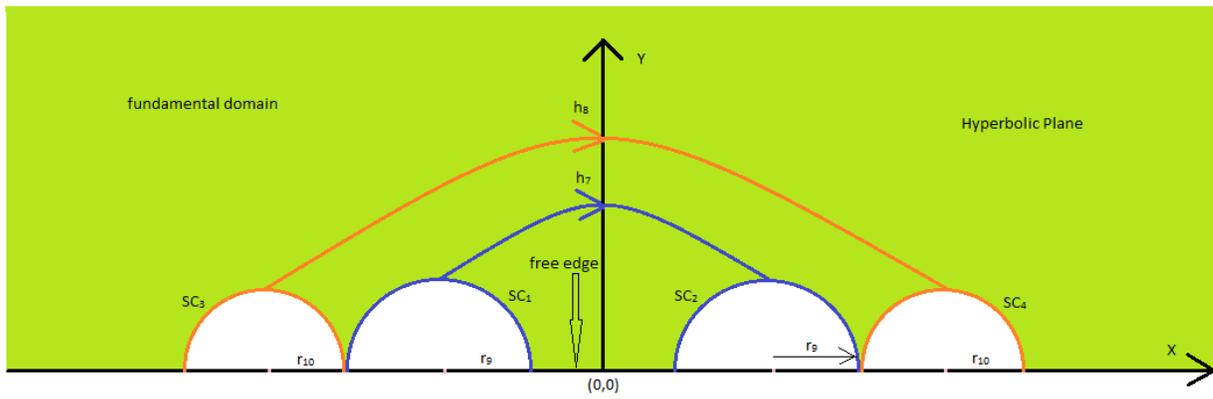
$Figure : 1(b)$
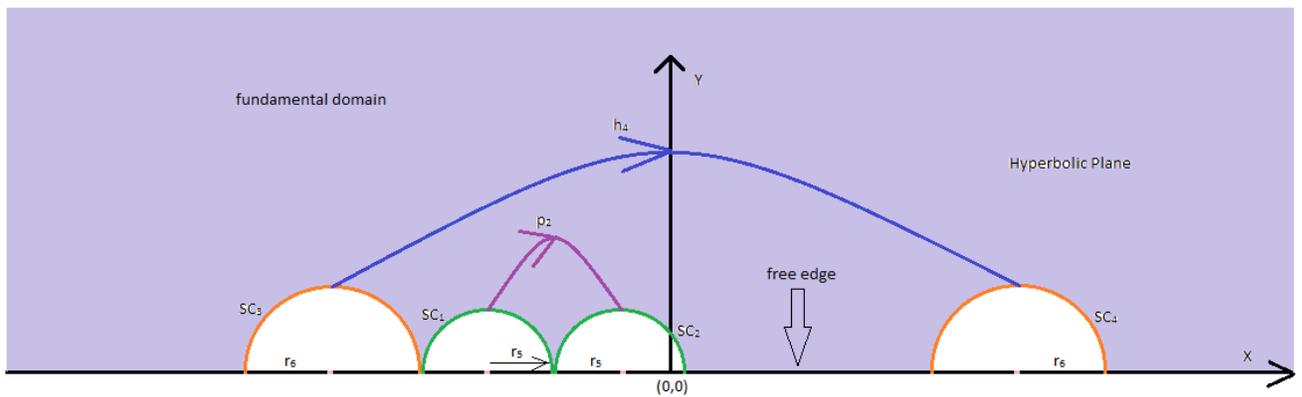
The overhead formation of a rank 2 semi-Fuchsian Schottky group supplies a (0,0,0,1,2) - group (see, Figure 2(b)).
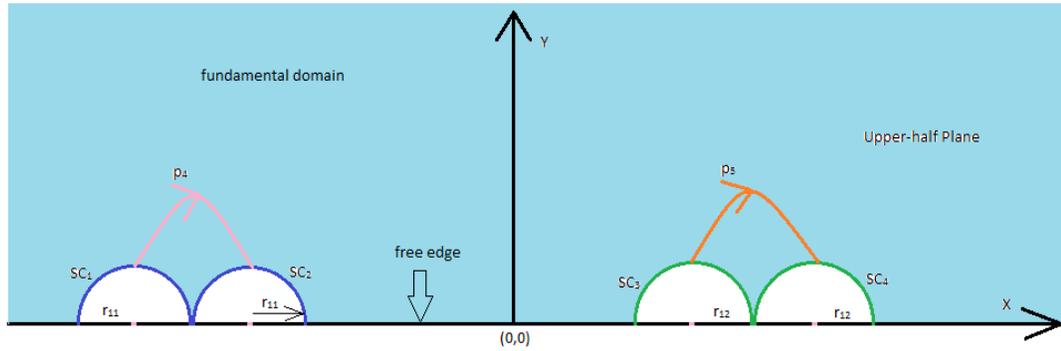
$Figure : 1(c)$

The foregoing framework of a rank 2 semi-Fuchsian Schottky group provides a (0,0,1,0,2) - group (see, Figure 2(f)) .
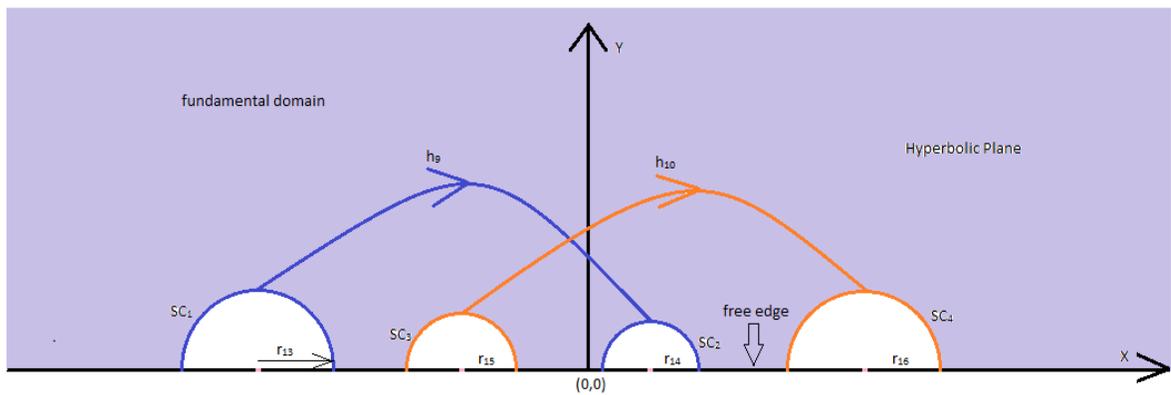
$Figure : 1(d)$

The aforementioned representation of a rank 2 semi-Fuchsian Schottky group delivers a (0,0,1,0,2) - group (see, Figure 2(f)).
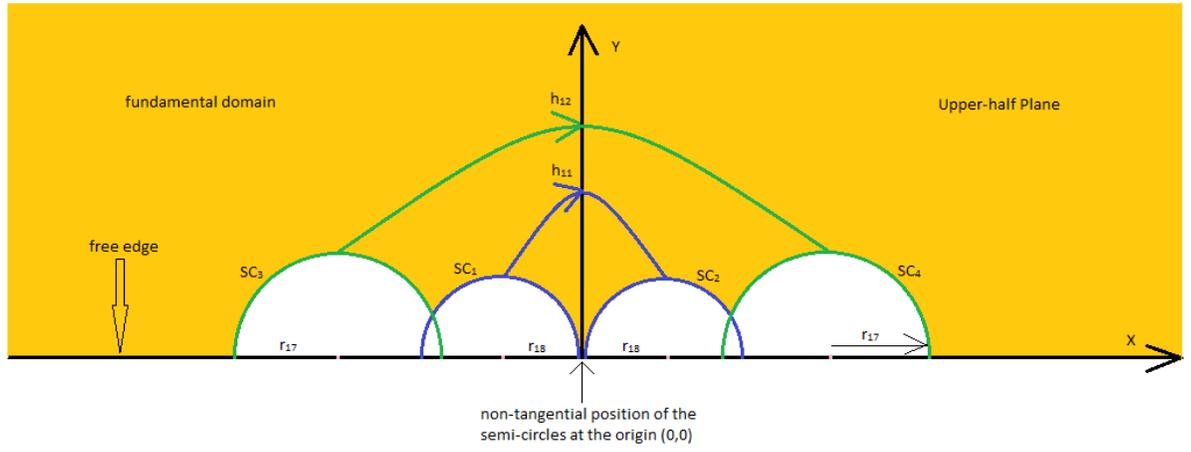
$Figure : 1(e)$

The overhead formation is not a structure of a rank 2 semi-Fuchsian Schottky group. In particular, the preceding structure will provide a (0,0,1,2,0) - group. More precisely, for any rank 2 semi-Fuchsian Schottky group, it is impossible to achieve a (0,0,0,2,1) - group. The only way it possible when we consider the semi-Fuchsian Schottky group of rank $\geq$ 3.

$$Figure : 1(f)$$

The above construction of a rank 2 semi-Fuchsian Schottky group supplies a (1,0,0,0,1) - group (see, Figure 2(e)).

$$Figure : 1(g)$$

The preceding portrait of a rank 2 semi-Fuchsian Schottky group furnishes a (0,0,2,0,1) - group (see, Figure 2(d)).

$Figure : 1(h)$

The overhead structure of a rank 2 semi-Fuchsian Schottky group indicates a (0,0,1,1,1) - group (see, Figure 2(d)).
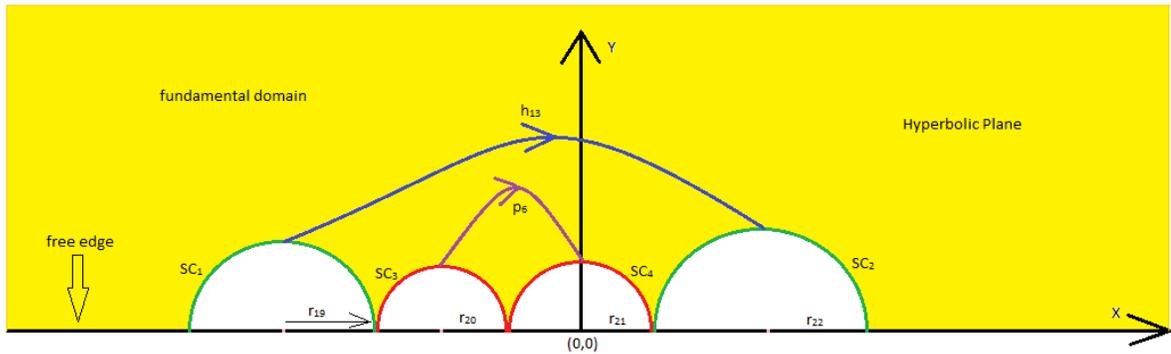
$Figure : 1(i)$

The above formation of a rank 2 semi-Fuchsian Schottky group supplies a (1,0,0,0,1) - group (see, Figure 2(e)).

$Figure : 1(j)$

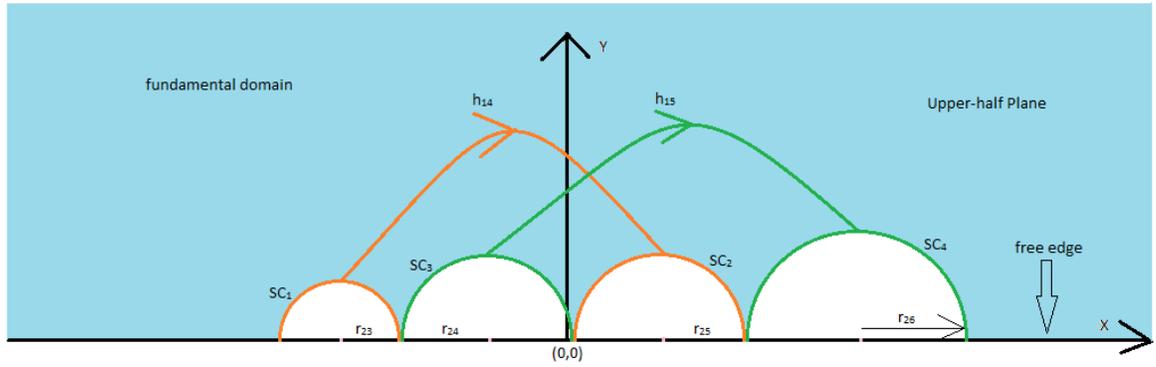
The preceding framework of a rank 2 semi-Fuchsian Schottky group provides a (1,0,0,0,1) - group (see, Figure 2(e)).

$Figure : 1(k)$

The overhead representation of a rank 2 semi-Fuchsian Schottky group indicates a (1,0,0,0,1) - group (see, Figure 2(e)).

$$Figure : 1(l)$$

The aforementioned image of a rank 2 semi-Fuchsian Schottky group may offer a (0,0,0,0,3) group or a (0,0,1,0,2) group (see, Figure 2(a), and Figure 2(f)).

$$Figure : 1(m)$$

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The preceding framework of a rank 2 semi-Fuchsian Schottky group supplies a (0,0,1,0,2) - group (see, Figure 2(f)).

$Figure: 1(n)$
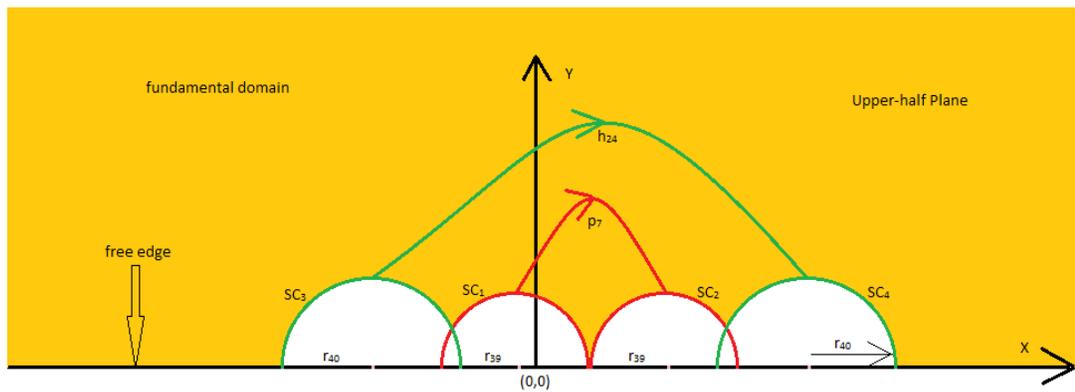
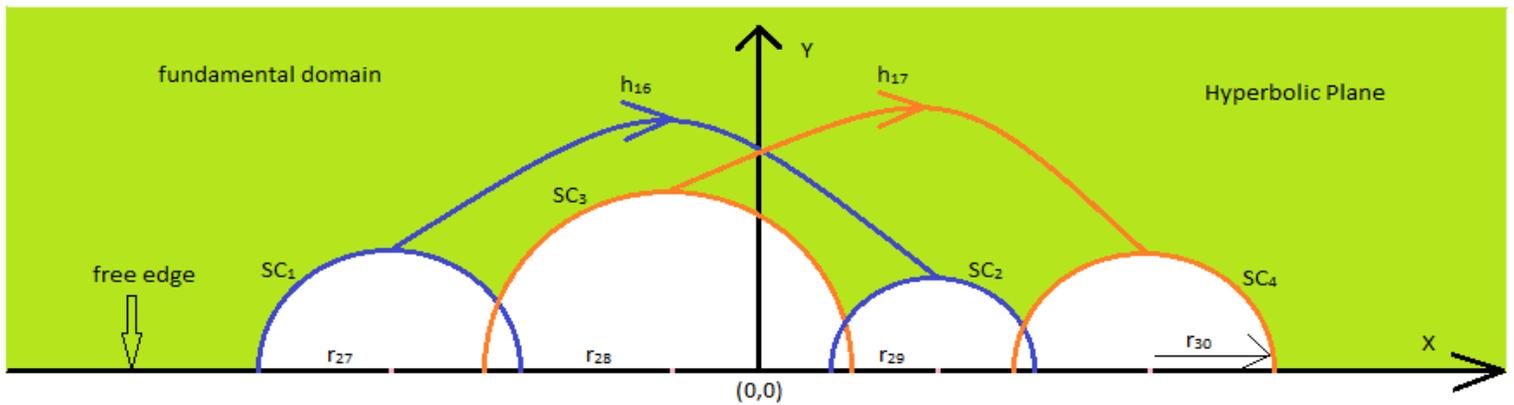The above structure of a rank 2 semi-Fuchsian Schottky group provides a (0,0,1,1,1) - group (see, Figure 2(d)).
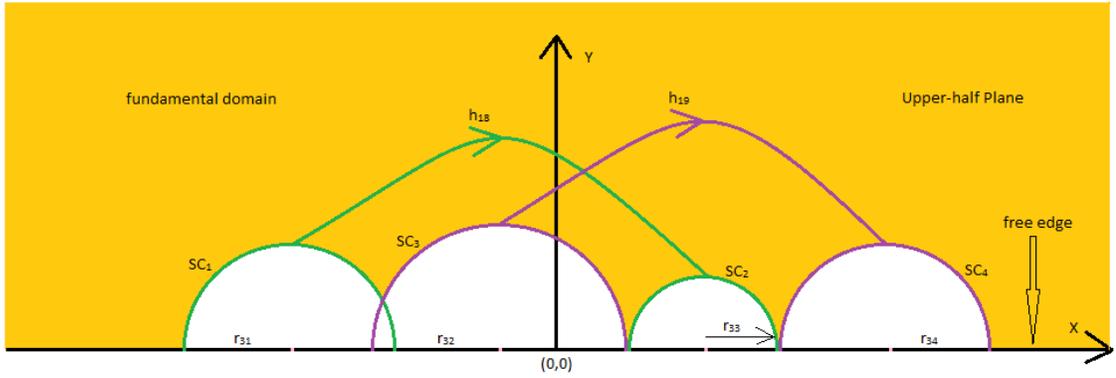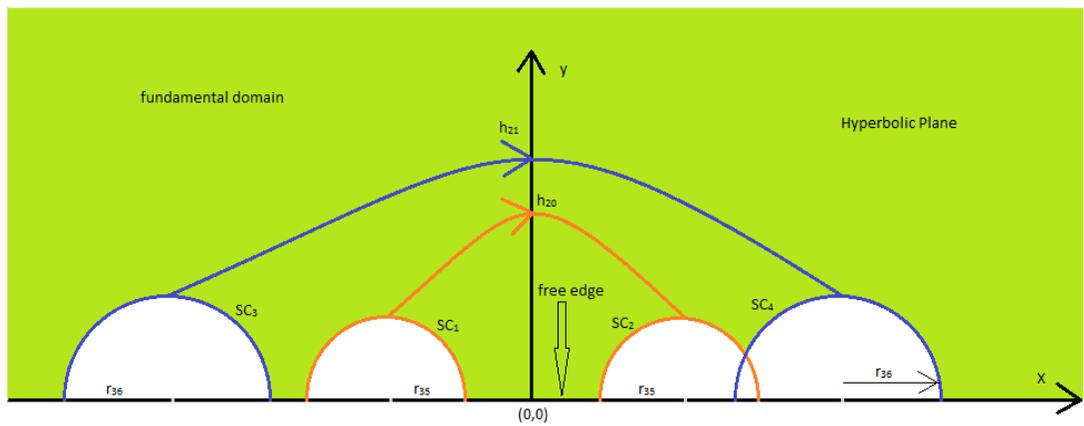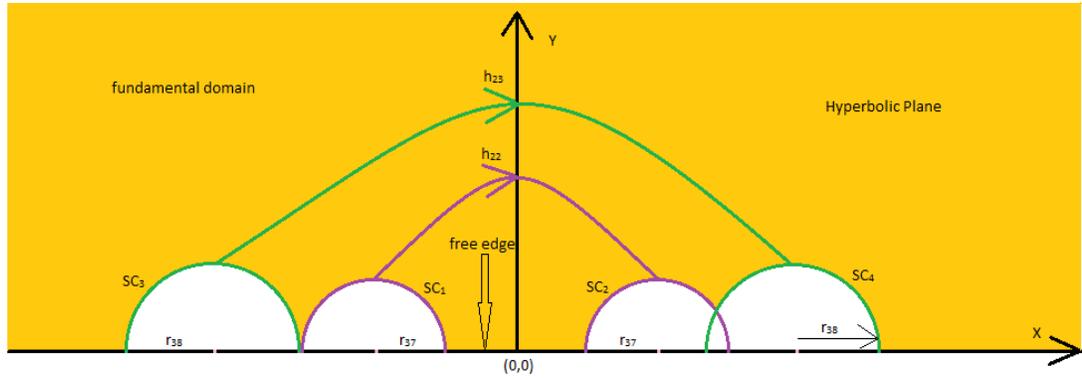
$Figure: 1(o)$

The overhead structure of a rank 2 semi-Fuchsian Schottky group may indicate a (0,0,1,1,1) - group or a (0,0,0,1,2) - group (see, Figure 2(d), and Figure 2(b)).

The aforementioned representation of a rank 2 semi-Fuchsian Schottky group supplies a (0,0,1,1,1) - group (see, Figure 2(d)).
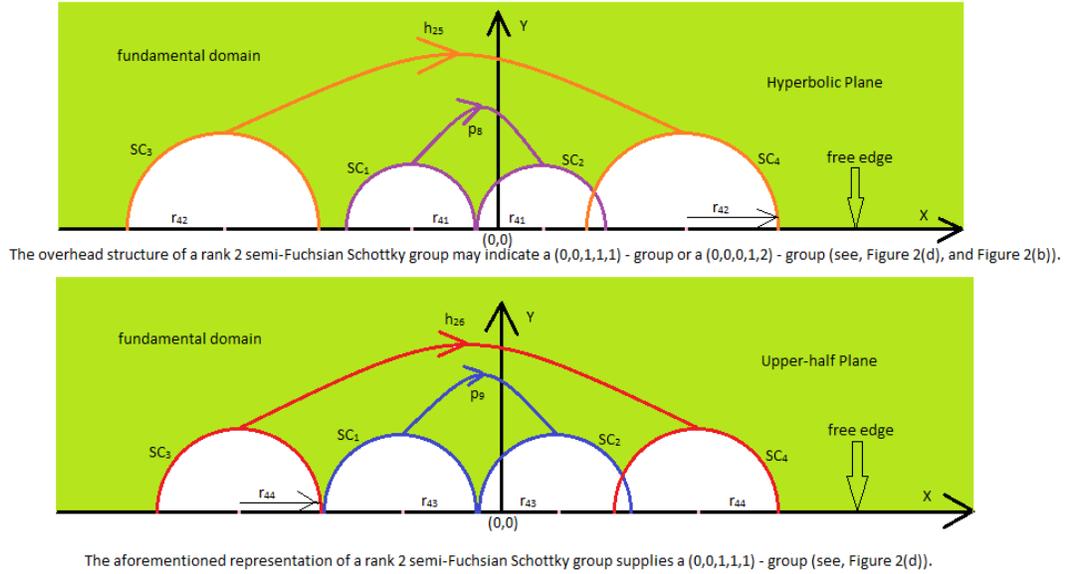
$Figure : 1(p)$

As we mentioned earlier, the preceding figures $1(a), 1(b), 1(c), 1(d), 1(e), 1(g), 1(h), 1(i), 1(j)$, $1(k)$, $1(l)$, $1(m)$, $1(n)$, $1(o)$, and $1(p)$ portray the various types of designs for the rank 2 classical semi-Fuchsian Schottky groups in the hyperbolic plane. Note that, for every diagram proposed in the above, $\exists$ at least one free edge is contained in the fundamental domain of the corresponding semi-Fuchsian Schottky (surface) group of rank 2. It signifies that the limit set of any rank 2 (hence any finite rank) semi-Fuchsian Schottky group is an uncountable set (not necessarily a Cantor set) with Hausdorff dimension situated in the open interval $(0, 1)$.

Now, the question is, why have we been interested in creating this kind of group? A Fuchsian (surface) group is of type (g, n, m) if the associated quotient surface is a compact hyperbolic Riemann surface of genus g with n points and m conformal holes removed. If the genus $g = 0$ and $m + n = 3$, we say that the associated surface (or, likewise, the Fuchsian group) is a pair of pants. Now, as we mentioned in Section 1, the area of a pair of pants (rising from a Fuchsian group) is π. Moreover, usually, a pair of pants emerging from two generators Fuchsian group is of finite area. However, in this paper, the pair of pants that arises from a rank 2 semi-Fuchsian Schottky group is of infinite area. Indeed, the associated hyperbolic surfaces (more precisely, a pair of pants or a torus) corresponding to the aforementioned constructed group (which we call rank 2 semi-Fuchsian Schottky group) contain at least one component that is an infinite (hyperbolic) area funnel attached to the closed geodesic boundary of its compact

core. The other components may be cusps. In particular, the free edges that appeared in the fundamental domain emerge as an infinite funnel in the hyperbolic surface. In essence, we define the semi-Fuchsian Schottky groups as follows.

A semi-Fuchsian Schottky (surface) group is of type (g, n, m, c, f) if the associated quotient surface is a semi-conformally compact hyperbolic Riemann surface of genus g with c numbers of cusp ends and f numbers of funnel ends, where n points and m conformal disks are removed from its compact core. We assert that g, n, m, and $c \geq 0$, but $f > 0$ always. Moreover, if $n = n_1, n_2, ..., n_k$, and $m = m_1, m_2, ..., m_l$, $n_i \neq n_j$, for $i \neq j$, and $m_i \cap m_j = \phi$, $i \neq j$, with $n_i \notin m_j$. Now, if the genus $g = 0$ and $m + c + f = 3$, we say that the quotient surface (or, likewise, the semi-Fuchsian Schottky group) is a pair of pants. In this article, we are mainly interested in the type $(g, 0, m, c, f)$, in particular, $m + c + f = 3$. For the basics of a hyperbolic manifold, pair of pants, funnel, and cusp ends, the readers may go through the resources, for instance, [12], [1], [3], [9], and [6]. In the following, we have discussed more about these groups precisely. We define a Riemann surface to be a semi-conformally compact hyperbolic Riemann surface if it is of infinite hyperbolic area, provides a finitely generated (non-abelian) fundamental group, with cusp and funnel (at least one) ends, with Gaussian curvature -1. Note that here, we treat the semi-conformally compact hyperbolic Riemann surface as a generalization of the conformally compact hyperbolic Riemann surface. For example, for the type $(0, 0, 0, 0, 3)$, the semi-Fuchsian Schottky group (which reduces to the Fuchsian Schottky group that type of group discussed in [12]) produces a conformally compact hyperbolic Riemann surface. Apart from that, for the type $(1, 0, 0, 0, 1)$, the semi-Fuchsian Schottky group provides a torus with one infinite funnel end, which is also a conformally compact hyperbolic Riemann surface. In [12], we have noticed that any finite rank Fuchsian Schottky groups give rise to a finite-type surface with an infinite hyperbolic area. Here, we refer to the term "finite type" to indicate the surfaces that provide the finitely generated fundamental groups. The interesting part is that, for any finite rank semi-Fuchsian Schottky groups, infinite area hyperbolic surfaces (with finitely generated fundamental groups) also originate here. But the difference between these two groups (from the point of view of the hyperbolic surface) is that the first one (i.e., Fuchsian Schottky) produces only funnel ends (see, Figure 2(a) and Figure 2(e)) in the corresponding associated surface whereas the other (i.e., semi-Fuchsian Schottky) may contain cusp ends also (see, Figure 2(a), 2(b), 2(c), 2(d), 2(e), and 2(f)) to the quotient surface. Indeed, all three types of groups (viz., real Schottky groups, Fuchsian Schottky groups, and semi-Fuchsian Schottky groups) are discrete subgroups of $PSL(2, \mathbb{R})$. In contrast, real Schottky groups provide closed

(maybe compact) hyperbolic Riemann surfaces, whereas Fuchsian Schottky groups indicate conformally compact hyperbolic Riemann surfaces. On the other hand, semi-Fuchsian Schottky groups offer semi-conformally compact hyperbolic Riemann surfaces. However, as previously discussed, in this manuscript, we treat a Fuchsian Schottky group as a special case of a semi-Fuchsian Schottky group. Therefore, to some extent, we can think that the conformally compact hyperbolic Riemann surfaces are included in the semi-conformally compact hyperbolic Riemann surfaces.

In [12], we have sketched a detailed construction (with a proof) of any finite rank classical Fuchsian Schottky group in the upper-half plane model (see Proposition 3.1 in [12]). Now, one can easily establish the formation (entitled just above, i.e., (a), (b), (c), and (d)) for any classical semi-Fuchsian Schottky group of rank 2 (and hence any finite rank) by utilizing Proposition 3.1 in [12]. So we omit this proof here and leave it to the reader. Henceforth, we proceed to the complicated case, non-classical type, in the next section, which is our main goal in this manuscript.

The subsequent figures describe the various types of funnel surfaces (viz., pair of pants with infinite hyperbolic area, and torus with one infinite end) corresponding to classical semi-Fuchsian Schottky groups.

$Figure : 2(a)$

A. A. SHAIKH AND U. ROY

infinite funnel

infinite area
pair of pants

closed geodesic
boundary of the
compact core

compact core
(blue color)

not a geodesic (closed)
boundary of the compact core

cusp

infinite
funnel

(0,0,0,1,2) - group

Figure : 2(*b*)

infinite funnel

geodesic boundary (closed)
of the compact core

infinite area
pair of pants

convex core
(blue + red color)

finite cusp

finite cusp

not a geodesic (closed)
boundary of
the compact core

(0,0,0,2,1) - group

Figure : 2(*c*)

funnels

geodesic boundary
attached with infinite
funnel

pair of pants

compact core
(blue color)

pair of pants
(look like)

convex core
(blue + yellow color)

cusp ends

(0,0,0,2,1) - group

(0,0,1,1,1) - group

geodesic boundary
components

(0,0,2,0,1) - group

$Figure : 2(d)$

infinite funnel

geodesic boundary
(of the convex/compact core)

torus with one
infinite end
(not a pair of pants)

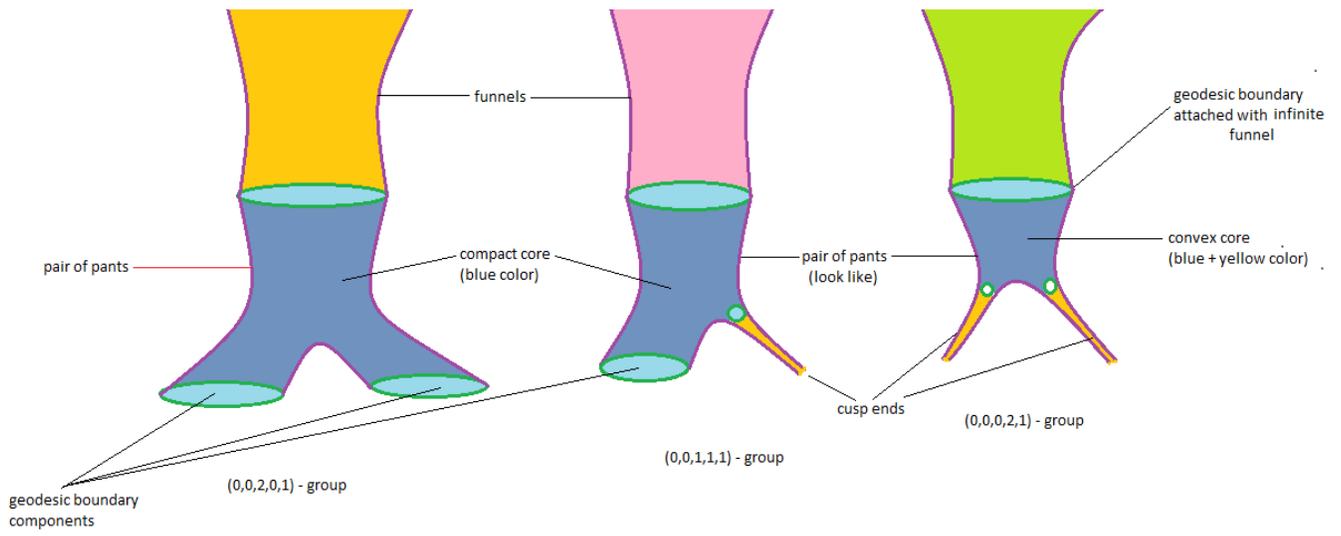convex core / compact
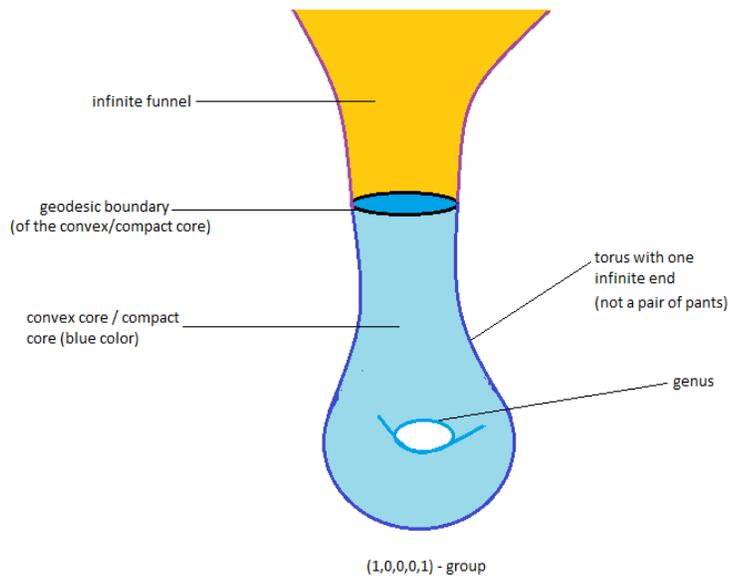core (blue color)
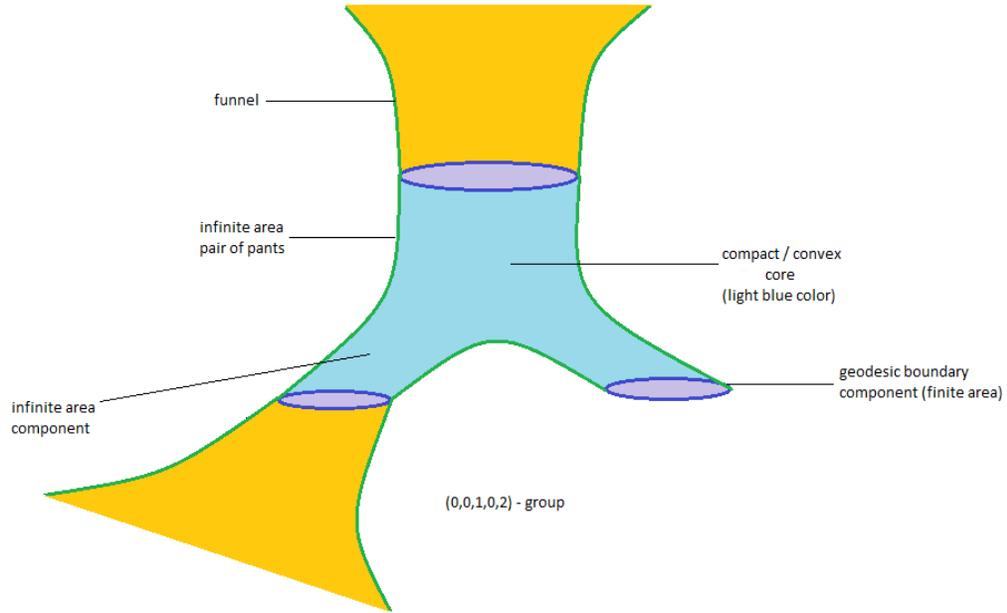
genus

(1,0,0,0,1) - group

$Figure : 2(e)$

Figure : 2(f)

3. SET UP FOR NON-CLASSICAL GENERATING SETS IN SEMI-FUCHSIAN SCHOTTKY GROUPS

For non-classical types, it is certainly not easy (or rather highly non-trivial) to formulate the construction of any Kleinian or Fuchsian group in hyperbolic space 3 or 2. However, in this paper, we have formulated the non-classical (Schottky) construction of the semi-Fuchsian Schottky group of rank 2 in a precise manner. It is well known that the defining curves operated in the non-classical Kleinian Schottky groups are Jordan curves except the Euclidean circles. But interestingly, till now in the literature, two examples of finitely generated non-classical Kleinian Schottky groups that Yamamoto [15] and Williams [14] arranged, the defining curves are two Euclidean circles and two rectangles lying in the Riemann sphere. We followed a similar situation (two semi-circular look-like arcs and two rectangles) with a slightly different structure within the upper-half plane (see, Figure 4 in the following). Indeed, it looks like the structure of the non-classical Schottky group mentioned in the paper [15] (also in [14]) is almost identical to the construction that we have delivered in this article within the hyperbolic plane, but, remarkably, it's not exactly that type of phenomenon. We describe this in a precise manner shortly. Furthermore, we believe that, although the construction that we have arranged in [13] and the structure that we have provided in this paper are completely different, some places in

this article have overlapped with our previous work [13], but for the sake of completeness, we have assembled those in this manuscript.

As previously stated, for the framework of a rank 2 semi-Fuchsian Schottky group with a non-classical structure, our approach to the configuration is slightly different from the work of Yamamoto [15]. Indeed, unlike the works of Yamamoto [15], we have presented the construction within the hyperbolic plane through upper verticle scaling by the value of sufficiently small positive real number $\frac{\tau}{2}$, where $\tau \to 0^+$. In particular, in this article, we aim to construct the design of the rank 2 semi-Fuchsian Schottky group with non-classical generating sets by using the following methodology. Two semi-circular arcs (defining curves) out of four occupy the position in the upper-half plane by the reflection of the upper vertical axis. The other two defining curves (not placed with the reflection of the upper imaginary axis) are situated by some (kind of) dilation in the hyperbolic plane. Note that, in the paper of Yamamoto [15], the positions of the small and big rectangles are endowed with some (type of) dilation (linked with a Möbius transformation) in the extended complex plane. But, within the upper-half plane, it is impossible to operate this type of thing for the rectangles (small and big) such that the transformation can be used as a generator of the group. This is the main challenging part of our construction. To overcome this kind of difficult situation, we have initiated a technique for rectangles suggested shortly in the following (see, (3) below for details). More precisely, in this manuscript, we have proposed four new innovative ideas to fabricate the non-classical Schottky structure of the rank 2 semi-Fuchsian Schottky group within the hyperbolic plane in a subsequent way:

(1) For the rank 2 Schottky group, Yamamoto [15] identified an appropriate positive real number (viz., $\sqrt{2}$) in the used hyperbolic element (one of the generators of the group out of two) so that the Schottky group became non-classical in the Riemann sphere. But in this study, for the rank 2 semi-Fuchsian Schottky group we have perceived that there do exist some positive real numbers but they don't fit for any hyperbolic element so the Schottky group becomes non-classical in the hyperbolic plane. The presence of complex numbers is identified. Note that the generators of our Schottky group need to be hyperbolic (not loxodromic). For the deficiency of the existence of such a positive real number, in this paper, we have taken the '*distance*' (in the sense of Euclidian geometry) between the origin and the particular number (i.e., the modulus values of the complex numbers to describe the position of that number in the upper half plane, without losing the generality of the construction) in the used hyperbolic element (count as the

generator of the group) to become this type of Schottky group non-classical in the hyperbolic plane.

(2) Yamamoto [15] utilized a sufficiently small positive number in one generator out of two. But, in this investigation, we have noted that using a sufficiently small positive number only with one generator is not adequate to initiate the non-classical structure in such a type of Schottky group in the upper-half plane model. To overcome this situation, we have exhibited sufficiently small positive numbers in both generators to make over the construction of that group.

(3) The sufficiently small positive number in Yamamoto's paper [15] tends to 0^+, on the other hand, here, we have taken the sufficiently small positive number tending to 0^+ in one generator (for converting semi-circular arcs equipped with their diameters lying on the horizontal axis to Jordan curves) and the other one we have used the sufficiently small positive number tends to 0^+ or 1^- (for converting rectangles to suitable Jordan curves). In this work, for operating the dilation of the rectangles, we have applied $\tau \to 0^+$ in the transformation for the lower part of the rectangles and utilized $\tau \to 1^-$ for the upper part of the same. But when we carried out the construction, we managed both the values of τ (see, in this section 3 below for more details).
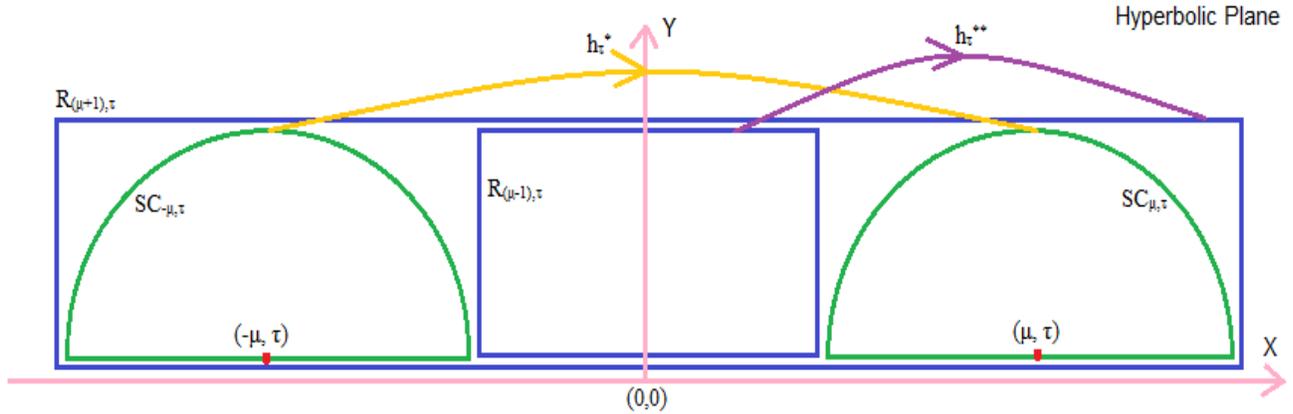
Figure : 4

The overhead Figure 4 provides the framework of the rank 2 semi-Fuchsian Schottky group equipped with a non-classical structure in the upper-half plane model.

(4) Yamamoto's construction [15] was situated by two rectangles and two circles on the complex plane where rectangles were built up with some kind of dilation (with rotation) and circles were placed by the reflection of the imaginary axis where the centers of the circles situated on the real axis and the midpoints of the rectangles are located on the origin (i.e., $(0,0)$) of the

axes. However, in our formation, the semi-circles (equipped with their diameters at the vertical height of τ from the circle at infinity) are created by the reflection of the upper imaginary axis, and, for the rectangles, all sides of both of them lie within the upper-half plane with midpoints of them positioned on the positive imaginary axis (see, Figure 4). Furthermore, the Jordan curves for the construction of the non-classical Schottky group [15] situated in the Riemann sphere (i.e., the boundary of the $3-$dimensional hyperbolic space) and the Jordan loops (i.e., the four hollow half-moons) for the framework of the rank 2 Fuchsian Schottky groups with non-classical generating sets [13] are lying on the circle at infinity (i.e., the boundary of the $2-$dimensional hyperbolic plane), but, for the structure of the rank 2 semi-Fuchsian Schottky group with a non-classical generating set, the Jordan curves (i.e., $SC_{\mu,\tau}$, $SC_{-\mu,\tau}$, $R_{(\mu-1),\tau}$, and $R_{(\mu+1),\tau}$) are situated within the upper-half plane without the boundary (i.e., extended real numbers).

In the following, for $\tau \to 0^+$, we define two semi-circles and two rectangles within the upper-half plane without the boundary (i.e., circle at infinity).

First, we have provided the details of the semi-circles $SC_{-\mu,\tau}$, and $SC_{\mu,\tau}$ subsequently.

$$SC_{\mu,\tau} = \left\{ \sqrt{(x-\mu)^2 + (y-\tau)^2} = 1 - \tau \right\},$$

$$\text{and } SC_{-\mu,\tau} = \left\{ \sqrt{(x+\mu)^2 + (y-\tau)^2} = 1 - \tau \right\}.$$

Also

$$h_\tau^*(SC_{-\mu,\tau}) = SC_{\mu,\tau},$$

where h_τ^* is defined by

$$h_\tau^* \left(\tau \to 0^+ \right) : z \longrightarrow \frac{\mu(1-\tau)^{-1}z + (1-\tau)\{\mu^2(1-\tau)^{-2} - 1\}}{(1-\tau)^{-1}z + \mu(1-\tau)^{-1}} \tag{1}$$

where, $z = (x+iy) \in \mathbb{H}^2$ and $\mu = \mod\left(\frac{1\pm i\sqrt{11}}{2}\right)$, the modulus value of the complex numbers $\frac{1}{2}(1 \pm i\sqrt{11})$. $\mathbb{H}^2$ denotes the hyperbolic plane.

Now, we have described the rectangles $R_{(\mu-1),\tau}$ and $R_{(\mu+1),\tau}$ successively.

The rectangle $R_{(\mu-1),\tau}$ with vertices:

$$(\mu - 1) + i\tau,$$

$$(\mu - 1) + i(1 - \tau),$$

$$(-\mu + 1) + i(1 - \tau),$$

$$\text{and } (-\mu + 1) + i\tau.$$

After that, we set the other rectangle $R_{(\mu+1),\tau}$ in the ensuing way.

(i) Right lower corner point (i.e., $\{(\mu-1)+i\tau\}$) of the rectangle $R_{(\mu-1),\tau}$ goes to the right lower corner point of the rectangle $R_{(\mu+1),\tau}$ under the transformation $\frac{(\mu^2-1)-2i\tau+\tau^2}{(\mu-1)^2+\tau^2}$.

(ii) Right upper corner point (i.e., $\{(\mu-1)+i(1-\tau)\}$) of the rectangle $R_{(\mu-1),\tau}$ proceeds to the right upper corner point of the rectangle $R_{(\mu+1),\tau}$ under the map $\frac{(\mu^2-1)-2i(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}$.

(iii) Left lower corner point (i.e., $\{(-\mu+1)+i\tau\}$) of the rectangle $R_{(\mu-1),\tau}$ reaches to the left lower corner point of the rectangle $R_{(\mu+1),\tau}$ under the function $\frac{(\mu^2-1)+2i\tau+\tau^2}{(\mu-1)^2+\tau^2}$.

(iv) Left upper corner point (i.e., $\{(-\mu+1)+i(1-\tau)\}$) of the rectangle $R_{(\mu-1),\tau}$ carries to the left upper corner point of the rectangle $R_{(\mu+1),\tau}$ under the transformation $\frac{(\mu^2-1)+2i(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}$.

So, for τ tends to 0^+, we have four functions viz., $\frac{(\mu^2-1)\pm2i\tau+\tau^2}{(\mu-1)^2+\tau^2}$ and $\frac{(\mu^2-1)\pm2i(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}$ for the criteria of rectangles. But our aim is to use only one transformation out of the four and manage it for the rest of the three.

Note that

$$\left(\frac{(\mu^2-1)\pm2i(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}\right)_{(\tau\to1^-)} = \left(\frac{(\mu^2-1)\pm2i\tau+\tau^2}{(\mu-1)^2+\tau^2}\right)_{(\tau\to0^+)}.$$

Henceforth, without losing generality, in this paper, we can utilize the function $\frac{(\mu^2-1)\pm2i(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}$, after that, we operate and control it for both $\tau\to0^+$ and $\tau\to1^-$ for the remaining part of this work.

We prepare

$$h_\tau^{**}(R_{(\mu-1),\tau}) = R_{(\mu+1),\tau},$$

where h_τ^{**} is defined by

$$h_\tau^{**}\ (\tau\to0^+\ \text{or}\ 1^-):z \longrightarrow \left(\frac{\{\mu^2-1\}+\ \text{mod}\ (\pm2i(1-\tau))+(1-\tau)^2}{\{\mu-1\}^2+(1-\tau)^2}\right)z. \quad (2)$$

The numerator of the preceding transformation contains the term "$\{\pm2i(1-\tau)\}$", but since h_τ^{**} needs to be hyperbolic, for simplicity, we take the modulus value of this term without losing the generality.

4. PROOF OF ALL THE NECESSARY LEMMAS

To prove Theorem 1, we have taken the assistance of five lemmas which we deduced throughout this section.

First of all, for the convenience of the readers, our goal is to display the existence of a fundamental domain for the rank 2 classical semi-Fuchsian Schottky group in the hyperbolic plane. The ensuing lemma will be helpful in this direction.

Lemma 4.1. *Suppose, Δ_{S_2} is a rank 2 (classical) semi-Fuchsian Schottky group generated by two Möbius transformations. Assume that, h_s is an element of group Δ_{S_2}. Then there exists a fundamental (maybe Dirichlet) domain $F_{\Delta_{S_2}}$ for Δ_{S_2} surrounded by four loops (all are semi-circular shapes lying on the circle at infinity), where one of the loops segregates the fixed points of the element h_s.*
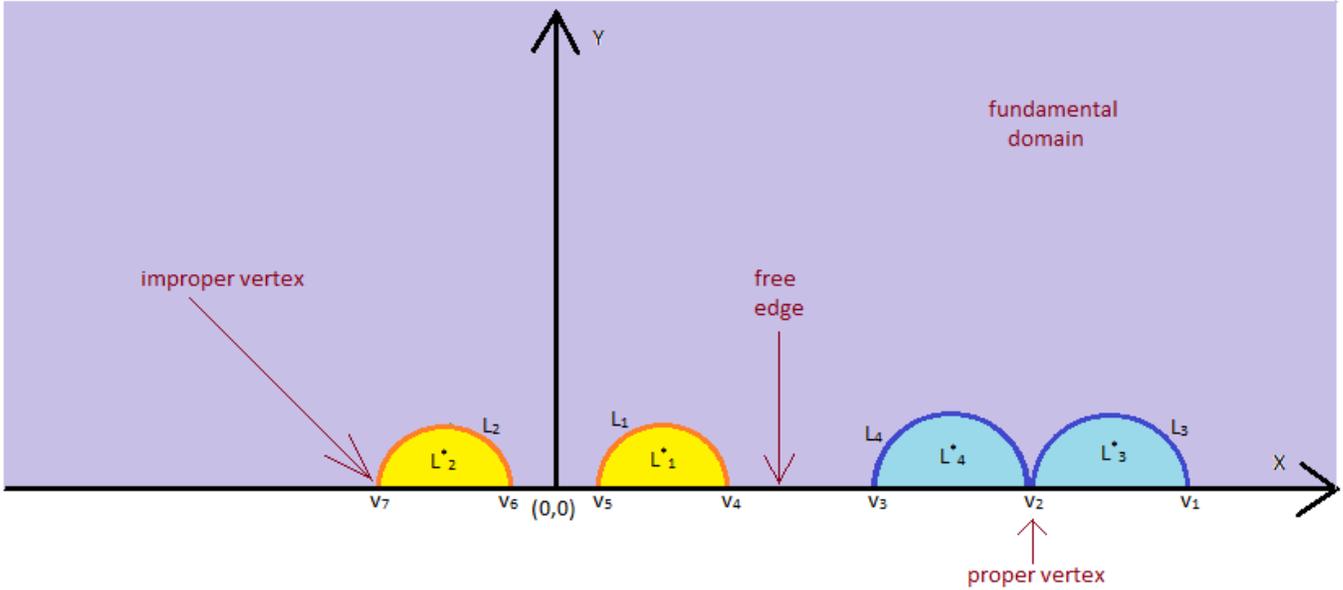
The preceding representation is a sample of designs of a fundamental domain for a rank 2 semi-Fuchsian Schottky group in the upper-half plane model. There may be other diagrams to (as mentioned in Section 1).

Figure : 5

Proof. Let, the fundamental domain $F_{\Delta_{S_2}}$ of Δ_{S_2} be enclosed by four loops created by L_1, L_2, L_3, and L_4, (L_2 twins with L_1, whereas L_3 is paired with L_4) containing at least one (maximum three) free edges. The polygon (look like) that exhibits the fundamental domain has seven vertices, viz., $v_1, v_2, v_3, v_4, v_5, v_6$, and v_7 (see Figure 5). Among those vertices, only v_2 is the proper vertex and the others (say, v_1, v_3, v_4, v_5, v_6, and v_7) are all improper vertices. It signifies that, here, all vertices are infinite vertices of the aforementioned polygon. Finite vertices will occur if we consider the cases for Figure 1(d), 1(h), 1(k), 1(l), 1(m), 1(o), or 1(p); for the definitions of various types of vertices, see [2]. Suppose the point '∞' is in the group Δ_{S_2} as an ordinary point. Clearly, $F_{\Delta_{S_2}}$ provides an unbounded region in the upper-half plane with an infinite hyperbolic area. Note that, Δ_{S_2} ($\subset PSL(2, \mathbb{R})$) is a discrete group acting in the hyperbolic plane $\mathbb{H}^2 = \{(x, y) : x \in \mathbb{R}, y > 0\}$. Let, L_1^*, L_2^*, L_3^*, and L_4^* be the 4 half disks (all are semi-circular shapes) in $\mathbb{H}^2$ with boundaries L_1, L_2, L_3, and L_4 respectively perpendicular to the circle at infinity. Then the unbounded region $F_{\Delta_{S_2}}^*$ of $\mathbb{H}^2$ is surrounded by the half disks

L_1^*, L_2^*, L_3^*, and L_4^* is a fundamental domain for Δ_{S_2} in $\mathbb{H}^2 \cup \{\infty\}$. Observe that, the semi-circular arcs L_1 and L_2 are linked with a hyperbolic Möbius transformation in the foregoing framework within the upper-half plane model. Now, since a hyperbolic element has an axis, we can consider a point 'ω' on the axis of that hyperbolic element, say, h_s in $\mathbb{H}^2$. Then there exists an element $\pi_s \in \Delta_{S_2}$ such that $\pi_s(\omega) \in F_{\Delta_{S_2}}^*$. Observe that $F_{\Delta_{S_2}}^*$ is geodesically convex in the hyperbolic plane. So, one of the disks L_1^*, L_2^*, L_3^*, and L_4^*, say L_1^*, separates the fixed points of $\{\pi_s h_s (\pi_s)^{-1}\}$. Hence, the boundary of L_1^*, i.e., the semi-circular curve L_1 will also separate these fixed points. Therefore, the required fundamental region of Δ_{S_n} is the domain encircled by $(\pi_s)^{-1}(L_1)$, $(\pi_s)^{-1}(L_2)$, $(\pi_s)^{-1}(L_3)$, and $(\pi_s)^{-1}(L_4)$, which is clearly the infinite area region $F_{\Delta_{S_2}}$ in the upper-half plane. So, the existence of the fundamental domain for the group Δ_{S_2} is established. $\qquad \square$

To prove the rest of the four lemmas, we are now aiming to set up the background notations that will be used throughout this section. Our goal is to prove Theorem 1 by the method of contradiction. Assume that for $\tau < 10^{-17}$, the group $\Gamma_{S_\tau}^N$ is classical. In the preceding lemma, we displayed the existence of a fundamental domain for the rank 2 semi-Fuchsian Schottky group. Let, $F_{\Gamma_{S_\tau}^N}$ be the fundamental domain for the group $\Gamma_{S_\tau}^N$ designed by four loops $SC_{-\mu,\tau}$, $SC_{\mu,\tau}$, $R_{(\mu-1),\tau}$, and $R_{(\mu+1),\tau}$, one of which, says $R_{(\mu-1),\tau}$, separates the fixed points '0' and '∞' of the transformation h_τ^{**}, and so does another boundary component of $F_{\Gamma_{S_\tau}^N}$ other than the loop $R_{(\mu-1),\tau}$. Now, we consider the set $SC_\tau^* = \{SC_\tau^1, SC_\tau^2, SC_\tau^3, ..., SC_\tau^{\mathbf{I}^+}\}$ which is the total list of the images of the loops $SC_{-\mu,\tau}$, $SC_{\mu,\tau}$, $R_{(\mu-1),\tau}$, and $R_{(\mu+1),\tau}$, applying by the group $\Gamma_{S_\tau}^N$ that are nested and intersect the open interval $(0, \mu+1)$ once, fulfilling one of the succeeding cases:

(i) Each semi-circle SC_τ^j keeps apart 0 from ∞,

(ii) Each semi-circle SC_τ^{j+1} differentiates the semi-circle SC_τ^j from 0, and

(iii) SC_τ^1, SC_τ^j, $j = 2, 3, ..., (\mathbf{I}^+ - 1)$ and $SC_\tau^{\mathbf{I}^+}$ meet $[\mu+1, \infty), (\mu-1, \mu+1)$ and $(0, \mu-1]$ respectively, where $\mu > 1$, $\mathbf{I}^+$ denotes the positive natural number.

Notice that, when two semi-circles in the set SC_τ^* (say, SC^m and SC^n, where $m, n \in \mathbf{I}^+$) are not segregated by other semi-circles in SC_τ^* then clearly, these two semi-circles situate the boundary of $F_{\Gamma_{S_\tau}^N}$. Also, the semi-circles SC_τ^j and SC_τ^{j+1} together with any two loops, one of which is from the set $\{p(SC_{-\mu,\tau}), p(SC_{\mu,\tau}) : p \in \Gamma_{S_\tau}^N\}$ and other from $\{p'(R_{(\mu-1),\tau}), p'(R_{(\mu+1),\tau}) : p' \in \Gamma_{S_\tau}^N\}$ surround a fundamental domain (say, $F_{\Gamma_{S_\tau}^N}$) for the group $\Gamma_{S_\tau}^N$. Let, $W_{jk\tau} = \{SC_\tau^j \cap i^{k-1}\mathbb{R}^+\}$, where $k = 1, 2, 3$, and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. In the upper-half plane, we set the largest distance between two consecutive semi-circles in the set SC_τ^* as

$Z_{j\tau} = \max_k\{|W_{jk\tau} - W_{(j+1)k\tau}|\}$. Suppose that $\Omega(\Gamma_{S_\tau}^N)$ is the domain of discontinuity of the group $\Gamma_{S_\tau}^N$.

After arranging the background notations above, our aim is now to prove Lemma 4.3. But before that, we need to establish the following Lemma 4.2. In fact, we have derived the subsequent Lemma 4.2 under certain assumptions when the value of τ tends to 0. In particular, this lemma is linked to the lengths of components of the domain of discontinuity for the rank 2 semi-Fuchsian Schottky group.

Lemma 4.2. *The length of each component of the segment $\{\Omega(\Gamma_{S_\tau}^N) \cap \mathbb{R}\}$ that meets the region fabricated by the union of closed intervals given as $[-(\mu+1), -(\mu-1)] \cup [(\mu-1), (\mu+1)]$ is less than the ensuing quantity*

$$\left\{\frac{(2.01)}{\sqrt{\mu}(\mu-1)^2} \times \sqrt{2} \times \sqrt{\tau} \times \sqrt{2\mu^5 - \mu^4 - 3\mu^3 + 6\mu^2 - 8\mu + 1}\right\},$$

where $\mu = |x| > 1$ and $|x|$ denotes the modulus value of the number x.

Proof. First, we consider two transformations defined by

$$h_\tau^*(z) = \frac{x(1-\tau)^{-1}z + (1-\tau)\{x^2(1-\tau)^{-2} - 1\}}{(1-\tau)^{-1}z + x(1-\tau)^{-1}}$$

and

$$h_\tau^{**}(z) = \left(\frac{\{x^2 - 1\} + \mod(\pm 2i(1-\tau)) + (1-\tau)^2}{\{x-1\}^2 + (1-\tau)^2}\right)z$$

where x is a positive real number greater than 1.

Let, $h_\tau^* h_\tau^{**} h_\tau^* (h_\tau^{**})^{-1}(z) = C(z)$ (say).

This yields the fixed points as z_i, for $i = 1, 2$, where

$$z_i = \frac{1}{x(x^2 - 2x + 2 - 2\tau + \tau^2)} \times [-\{x^3 - x - (x^2 - 2x - 1)\tau - (2+x)\tau^2 + \tau^3\} \pm \{x^2(x^6 - 2x^5 + 4x^4$$

$$-2x^3 - 2x^2 + 4x - 3) + 2x\tau(-2x^5 + x^4 + 3x^3 - 6x^2 + 8x - 1) + x^2\tau^2(x^4 - 7x^2 - 40)$$

$$+2\tau^3(2x^4 - 4x^3 + 21x^2 - 6x - 2) + \tau^4(-x^4 - 2x^3 - 26x^2 + 8x + 6) + 2\tau^5(4x^2 - x - 2) + \tau^6(1 - x^2)\}^{\frac{1}{2}}].$$

Without loss of generality, we let $x^6 - 2x^5 + 4x^4 - 2x^3 - 2x^2 + 4x - 3 = 0$.

From this equation, we get the following roots.

$x = 1, -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, and $\frac{1}{2}(1 \pm i\sqrt{11})$.

Now, suppose $\mu = |x|$, where $|x|$ denotes the modulus value of the complex numbers $\frac{1}{2}(1 \pm i\sqrt{11})$, i.e., $\mu = \sqrt{3}$ (> 1).

Although we have presumed that x is a positive real number greater than 1, the equation $x^6 - 2x^5 + 4x^4 - 2x^3 - 2x^2 + 4x - 3 = 0$ is unable to yield a root that corresponds to such a

real number. On the other hand, we must set $x^6 - 2x^5 + 4x^4 - 2x^3 - 2x^2 + 4x - 3$ equal to zero if we want to apply the technique from Yamamoto's work ([15]). Instead, we were unable to locate the necessary, small positive delta. As a result, without compromising generality, we have taken the modulus values (> 1) of the equation's roots.

Let, χ_1 be the component of $\{\Omega(\Gamma^N_{S_\tau}) \cap \mathbb{R}\}$ bounded by the fixed points of the function $\{h^*_\tau h^{**}_\tau h^*_\tau (h^{**}_\tau)^{-1}\}$. All components of $\{\Omega(\Gamma^N_{S_\tau}) \cap \mathbb{R}\}$ are analogous to one another for the group $\Gamma^N_{S_\tau}$. Now, we set $\chi_2 = (h^*_\tau)^{-1}(\chi_1)$, $\chi_3 = (h^{**}_\tau)^{-1}(\chi_2)$, and $\chi_4 = (h^{**}_\tau)^{-1}(\chi_1)$. Assume that X is a component of $\{\Omega(\Gamma^N_{S_\tau}) \cap \mathbb{R}\}$ included in the semi-circle $SC_{2,\tau}$ or $SC_{4,\tau}$.

So, X can be expressed in the following way.

$$\alpha_{2g}(X) = \{(h^*_\tau)^{c_{2g}}(h^{**}_\tau)^{c_{2g-1}}.(h^*_\tau)^{c_{2g-1}}(h^{**}_\tau)^{c_{2g-2}}...(h^*_\tau)^{c_3}(h^{**}_\tau)^{c_2}.(h^*_\tau)^{c_2}(h^{**}_\tau)^{c_1}\}(X),$$

where, X is either χ_1, χ_2, χ_3, or χ_4 and $c_{2g}c_{2g-1}...c_2 \neq 0$.

Let us denote $l(X)$ as the length of X.

Then we get

$$l(X) < \left\{ \frac{(2.01) \times \sqrt{2} \times \sqrt{\tau} \times \sqrt{-2\mu^5 + \mu^4 + 3\mu^3 - 6\mu^2 + 8\mu - 1}}{\sqrt{\mu}(\mu^2 - 2\mu + 2 - 2\tau + \tau^2)} \right\}$$

$$< \left\{ \frac{(2.01)}{\sqrt{\mu}(\mu-1)^2} \times \sqrt{2} \times \sqrt{\tau} \times \sqrt{2\mu^5 - \mu^4 - 3\mu^3 + 6\mu^2 - 8\mu + 1} \right\},$$

since the length of 'i' is 1.

In this situation, our goal is to derive that the length of any image of X under α_{2g} is less than the length of X, which we have shown by proving that $|\alpha'_{2g}(x)| < 1, \forall\, x \in X$. In the following, we have established this by using the method of induction concerning g.

First of all, when $g = 1$, we have $\alpha_2 = (h^*_\tau)^{c_2}(h^{**}_\tau)^{c_1}$.

Now, we have studied three cases for the value of c_1 in the following way:

(i) Case I, for $c_1 = 0$, (ii) Case II, when $c_1 < 0$, and (iii) Case III, for $c_1 > 0$.

Obviously,

$$|(h^*_\tau)'(x)| = \left| \frac{(1-\tau)^2}{(x+\mu)^2} \right|,$$

$$\text{and } |(h^{**}_\tau)'(x)| = \frac{\mu^2 - 1 + 2(1-\tau) + (1-\tau)^2}{(\mu-1)^2 + (1-\tau)^2}.$$

Inside the semi-circle $SC_{-\mu,\tau}$, we obtain $|(h^*_\tau)'(x)| > 1$ since $|x + \mu| < 1 - \tau$, and outside the semi-circle $SC_{-\mu,\tau}$, we have $|(h^*_\tau)'(x)| < 1$ since $|x + \mu| > 1 - \tau$.

(i) Case I, when $c_1 = 0$: Our goal is to show that $|\{(h^*_\tau)^{c_2}\}'(x)| < 1 \,\forall\, x \in X$ by utilizing the method of induction on c_2. Suppose we look at the case when $c_2 > 0$, for $c_2 = 1$ the induction

becomes trivial. For $c_2 = 2$, we have the following outcome.

$$
\begin{aligned}
|\{(h_\tau^*)^2\}'(x)| &= \frac{d}{dx}\left[\left(\frac{\mu x + \{\mu^2 - (1-\tau)^2\}}{x+\mu}\right)^2\right] \\
&= \frac{d}{dx}\left[\left(\mu - \frac{(1-\tau)^2}{x+\tau}\right)^2\right], \text{ since } x \neq -\mu \\
&= 2\left(\mu - \frac{(1-\tau)^2}{x+\mu}\right)\left(\frac{1-\tau}{x+\mu}\right)^2 \\
&< 2\left(\mu - \frac{(1-\tau)^2}{x+\mu}\right),
\end{aligned}
$$

since outside the semi-circle $SC_{-\mu,\tau}$, we get $(1-\tau)^2 < (x+\mu)^2$.

So, if we consider

$$
\begin{aligned}
x &< \frac{1}{4}\{(-2\mu+1) - \sqrt{(2\mu+1)^2 + 16(1-\tau)^2}\}, \\
\text{and } x &> \frac{1}{4}\{(-2\mu+1) + \sqrt{(2\mu+1)^2 + 16(1-\tau)^2}\},
\end{aligned}
$$

we obtain

$$
|\{(h_\tau^*)^2\}'(x)| < 1.
$$

Observe that, if we take outside the restriction of x, i.e.,

$$
\frac{1}{4}\{(-2\mu+1) - \sqrt{(2\mu+1)^2 + 16(1-\tau)^2}\} < x < \frac{1}{4}\{(-2\mu+1) + \sqrt{(2\mu+1)^2 + 16(1-\tau)^2}\},
$$

the position of χ_1 will be on the right of this region, whereas χ_2 is located to the left of that zone, and χ_3 is situated to the right of this region. Furthermore, for a very small positive number τ, χ_4 is also outside that zone. Hence, these yield $\forall\, x \in X$,

$$
|\{(h_\tau^*)^{c_2}\}'(x)| < 1.
$$

Now, our objective is to show that the condition $|\{(h_\tau^*)^{c_2}\}'(x)| < 1$ is true for $c_2 = n+1$ when it is true for $c_2 = n$.

This can be easily verified from the subsequent deduction:

$$
\begin{aligned}
|\{(h_\tau^*)^{n+1}\}'(x)| &= \left|\frac{d}{dx}\{(h_\tau^*)^{n+1}(x)\}\right| \\
&= \left|\frac{d\{(h_\tau^*)^{n+1}(x)\}}{d\{(h_\tau^*)^n(x)\}}\right| \times \left|\frac{d\{(h_\tau^*)^n(x)\}}{dx}\right| \\
&< \left|\frac{d\{(h_\tau^*)^{n+1}(x)\}}{d\{(h_\tau^*)^n(x)\}}\right|, \text{ since from our supposition, we have } \left|\frac{d\{(h_\tau^*)^n(x)\}}{dx}\right| < 1 \\
&= \left|\frac{d\{(h_\tau^*)(x)\}}{dx}\right|, \text{ replacing x by } (h_\tau^*)^n(x) \\
&= \left|\frac{(1-\tau)^2}{(x+\mu)^2}\right| \\
&< \left|\frac{1}{(x+\mu)^2}\right| \\
&= \frac{1}{|(h_\tau^*)^n(x)+\mu|^2}.
\end{aligned}
$$

Now, if we utilize the modulus values of the coordinates which are coming from $(h_\tau^*)^n(x)$ when $x \in \chi_i$, for $i = 1, 2, 3$, and 4, that run inside the semi-circle $SC_{\mu,\tau}$, and use these modulus values in $\frac{1}{|(h_\tau^*)^n(x)+\mu|^2}$, we get

$$
|\{(h_\tau^*)^{n+1}\}'(x)| < 1.
$$

So, we are done.

The situation for $c_2 < 0$ is very much similar to the previous one. At first, we prove it for $c_2 = -2$, and after that, our aim will be to show that assuming $|\{(h_\tau^*)^{c_2}\}'(x)| < 1$ holds for $c_2 = n$, $n < 0$, then it also holds for $c_2 = n - 1$. In this way to prove this case one can follow the exact method as we have studied in the previous one. Therefore, Case I is established.

(ii) Case II, for $c_1 < 0$: We have shown this case by utilizing an analogous type of induction method as in the foregoing case, where we can write the derivative of α_2 in a subsequent way:

$$
\begin{aligned}
|(\alpha_2)'(x)| &= |\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}\}'(x)| \\
&= \left|\frac{d}{dx}\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}(x)\}\right| \\
&= \left|\frac{d\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}(x)\}}{d\{(h_\tau^{**})^{c_1}(x)\}}\right| \times \left|\frac{d\{(h_\tau^{**})^{c_1}(x)\}}{dx}\right|.
\end{aligned}
$$

$$i.e., |(\alpha_2)'(x)| \quad < \quad \left| \frac{d\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}(x)\}}{d\{(h_\tau^{**})^{c_1}(x)\}} \right|, \text{ since for } c_1 < 0, \text{ we have } \left| \frac{d\{(h_\tau^{**})^{c_1}(x)\}}{dx} \right| < 1$$

$$= \left| \frac{d\{(h_\tau^*)^{c_2}(v)\}}{dv} \right|, \text{ replacing } v \text{ by } \{(h_\tau^{**})^{c_1}(x)\}$$

$$= \left| \frac{(1-\tau)^2}{(v+\mu)^2} \right|$$

$$< \quad \frac{1}{|v+\mu|^2}$$

$$< \quad 1,$$

by utilizing Case I when $c_1 = 0$ and applying all the domains (when $c_1 < 0$) that are outside the semi-circle $SC_{-\mu,\tau}$.

Hence, Case II follows.

(iii) Case III, when $c_1 > 0$: In this case, for the first step of the induction concerning g the objective is to establish that the image of X under the transformations $\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}\}$ is less than X. Now, for the necessary subcases, we first investigate the case for $c_2 > 0$. Note that if $c_1 = 1$, then from our assumption we can assert that the transformation h_τ^{**} sends χ_3 to χ_2 and χ_4 to χ_1, and also we already have $|\{(h_\tau^*)^{c_2}\}'(x)| < 1 \ \forall \ x \in X$. Furthermore, for $c_1 > 1$, the images of χ_1 and χ_2 under the transformation $(h_\tau^{**})^{c_1}$ are outside the semi-circle $SC_{\mu,\tau}$. Also, outside the semi-circle $SC_{-\mu,\tau}$, the Möbius transformation h_τ^* is contracting, since we reach the following argument:

$$|(h_\tau^*)'(x)| \quad = \quad \left| \frac{(1-\tau)^2}{(x+\mu)^2} \right|$$

$$< \quad 1,$$

for $|x + \mu| > (1 - \tau)$, those points which are outside the semi-circle $SC_{-\mu,\tau}$. Therefore, when $c_2 = 1$, the function $\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}\}$ gives the longest interval. In this circumstance, our aim is now to prove that the transformation $\{(h_\tau^*)(h_\tau^{**})^{c_1}\}(X)$ is shorter than X, $\forall \ c_1$. To show that our goal is now to deduce that the values of A_1 and A_2 are less than $|\chi_{1a} - \chi_{1b}|$, where

$$A_1 \quad = \quad \left| h_\tau^* \left\{ \left(\frac{(\mu^2 - 1) + 2(1 - \tau) + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2} \right)^{c_1} \chi_{1a} \right\} - h_\tau^* \left\{ \left(\frac{(\mu^2 - 1) + 2(1 - \tau) + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2} \right)^{c_1} \chi_{1b} \right\} \right|,$$

$$A_2 \quad = \quad \left| h_\tau^* \left\{ \left(\frac{(\mu^2 - 1) + 2(1 - \tau) + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2} \right)^{c_1} \chi_{2a} \right\} - h_\tau^* \left\{ \left(\frac{(\mu^2 - 1) + 2(1 - \tau) + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2} \right)^{c_1} \chi_{2b} \right\} \right|,$$

and $\chi_{1a}, \chi_{1b}, \chi_{2a}, \chi_{2b}$ are the end points of χ_1 and χ_2.

Now, we have investigated 4 cases in the following way (for τ tends to be 0^+ or 1^-):

(i) $c_1 = 1$,

$$A_1 = \frac{(1-\tau)^2(\chi_{1a} - \chi_{1b})}{\frac{(\mu^2-1)+2(1-\tau)+(1-\tau)^2}{(\mu-1)^2+(1-\tau)^2}\chi_{1a}\chi_{1b} + \mu(\chi_{1a} + \chi_{1b}) + \frac{(\mu-1)^2+(1-\tau)^2}{(\mu^2-1)+2(1-\tau)+(1-\tau)^2}}$$

$$< (\chi_{1a} - \chi_{1b}).$$

(ii) $c_1 \to$ absolute value less than 1, $A_1 \to 0$.

(iii) $c_1 \to 0$, $A_1 \to 0$.

(iv) $c_1 \to \infty$, $A_1 \to 0$.

For A_2, the situation is very similar if we replace χ_{1a}, and χ_{1b} with χ_{2a} and χ_{2b}. Hence, any image of X under the transformation $\{(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}\}$ is less than X, for $c_1 > 0$. Therefore, we have established the first step of the induction concerning g. Now, we aim to deliver the next step of the induction. In particular, our purpose is to derive that for any combination of c_1 and c_2, we have $|(\alpha_2)'(x)| < 1$ for all $x \in X$. In fact, assuming

$$|\{(h_\tau^*)^{c_{2n+2}}(h_\tau^{**})^{c_{2n+1}}(h_\tau^*)^{c_{2n}}(h_\tau^{**})^{c_{2n-1}}...(h_\tau^*)^{c_2}(h_\tau^{**})^{c_1}\}| < 1, \forall \text{ x} \in \text{X},$$

our goal is to prove that it is true for $g = (n+1)$, i.e., the image of X under the transformation $\alpha_{2(n+1)}$ is not greater than the image of X under α_{2n}. We have succeeded by proving the two subcases as follows.

(i) Subcase I, for $c_{2n+1} < 0$, and (ii) Subcase II, when $c_{2n+1} > 0$.

(i) Subcase I, $c_{2n+1} < 0$: It is known that the image of the length of X under the function α_{2n} is less than the length of X. Now, since $c_{2n+1} < 0$, we have that the image of the length of X under the map $(h_\tau^{**})^{c_{2n+1}}\alpha_{2n}$ is again less than the length of X. Again, the image of X corresponding to the function α_{2n} can be inside either the semi-circle $SC_{-\mu,\tau}$ or $SC_{\mu,\tau}$. So, the image under the transformation $\{(h_\tau^{**})^{c_{2n+1}}\alpha_{2n}\}$ is inside the loop $R_{(\mu-1),\tau}$, i.e., outside both the semi-circles $SC_{-\mu,\tau}$ and $SC_{\mu,\tau}$. Also, the transformation h_τ^* is contracting outside the region $SC_{\mu,\tau}$, and hence $(h_\tau^*)^{-1}$ is also contracting outside the region $SC_{-\mu,\tau}$. Hence, the image corresponding to the function $\{(h_\tau^*)^{c_{2n+2}}(h_\tau^{**})^{c_{2n+1}}\alpha_{2n}\}$ is less than the image under the transformation $\{(h_\tau^{**})^{c_{2n+1}}\alpha_{2n}\}$. This completes the proof of the subcase I.

(ii) Subcase II, $c_{2n+1} > 0$: To derive this case, we have discussed two more subcases, one is when $f \le |\alpha_{2n}(x)| < \mu + 1$ and the other for $\mu - 1 \le |\alpha_{2n}(x)| < f$, where $f = \sqrt{\mu^2 - (1-\tau)^2}$ is the attractive fixed point of the function h_τ^*.

Subcase II(a) : For this case, we can write the transformation $\alpha'_{2g+2}(x)$ in the following way:

$$
\begin{aligned}
|\alpha'_{2g+2}(x)| &= \left|\frac{d}{dx}\{\alpha_{2g+2}(x)\}\right| \\
&= \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}\right| \times \left|\frac{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{\alpha_{2g}(x)\}}\right| \times \left|\frac{d\{\alpha_{2g}(x)\}}{dx}\right| \\
&= \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}\right| \times \left|\frac{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{\alpha_{2g}(x)\}}\right| \times |\alpha'_{2g}(x)| \\
&< \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}\right| \times \left|\frac{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{\alpha_{2g}(x)\}}\right| \\
&= \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}\right| \times \left|\frac{d\{(h^{**}_\tau)^{c_{2g+1}}(y)\}}{dy}\right|, \text{ replacing } y \text{ by } \{\alpha_{2g}(x)\} \\
&< \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}{d\{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\}}\right|(3.75)^{c_{2g+1}}, \text{ for } \tau \text{ tends to } 0^+ \text{ or } 1^- \\
&= \left|\frac{d\{(h^*_\tau)^{c_{2g+2}}(w)\}}{dw}\right|(3.75)^{c_{2g+1}}, \text{ replacing } w \text{ by } \{(h^{**}_\tau)^{c_{2g+1}}\alpha_{2g}(x)\} \\
&< \left|\frac{(1-\tau)^2}{\{(3.75)^{c_{2g+1}}+\mu\}^2}\right|(3.75)^{c_{2g+1}}, \text{ replacing x by } (h^*_\tau)^{c_{2g+1}}(w) \text{ in } |(h^*_\tau)'(x)| = \left|\frac{(1-\tau)^2}{(x+\mu)^2}\right| \\
&< \frac{(3.75)^{c_{2g+1}}}{\{(3.75)^{c_{2g+1}}+\mu\}^2} \\
&< 1.
\end{aligned}
$$

So, the proof of the subcase II(a) is done.

Subcase II(b) : Since $\mu-1 \leq |\alpha_{2n}(x)| < f$, we have $|(h^{**}_\tau)^{c_{2g-1}}\alpha_{2g-2}(x)| < (\mu-1)$. So, $c_{2g-1} < 0$. Now, for this case, one can easily estimate the value of $|\alpha'_{2g-2}(x)|$ by utilizing the technique used to evaluate the value of $|\alpha'_{2g+2}(x)|$. Apart from that, the situation is similar to the preceding subcase II(a). So, we have shown the required one, i.e., $|\alpha'_{2g+2}(x)| < 1$. Consequently, we have completed the induction.

Hence, the lemma is established. $\qquad\square$

After proving the overhead lemma, we are now prepared to show Lemma 4.3. This lemma is also related to the lengths of components of the domain of discontinuity for the semi-Fuchsian Schottky group $\Gamma^N_{S_\tau}$ which intersects the upper imaginary axis in the hyperbolic plane.

Lemma 4.3. *The length of each component of $\{\Omega(h^{**}_\tau \Gamma^N_{S_\tau}(h^{**}_\tau)^{-1}) \cap i\mathbb{R}^+\}$ which meets the vertical segment $i[0, (\mu+1)^{(3.7)}]$ is less than the following quantity*

$$
\left\{\frac{(2.01)}{\sqrt{\mu}(\mu-1)^2} \times (\mu+1)^{(2.7)} \times \sqrt{2} \times \sqrt{\tau} \times \sqrt{2\mu^5 - \mu^4 - 3\mu^3 + 6\mu^2 - 8\mu + 1}\right\},
$$

where $\mu = |x| > 1$ and $|x|$ denotes the modulus value of the number x .

Proof. It is obvious that, $\left(h_\tau^{**}\right)^{-1}(z) = \left(\frac{\{\mu-1\}^2+(1-\tau)^2}{\{\mu^2-1\}+ \text{ mod }(\pm 2i(1-\tau))+(1-\tau)^2}\right)z.$

Therefore, $\forall\, z$, the value of the function $\left(h_\tau^{**}\right)^{-1}$ is less than 1.

So, the transformation $\left(h_\tau^{**}\right)^{-1}$ is contracting.

Note that

$$
\begin{aligned}
h_\tau^{**}(z) &= \left(\frac{\{\mu^2-1\}+ \text{ mod }(\pm 2i(1-\tau))+(1-\tau)^2}{\{\mu-1\}^2+(1-\tau)^2}\right)z \\
&\leq \{(\mu+1)^{(1.35)}\}z, \text{ for } \tau \to 0^+ \text{ or } 1^-.
\end{aligned}
$$

Let us define a map, $h : z \to \{(\mu+1)^{(1.35)}\}z$, $z \in \mathbb{H}^2$, where $\mathbb{H}^2$ represents the hyperbolic plane. Now, for simplicity, we intend to move forward to the function h instead of the transformation h_τ^{**}, without loss of generality. So, if we use the map h_τ^{**} twice on the region mentioned in the Lemma 4.2, i.e., the segment $[-(\mu+1), -(\mu-1)] \cup [(\mu-1), (\mu+1)]$, the component $\{\Omega(h_\tau^{**}\Gamma_{S_\kappa}^N(h_\tau^{**})^{-1}) \cap i\mathbb{R}^+\}$ will touch the vertical piece $i[1, (\mu+1)^{(3.7)}]$. Since the function $\left(h_\tau^{**}\right)^{-1}$ is contracting, one can utilize the transformation $\left(h_\tau^{**}\right)^{-1}$ several times to extend the standing fragment $i[1, (\mu+1)^{(3.7)}]$ to the whole upright portion $i[0, (\mu+1)^{(3.7)}]$. Notice that, this act doesn't affect the size of the interval. This proves the lemma. $\qquad\square$

Now, we aim to measure the distance between the curves in the set SC_τ^*, which we have obtained from the images of the loops $SC_{\mu,\tau}, SC_{-\mu,\tau}, R_{(\mu+1),\tau}$, and $R_{(\mu-1),\tau}$ applying by the group $\Gamma_{S_\tau}^N$. The ensuing lemma will be effective in the sequel.

Lemma 4.4. *Let $\zeta > 0$. If $|W_{jk\tau}-W_{(j+1)k\tau}| < \zeta$ for at least two values of k, where $k \in \{1,2,3\}$, then $Z_{j\tau} < \left[8 \times \frac{3\mu+1}{\mu-1} \times \{1+(\mu+1)^{(1.35)}+(\mu+1)^{(2.7)}\}^2\left(1+\frac{3\mu-1}{2\sqrt{2}\sqrt{\mu^2-1}}\right)+1\right]\zeta.$*

Proof. Suppose $O_j = x_j + i.0.y_j$ and r_j are the center and the radius of the semi-circle SC_τ^j respectively. We also let, d_j be the distance between the centers of the semi-circles SC_τ^j and SC_τ^{j+1} where $O_{j+1} = x_{j+1} + i.0.y_{j+1}$ and r_{j+1} are the center and the radius of the semi-circle SC_τ^{j+1} respectively. We define a semi-circle SC_τ, which is concentric to the semi-circle SC_τ^{j+1} and tangent to SC_τ^j at a point Q. Obviously, the point Q is situated on the x-axis. So, the semi-circle SC_τ can be expressed by the equation $\sqrt{(x-x_{j+1})^2+(y-0)^2} = (r_j-d_j)$. We set $W_k' = SC_\tau \cap i^{k-1}\mathbb{R}^+$. Observe that, $W_3'Q$ is the diameter of the semi-circle SC_τ (see, Figure 6). Let, k_1 and k_2 denote the two values of k for which $|W_{jk_i\tau} - W_{(j+1)k_i\tau}| < \zeta$. Now, without loss of generality, we assume that

$$
\angle W_2'O_{j+1}O_j \leq \angle W_2'O_{j+1}W_1' \quad \text{or} \quad \angle SO_{j+1}W_2' \leq \angle QO_{j+1}W_2'.
$$

Suppose $\theta = \angle SO_{j+1}W_2'$. At first, we consider the case when $\frac{\pi}{2} \leq \theta \leq \pi$, after that we comment on the situation when $0 \leq \theta \leq \frac{\pi}{2}$. Now, considering the triangle $\triangle SQW_2'$ with $\angle W_2'QS = 90°$, we have

$$\sin\theta > \sin\angle W_2'SQ \quad \text{and} \quad \sin\angle W_2'SQ = \frac{|W_2' - Q|}{2(r_j - d_j)}.$$

Hence

$$\sin\theta > \frac{|W_2' - Q|}{2(r_j - d_j)}. \tag{3}$$

Also, $|W_1'| + |W_2'| + |W_3'| \geq \text{Diameter of } SC_\tau$.

$$i.e., \frac{1}{2(r_j - d_j)} \geq \frac{1}{\sum_{k=1}^{3}|W_k'|}. \tag{4}$$

Again, by triangle inequality we get

$$\begin{aligned} |W_{k_i}' - Q| + |W_{k_j}' - Q| &\geq |W_{k_i}' - W_{k_j}'| \\ &> |W_{k_i}'|, \end{aligned} \tag{5}$$

for $i \neq j$, where $i, j = 1, 2$ or 3.

So, equation (3), (4), and (5) together yield

$$\sin\theta > \frac{|W_3'|}{2(\sum_{k=1}^{3}|W_k'|)}. \tag{6}$$

Since the transformation h_τ^{**} has fixed points at 0 and ∞, and the semi-circle SC_τ separates these fixed points, any image of SC_τ under h_τ^{**} will also separate these fixed points '0' and '∞'. These fixed points are placed in either one of the semi-circular curves in the set SC_τ^* or outside the semi-circle $SC_{1,\tau}$. The image can not be inside the semi-circle SC_τ^j (where $j = 1, 2, 3, ...,$ $\mathbf{I}^+$) since the transformation h_τ^{**} has '0' as a repulsive fixed point. On the other hand, the image can not be the whole semi-circle SC_τ^j either, since SC_τ^j is tangential to the semi-circle SC_τ at the point Q. So it must be outside the semi-circle SC_τ^j. Therefore the images of W_k' for the transformation h_τ^{**} must be outside the W_k' themselves.

This gives the following.

$$\begin{aligned} |W_{k+1}'| &< \left(\frac{(\mu^2 - 1) + 2(1 - \tau) + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2}\right)|W_k'| \\ &< (\mu + 1)^{(1.35)}|W_k'|. \end{aligned}$$

Hence

$$\sum_{k=1}^{3}|W_k'| < |W_{k_1}'| + (\mu + 1)^{(1.35)}|W_{k_1}'| + (\mu + 1)^{(2.7)}|W_{k_1}'|.$$

Therefore, from equation (6) we obtain

$$\sin\theta \; > \; \frac{1}{2\{1 + (\mu+1)^{(1.35)} + (\mu+1)^{(2.7)}\}}.$$

So

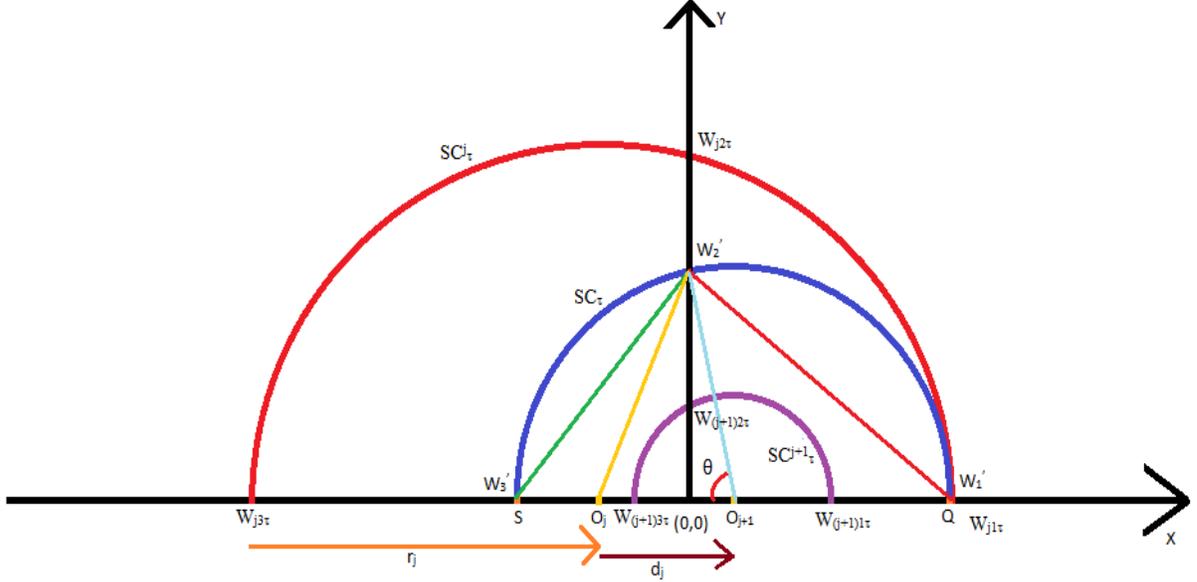$$(1 + \cos\theta)^{-1} < 8\{1 + (\mu+1)^{(1.35)} + (\mu+1)^{(2.7)}\}^2. \tag{7}$$

$Figure : 6$

Since the semi-circles SC_τ^j and SC_τ^{j+1}, along with two other curves in the sets $\{p(SC_i) : p \in \Gamma_{S_\tau}^N, i = 1, 2\}$ and $\{p'(R_i) : p' \in \Gamma_{S_\tau}^N, i = 1, 2\}$ bound a fundamental domain for the group $\Gamma_{S_\tau}^N$, where $\Gamma_{S_\tau}^N = \; < h_\tau^*, h_\tau^{**} >$, we claim that $h_\tau^{**}(SC_\tau^j) \cap SC_\tau^{j+1} = \phi$. Assume that M and N are two points situated on the semi-circle SC_τ^j. Observe that the function h_τ^{**} preserves any line through the origin. Hence, if we take a line H_τ^{**} through the origin $(0,0)$ and the point $(x_j, 0)$ we get the distance between the point M and O_j is $(r_j - |O_j|)$, which is away from the origin. However, the distance between the point N and O_j is $(r_j + |O_j|)$. Again, the image of point M for the map h_τ^{**} is placed on the line H_τ^{**} and further from the origin $(0,0)$ than the point 'N'.

Hence

$$\left(\frac{(\mu^2 - 1) + |\pm 2i(1 - \tau)| + (1 - \tau)^2}{(\mu - 1)^2 + (1 - \tau)^2}\right)(r_j - |O_j|) > \frac{1}{2}(r_j + |O_j|).$$

This inequality gives the subsequent result for $\tau \to 0^+$ or 1^-.

$$|O_j| < \left(\frac{\mu^2 + 2\mu - 3}{3\mu^2 - 2\mu - 1}\right)r_j. \tag{8}$$

Again

$$d_j \le |O_j| + |O_{j+1}|. \tag{9}$$

Since the origin is placed inside the semi-circle SC_τ^{j+1}, it is also located inside the semi-circle SC_τ. Thus, we have

$$|O_{j+1}| < r_j - d_j. \tag{10}$$

Combining equations $(8), (9)$, and (10) we get the following.

$$d_j < \frac{2(\mu^2 - 1)}{(3\mu^2 - 2\mu - 1)} r_j. \tag{11}$$

Suppose $\zeta' = |W_{jk_i\tau} - W'_{k_i}| < \zeta$, where $k_i = i$, for $i = 1, 2,$ and 3. Observe that, for $k_i = 1, \zeta' = 0$ and $k_i = 2$ or 3, $\zeta' > 0$. Obviously for $k_i = 1, r_j = |O_j - W'_{k_i}| + \zeta'$ and for $k_i = 2$ or 3, $r_j < |O_j - W'_{k_i}| + \zeta'$, where $\zeta' > 0$. Now, if we look at the real axis for $k_i = 1$ and $k_i = 3$, we obtain $(r_j - \zeta') = |O_j - W'_{k_1}|$ and $(r_j - \zeta') < |O_j - W'_{k_3}|$ respectively. Again, if we study the triangle $\triangle W'_{k_2} O_{j+1} O_j$, for $k_i = 2$, it produces $(r_j - \zeta') < |O_j - W'_{k_2}|$. So, if we apply the cosine formula on the triangle $\triangle W'_{k_2} O_{j+1} O_j$, we deduce the subsequent result:

$$\begin{aligned}
(r_j - \zeta')^2 &< |O_j - W'_{k_2}|^2 \\
&= d_j^2 + (r_j - d_j)^2 - 2d_j(r_j - d_j)\cos\theta.
\end{aligned}$$

Consequently, for $k_i = i$, where $i = 2$, we obtain the following:

$$\begin{aligned}
d_j(r_j - d_j)(1 + \cos\theta) &< r_j\zeta' \\
\text{i.e., } d_j &< r_j\zeta(r_j - d_j)^{-1}(1 + \cos\theta)^{-1}. \tag{12}
\end{aligned}$$

Again, equation (11) entails

$$(r_j - d_j)^{-1} < \frac{(3a + 1)}{(a - 1)r_j}.$$

Now, putting the values of $(r_j - d_j)^{-1}$ and $(1 + \cos\theta)^{-1}$, equation (12) converts to the subsequent form:

$$d_j < 8 \times \frac{(3\mu + 1)}{(\mu - 1)} \times \{1 + (\mu + 1)^{(1.35)} + (\mu + 1)^{(2.7)}\}^2 \zeta. \tag{13}$$

Observe that, when $0 \leq \theta \leq \frac{\pi}{2}$, the value of $(1 + \cos \theta)^{-1}$ is easily included in the equation (7). Now, our objective is to evaluate the value of $Z_{j\tau}$, which we have delivered in the following.

$$
\begin{aligned}
Z_{j\tau} &= \max_{k}\{|W_{jk\tau} - W_{(j+1)k\tau}|\} \\
&< \sum_{k=1}^{3} |W_{jk\tau} - W_{(j+1)k\tau}| \\
&= |2r_j - 2r_{j+1}| + \left|\sqrt{r_j^2 - x_j^2} - \sqrt{r_{j+1}^2 - x_{j+1}^2}\right| \\
&= 2(r_j - r_{j+1}) + \frac{(r_j^2 - x_j^2) - (r_{j+1}^2 - x_{j+1}^2)}{\sqrt{r_j^2 - x_j^2} + \sqrt{r_{j+1}^2 - x_{j+1}^2}}.
\end{aligned}
$$

Equation (8) implies that

$$
|x_j| < \left(\frac{\mu^2 + 2\mu - 3}{3\mu^2 - 2\mu - 1}\right) r_j.
$$

Therefore

$$
r_j^2 - |x_j|^2 > r_j^2 - \left(\frac{\mu^2 + 2\mu - 3}{3\mu^2 - 2\mu - 1}\right)^2 r_j^2,
$$

i.e.,

$$
\frac{1}{\sqrt{r_j^2 - |x_j^2|}} < \frac{(3|\mu| + 1)}{2\sqrt{2}\sqrt{|\mu|^2 - 1}} \times \frac{1}{r_j}.
$$

Hence

$$
\begin{aligned}
Z_{j\tau} &< 2(r_j - r_{j+1}) + \frac{|r_j^2 - r_{j+1}^2 - x_j^2 + x_{j+1}^2|}{\frac{2\sqrt{2}\sqrt{\mu^2-1}}{(3\mu+1)}|r_j + r_{j+1}|} \\
&< 2(r_j - r_{j+1}) + \frac{|r_j + r_{j+1}||r_j - r_{j+1}| + |x_j - x_{j+1}||r_j + r_{j+1}|}{\frac{2\sqrt{2}\sqrt{|\mu|^2-1}}{3|\mu|+1}|r_j + r_{j+1}|} \\
&= 2(r_j - r_{j+1}) + \frac{3\mu + 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}\left[|r_j - r_{j+1}| + |x_j - x_{j+1}|\right] \\
&= |r_j - r_{j+1}|\left[2 + \frac{3\mu + 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}\right] + \frac{3\mu - 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}|x_j - x_{j+1}|.
\end{aligned}
$$

Now, if we consider a line through the centers of the semi-circles SC_τ^j and SC_τ^{j+1} and use the equation (13), we can get the following result:

$$
r_j - r_{j+1} < d_j + \zeta.
$$

Again, from our supposition we have

$$
|x_j - x_{j+1}| < d_j.
$$

Therefore

$$Z_{j\tau} \; < \; (d_j + \zeta)\Big[2 + \frac{3\mu + 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}\Big] + \frac{3\mu - 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}d_j.$$

Finally, from equation (13), if we put the value of d_j, the above result reduces to

$$Z_{j\tau} \; < \; \Big[8 \times \frac{3\mu + 1}{\mu - 1} \times \{1 + (\mu + 1)^{(1.35)} + (\mu + 1)^{(2.7)}\}^2 \Big(1 + \frac{3\mu - 1}{2\sqrt{2}\sqrt{\mu^2 - 1}}\Big) + 1\Big]\zeta,$$

which is the required one.

So, Lemma 4.4 is proven. □

The following lemma deals with the value of sufficiently very small positive τ and the upper bound of the gaps between the semi-circles in the set SC_τ^*.

Lemma 4.5. *For every sufficiently small positive real number $\tau < 10^{-17}$ and for every positive integer, where $1 \leq j < I^+$, the value of $Z_{j\tau}$ is less than $\frac{1}{50}$.*

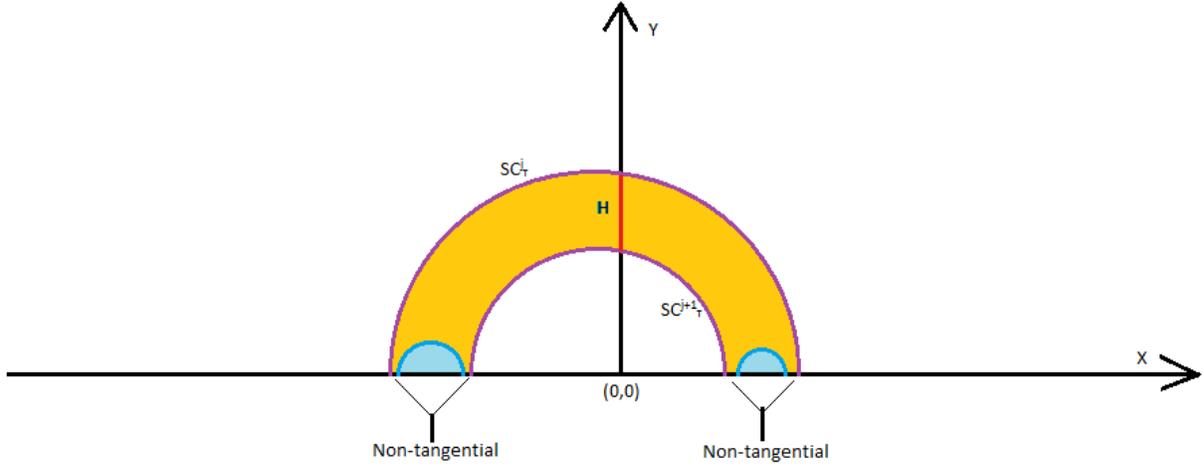*Figure : 7*

Proof. Assume that $F^j_{\Gamma^N_{S_\tau}}$ (see the gold-yellow color region in Figure 7) is a fundamental domain for the group $\Gamma^N_{S_\tau}$. Let, $V_{j\tau}$ (see, between the purple color semi-circular arcs, i.e., the gold-yellow + turquoise color areas) be the doubly connected domain encircled by the semi-circles SC^j_τ and SC^{j+1}_τ within the upper-half plane. Note that, apart from the two semi-circular curves SC^j_τ and SC^{j+1}_τ, there are also two semi-circles that are designed to form the fundamental domain $F^j_{\Gamma^N_{S_\tau}}$. Therefore, we are getting exactly one fragment, viz., H (see the red color vertical segment) of $\{V_{j\tau} \cap i\mathbb{R}^+\}$ that overlap in the segment $\{F^j_{\Gamma^N_{S_\tau}} \cap i\mathbb{R}^+\}$, where H is placed within the hyperbolic plane (between the two semi-circles SC^j_τ and SC^{j+1}_τ). So, the vertical piece H is also located in the component of $\{\Omega(h_\tau^{**}\Gamma^N_{S_\tau}(h_\tau^{**})^{-1}) \cap i\mathbb{R}^+\}$ meeting the vertical closed interval $i[0, (\mu + 1)^{(3.7)}]$.

However, from Lemma 4.3, we obtain that the length of the piece H is less than the subsequent quantity

$$\left\{ \frac{(2.01)}{\sqrt{\mu}(\mu - 1)^2} \times (\mu + 1)^{(2.7)} \times \sqrt{2} \times \sqrt{\tau} \times \sqrt{2\mu^5 - \mu^4 - 3\mu^3 + 6\mu^2 - 8\mu + 1} \right\}.$$

Now, set $\tau < 10^{-17}$, this yields $Z_{j\tau} < \frac{1}{50}$, which is our required value of $Z_{j\tau}$.

This completes the proof of Lemma 4.5. $\square$

5. PROOF OF THE MAIN THEOREM

After proving all the necessary lemmas in Section 4, we are now ready to establish the main theorem of this article.

Proof of Theorem 1. Recall that our plan is to prove this theorem by the method of contradiction. Let SC_{τ^*} be a semi-circular curve meeting the interval $[-(\mu + 1)^{(3.7)}, -(\mu + 1)]$ that is identical to the loop $R_{(\mu-1),\tau}$ for the group generated by the function $(h_\tau^{**})^2$. The loop $R_{(\mu-1),\tau}$ segregates the fixed points '0' and '∞' of the transformations h_τ^{**}. Hence, the semi-circular curve SC_{τ^*} also separates these fixed points. Since the semi-circular curve SC_{τ^*} is outside the semi-circle $SC_{-\mu,\tau}$, its image $h_\tau^*(SC_{\tau^*})$ will touch the interval $[\mu - 1, \mu + 1]$ twice. Again, the fixed point ∞ and the points $\left\{ -(\mu + 1)^{(3.7)} \right\}$ are outside of the semi-circular curve SC_{τ^*}. Therefore, the points $h_\tau^*(\infty)$ and $h_\tau^*\left\{ -(\mu + 1)^{(3.7)} \right\}$ will be kept apart by the curve $h_\tau^*(SC_{\tau^*})$ and situated within this semi-circle by lying on the real axis. Consequently, the diameter of $h_\tau^*(SC_{\tau^*})$ will be greater than $\frac{1}{50}$, since

$$
\begin{aligned}
\left| h_\tau^*\left\{ -(\mu + 1)^{(3.7)} \right\} - h_\tau^*(\infty) \right| &= \left| \frac{(1 - \tau)^2}{(\mu + 1)^{(3.7)} + \mu} \right| \\
&< \left| \frac{1}{(\mu + 1)^{(3.7)} + \mu} \right|_{(\tau \to 0^+ \text{ or } 1^-)} \\
&> \frac{1}{50}.
\end{aligned}
$$

However, Lemma 4.5 entails that the gaps between the semi-circular curves in the set SC_τ^* are less than $\frac{1}{50}$. Consequently, $h_\tau^*(SC_{\tau^*})$ meets some of the SC_τ^j in the set SC_τ^*. Hence, all images of the curves $SC_{-\mu,\tau}$, $SC_{\mu,\tau}$, $R_{(\mu-1),\tau}$, and $R_{(\mu+1),\tau}$ are not disjoint, which is not possible. So our assumption is wrong. Therefore, the Schottky group $\Gamma_{S_\tau}^N = <h_\tau^*, h_\tau^{**}>$ is a non-classical semi-Fuchsian Schottky group equipped with our presumed generating set. This establishes the main goal of this article. $\square$

6. AN EXAMPLE OF A SEMI-FUCHSIAN SCHOTTKY GROUP WITH A NON-CLASSICAL GENERATING SET

In Sections 3, 4, and 5, we presented the structure of the rank 2 semi-Fuchsian Schottky group endowed with a non-classical structure by establishing Theorem 1. Now, in Theorem 1 if we put $\mu = \sqrt{3}$ in the hyperbolic elements (1) and (2) in Section 1, we get the following Möbius transformations:

$$h_{\tau^*}^1 \ (\tau^* \to 0^+) : z \ \longrightarrow \ \frac{\sqrt{3}(1-\tau^*)^{-1}z + (1-\tau^*)\{3(1-\tau^*)^{-2}-1\}}{(1-\tau^*)^{-1}z + \sqrt{3}(1-\tau^*)^{-1}} \tag{14}$$

$$\text{and } h_{\tau^*}^2 \ (\tau^* \to 0^+ \text{ or } 1^-) : z \ \longrightarrow \ \Big(\frac{2 + |\pm 2i(1-\tau^*)| + (1-\tau^*)^2}{(\sqrt{3}-1)^2 + (1-\tau^*)^2}\Big)z. \tag{15}$$

For $\mu = \sqrt{3}$, we have already examined in Theorem 1 that if we take the value of τ^* to be less than 5×10^{-18}, then the semi-Fuchsian Schottky group $\Gamma_{S_{\tau^*}}^N$ generated by the hyperbolic transformations $h_{\tau^*}^1$ and $h_{\tau^*}^2$ is non-classical in the upper-half plane model. Hence, the proof of Corollary 1.0.1 is done.

Therefore, Corollary 1.0.1 leads to a non-trivial example of the rank 2 semi-Fuchsian Schottky group equipped with a non-classical Schottky structure within the hyperbolic plane without the boundary $\mathbb{R} \cup \{\infty\}$.

Open Problem 1. Does there exist any rank 2 semi-Fuchsian Schottky group with non-classical generating sets where the two generators, one is parabolic and the other is hyperbolic?

Open Problem 2. Are the geometries of hyperbolic surfaces coming out from the diagram of certain semi-Fuchsian Schottky groups proposed in Section 1, viz., Figure $\{1(b), 1(c), \text{and} 1(p)\}$; Figure $\{1(d), 1(e), 1(m), \text{and } 1(n)\}$ analogous or different?

Open Problem 3. Do all fundamental domains (mentioned in Section 1, Figure $1(a), 1(b), 1(c), ..., 1(p)$) give rise to Dirichlet domains in the hyperbolic plane?

Open Problem 4. Do the hyperbolic torus with one infinite end appearing in Figure $1(g)$, Figure $1(j)$, Figure $1(k)$, and Figure $1(l)$ (see Section 1) resemble or are distinct?

7. DATA AVAILABILITY STATEMENT

Our manuscript has no associated data.

8. STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest. The authors have no relevant financial or non-financial interests to disclose.

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