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A Primitive Notion Versus the non-Archimedean Paradigm: Introducing Homogeneous Infinitesimals

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Abstract

Infinitesimals have long been considered to be non-Archimedean, in that any number times an infinitesimal cannot equate to a large real number. In this paper I present a proof by contradiction that infinitesimals, if defined as homogeneous, can falsify this claim.

Key words: keyword1, Keyword2, Keyword3, Keyword4

Introduction

The Archimedean axiom[1], often written as

$$na > b, \tag{1}$$

is a fundamental algebraic property, where a multiple n of a is said to exceed b. However, this property is said not to apply to infinitesimals (i.e., $a = dx$), making them "non-Archimedean". This has been said to change the inequality so that it is sometimes described using[2]

$$ndx \leq 1, \tag{2}$$

which comes from

$$(\exists dx > 0)(\forall n \in \mathbb{N})[dx \leq \frac{1}{n}]. \tag{3}$$

This has been also described as "the sum (of infinitesimals) is always infinitely small" [3]. In simpler words, an equation such as

$$ndx = 12 \tag{4}$$

is not a valid equation in any known mathematics. If infinitesimals are still desired, then it has been said that they "enrich" real numbers, and this is sometimes called a "non-Archimedean continuum". In this paper, I dispute Equation 2 via a new homogeneous concept of infinitesimals in which their magnitudes and cardinality are only defined relative to each other. This will give us a direct counterargument to Equation 2.

[1] see Equation 2.1 in [2]

[2] from Equation 4 in [2]

Definition 1 Let

$$n_a dx_a \leq n_b dx_b, \tag{5}$$

define *finite length* A relative to B, written as

$$A = n_a dx_a \leq n_b dx_b = B. \tag{6}$$

Here, n is a transfinite cardinality and dx is defined as a primitive notion of infinitesimal length[3] for postulates. From this definition let us define not only length but also area and volume.

Definition 2 Let

$$n_a dx_1 = \textbf{finite length}, \tag{7}$$

$$n_a(dx_1 dx_2) = \textbf{finite area}, \tag{8}$$

and

$$n_a(dx_1 dx_2 dx_3) = \textbf{finite volume} \tag{9}$$

and so forth.

This allows new mereological and conceptual insights[4]. I can now prove, through an example provided by Evangelista Torricelli[5], that Leibniz's transcendental law of homogeneity [10] is fundamentally flawed. Simply put, Leibniz's technique gives an incorrect answer because area must be homogeneous (meaning that a finite area must be composed of a transfinite number of elements of area) when using infinitesimals. His equation, written as ([12] p. 147)

$$\boldsymbol{\Delta}\textbf{area} = d(xy) = xdy + ydx + dxdy, \tag{10}$$

gives the incorrect third term $dxdy$, whereas the proposed concept gives only the correct two,

$$\boldsymbol{\Delta}\textbf{homogeneous area} = d(n_a(dxdy)) = n_x dx(dy) + n_y dy(dx). \tag{11}$$

This concept also explains why both relative magnitudes, such as dx and the cardinality n, are required to describe the relationship between finite measures. We should not count the number of points; instead, we should count the relative number of elements of length, area, and volume. Whereas set theory can not get past Equation 3, this paper will demonstrate particular new solutions for a set composed of infinitesimals.

Definition 3 Let

$$S = \{\{dx_1, dx_2, dx_m\}_1, \{dx_1, dx_2, dx_m\}_2, \{dx_1, dx_2, dx_m\}_n\}, \tag{12}$$

such that the

$$\sum_{dx_m \in S}^{n} dx_m \tag{13}$$

equals length for $m = 1$, area for $m = 2$, volume for $m = 3$, and so forth.

The n (relative cardinality) is required because (if all the elements dx_m are equal) the sum with the larger n is the greater length, area, volume, etc.

It may seem logically out of order to present Definition 1 before Definition 3 but the conceptual chain and proof that starts from Equation 2 needs to be presented first.

[3] Although it is often stated that primitive notions should be obvious, I argue that the result of this paper demonstrates that this is a false assumption. See Section A

[4] It may be helpful to think of infinitesimals as either having no shape or being able to conform to any shape.

[5] This example is well known enough to be the very first figure of [6].

Background

Infinitesimals, which are no longer in vogue in mathematics due to the success of real analysis (and in spite of non-standard analysis [14]), have always been associated with some paradoxes. Some say they have been "purged" [7] and "banished" [11] in part due to these unanswered questions. One of the original paradoxes was the homogeneous/heterogeneous viewpoint of the 17th century. This argument centered on whether geometrical sums were made up of elements of the same dimension (homogeneous) or one dimension less (heterogeneous). The simplest example was whether lines were made up of infinitesimal one-dimensional segments of lines (homogeneous ([9] p.4)) or non-dimensional points (heterogeneous)[6]. Analogously, it was also debated whether an area is composed of infinitesimally thin slices of area or one-dimensional stacked lines and whether a volume is composed of infinitesimally thin sheets of volume or stacked two-dimensional planes. Evangelista Torricelli's (1608–1647) analysis of the heterogeneous/homogeneous debate [5] landed him firmly on the infinitesimal segment side as recent authors have pointed out ([1] and [9] p. 125).

Francois De Gandt, recognized as a leading expert on Torricelli's work, has called one particular example of Torricelli's a "condensed" "fundamental example" for his view on the heterogeneous/homogeneous paradox ([5] p.164). Therefore let us take his example and start by examining the historical context so that we have a firm footing for the homogeneous/heterogeneous argument from which we can develop a proof by contradiction.

Historical viewpoint of Torricelli's parallelogram paradox

We could initially recreate the historical explanation for Torricelli's parallelogram involving lines of non-zero infinitesimal width, but let us first consider the simplest observation: how to determine the number of points in a line.

Proposition 1 *Assume that a line is made of points (instead of a line having points on it) and that the number of points in a line determines its length. Two lines that are of the same length have the same number of points. A shorter line has fewer points and a longer line has more points.*

Proof By Contradiction

Proof Take a parallelogram with the four corner points labeled A, B, C, and D, as shown in Figure 1. Draw a line BD down the diagonal, and create a point E on the diagonal line BD. Now draw perpendicular lines from E to a point F on AD, and a second line to a point G on CD. Move these two lines point by point simultaneously so that E moves toward B until they meet, keeping the lines EF and EG always parallel to AB and BC, respectively. When we move the lines EF and EG, we are moving their ends simultaneously from point to point on AD, CD, and BD.

If line AD is shorter than line CD, the number of points that line AD contains is less than the number of points that line CD contains. However, this creates a paradox. Since we moved the lines point by point and both points F and G ended up together at point B, this shows that lines AD and CD must also have the same number of points, as shown in the equations in Figure 2. □

The simplest interpretation of Torricelli's meaning is that the lines AD and CD must be made up of infinitesimal segments (and not dimensionless points) and that there must be an equal number of these segments in each line, even if they are not of the same magnitude.

The most advanced current explanation ([9] p. 125) for this paradox is the "understanding that the amount of elements of an infinite set, that is its cardinality, is different from its measure." While I very much agree with these properties, let us introduce a new tool to gain more insight.

[6] Note that this is in contrast to the philosophical view that a line exists and a point "lies on" that line.

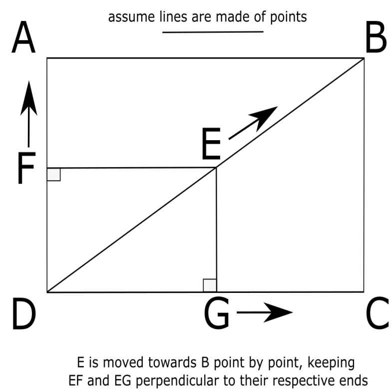

Fig. 1: Torricelli: moving perpendicular lines point by point.

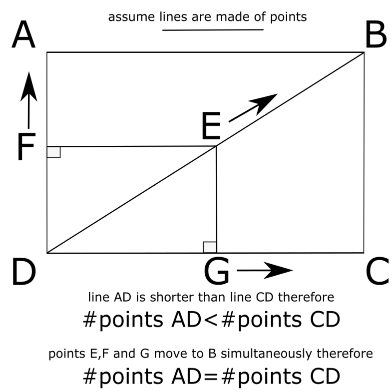

Fig. 2: Torricelli: equal numbers of points or not?

Choice of cardinality and magnitude

First, let us define a common starting paradigm. Imagine a single one-dimensional finite line segment composed solely of adjacent infinitesimal line segments SEG. The sum of their magnitudes $|SEG|$ is defined as the "length" of the finite line segment. This is very similar to the original geometric concept in calculus of taking a singular line segment and dividing it into smaller and smaller segments an "infinite" number of times until these segments of length become "infinitesimals" (naturally, their sum is still the original finite length). We impose that the magnitude of the infinitesimals of this line is consistent with Eudoxus' theory of proportions in that their *magnitudes are only measurable relative to another infinitesimal* (see Equations 15 and 16). Thus, their sums, or finite line lengths, are also in agreement with Bernhard Riemann's definition [13] that the length of every line is "measurable by every other line" (although his definition does not include any mention of the infinitesimals that we are considering), and we can write

$$\sum |SEG| \equiv \textbf{line length}. \tag{14}$$

In the interest of utilizing a graphical teaching method, let us use Figure 3 to represent this line segment.

Suppose that the magnitude $|SEG^n|$ of a segment n could either be of equal relative magnitude (as in Equation 15) to another segment $n-1$ or could have a different value (as in Equation 16) *even within the line itself.*

Fig. 3: Graphical segments of length representing magnitudes of infinitesimals of length: Intrinsically flat.

Definition 4 Let

$$|SEG^n| - |SEG^{n-1}| = 0, \tag{15}$$

be defined as *intrinsically flat.*

Let us define a "point" as simply an infinitesimal SEG that is of null magnitude in the direction along the line, so that we can also understand that Euclid's definition of a straight line (Euclid's Elements, Book I, Definition 4) is one that "lies evenly with the points upon itself". In this case, both terms are equal or even (with a point between the two segments and at their respective ends).

We can then propose that infinitesimals can have non-equal relative magnitudes.

Definition 5 Let

$$|SEG^n| - |SEG^{n-1}| \neq 0 \tag{16}$$

be defined as *intrinsically curved.*

The points are no longer equally spaced, as represented by Figure 4.

Fig. 4: Graphical segments of unequal length representing unequal infinitesimal magnitudes: Intrinsically curved.

Now also suppose that we can examine the number of segments in one line versus another. We call this number the *relative cardinality* (n) so that we can write (in the simplifying case that the line is intrinsically flat) the equation

$$\sum |SEG| = n * |SEG| \equiv \textbf{line length}. \tag{17}$$

Let us assume that n has similar properties to Cantor's transfinite numbers [8], in that n is an "infinite" cardinal number, but that cardinality can be greater than, equal to, or less than the value of another n. Let us now rename $|SEG|$ as dx,

$$dx \equiv |SEG|, \tag{18}$$

so that we have

$$\sum_{1}^{n} |SEG| = n * dx \equiv \textbf{line length}. \tag{19}$$

This is where we arrive at the discrepancy with the concept of infinitesimals being non-Archimedean. I am not aware of any research that has considered the ramifications of letting n be transfinite. By this, I mean the following equations can be proven to be logically true for mapping real numbers onto sums of homogeneous infinitesimals. This means we can write

$$n_a dx_a \leq n_b dx_b, \tag{20}$$

which we will assume for now can be used to represent the real number equations

$$5 = 5 \tag{21}$$

or

$$4.5 < 9. \tag{22}$$

Example 1 *Let*

$$n_a dx_a = n_b dx_b = 5 \tag{23}$$

with $n_a = n_b$ *and* $dx_a = dx_b$.

If we double n_a *but cut* dx_a *in half, then*

$$\frac{n_a}{n_b} = 2 \tag{24}$$

and

$$\frac{dx_a}{dx_b} = \frac{1}{2} \tag{25}$$

and the equation $n_a dx_a = n_b dx_b = 5$ *still holds true.*

It is important to understand that if we instead wrote Equation 20 as

$$A = n_a dx_a, \tag{26}$$

$$B = n_b dx_b, \tag{27}$$

$$A \leq B, \tag{28}$$

this would mask the relativity of their constituent infinitesimals, which I will call a new axiom.

Axiom of N–M Choice

Definition 6 Let the *Axiom of N–M Choice* (ANMC) define that there is an inherent choice made when two finite sums are compared. This is the relative choice of the number n of elements and the magnitude M of those elements.

Note that I say *elements* since there is not only the primitive notion (see Section A) of "infinitesimal length" but also "infinitesimal area", "infinitesimal volume", and so forth. What is commonly considered to be a line[7] is defined as the sum of infinitesimal elements of length, area is defined as the sum of infinitesimal elements of area, volume is defined as the sum of infinitesimal elements of volume, etc. The sums (length, area, volume) are homogeneous in that their composition is strictly made of elements of the same "dimension". Area cannot be composed of elements of length, as will be logically proven from the primitive notions and postulates. This is what defines a sum as being "homogeneous". Thus, we have

$$n_a dx_a \leq n_b dx_b \tag{29}$$

for sums of elements of length (there are n_a elements of length on the left and n_b elements of length on the right),

$$n_a(dx_{1a}dx_{2a}) \leq n_b(dx_{1b}dx_{2b}) \tag{30}$$

for sums of elements of area, and

$$n_a(dx_{1a}dx_{2a}dx_{3a}) \leq n_b(dx_{1b}dx_{2b}dx_{3b}). \tag{31}$$

for sums of volumes (there are n_a elements of volume on the left and n_b elements of volume on the right). If the elements are all of the same magnitude, then the sum with the most elements has the longest length, the largest area, the largest volume, etc. I will limit this scenario to three terms for now (due to the nature of the problem at hand), but it is by no means limited to only three. Note that in each case, n_a represents the

[7] I say "commonly" because I will introduce "lineal" lines, "areal" lines, "voluminal" lines, etc.

total number of elements and not the constituent elemental directional magnitudes dx. Thus, we can write for the set

$$\begin{aligned} S = \{&\{dx_1, dx_2, dx_m\}_1, \\ &\{dx_1, dx_2, dx_m\}_2, \\ &\{dx_1, dx_2, dx_m\}_n\} \end{aligned} \tag{32}$$

that

$$\sum_{dx_m \in S}^{n} dx_m \tag{33}$$

equals length for $m = 1$, area for $m = 2$, volume for $m = 3$, and so forth. The n is required since, for example, if all the elements are equal, then the sum with the larger n is the greater length, area, volume, etc.

Example 2 *For a square, we could write*

$$n_a(dx_{1a}dx_{2a}) = (n_{1a}dx_{1a})(n_{2a}dx_{2a}) = (n_{1a2a}dx_a)^2 \tag{34}$$

since

$$n_a = (n_{1a})(n_{2a}) = (n_{1a2a})^2. \tag{35}$$

The two terms $(n_{1a2a}dx_a)$ *would represent the length of the sides of the square since*

$$(n_{1a}dx_{1a}) = (n_{2a}dx_{2a}). \tag{36}$$

Since there does not appear to be prior research equivalent to Equation 3 to draw from I have named this overall concept and the resulting philosophy the *Calculus, Philosophy, and Notation of Axiomatic Homogeneous Infinitesimals* (CPNAHI). Although it is a mouthful, CPNAHI seems to adequately cover the breadth of the concept. Also note that I do not use standard mathematical notation such as R^n for "real space" since there appears to be a conceptual distinction between CPNAHI and real analysis that I have only briefly explored.

Definition of lines and points

I will now briefly define the concepts of lineal lines, areal lines, voluminal lines, etc.

Definition 7 A lineal line is defined as a path of adjacent infinitesimal elements of length, and a lineal line point is just a null infinitesimal (zero dimensions).

Definition 8 An areal line is defined as a path of adjacent infinitesimal elements of area. An areal line point is defined as an element of area that is null perpendicular to the path (one dimension).

Definition 9 A voluminal line is defined as a path of adjacent infinitesimal elements of volume. A voluminal line point is an element that is null perpendicular to the path (two dimensions).

These definitions provide a new method for examining Evangelista Torricelli's parallelogram paradox ([1] and [9]).

Analysis of lines in Torricelli's parallelogram using the ANMC

Intrinsically flat lineal lines curved with respect to another line

Using the ANMC, we can assign equations to describe Torricelli's parallelogram.

Proposition 2 *Torricelli's example uses a rectangle so we will define relative line lengths as*

$$BD > CD > AD. \tag{37}$$

Proposition 3 *Since Torricelli's example states that lines BD, CD, and AD are each traversed one segment at a time, then the number of segments in each can be thought of as equal on a one-to-one basis* [8]*, and we can write*

$$n_{BD} = n_{CD} = n_{AD}. \tag{38}$$

Proposition 4 *Since the segments in any line are all equal, all adjacent segments in each line are equal, and we can write*

$$dx^n_{BD} - dx^{n-1}_{BD} = 0 \tag{39}$$

$$dx^n_{CD} - dx^{n-1}_{CD} = 0 \tag{40}$$

$$dx^n_{AD} - dx^{n-1}_{AD} = 0. \tag{41}$$

Proof Since

$$n_{BD} dx_{BD} = BD > n_{CD} dx_{CD} = CD > n_{AD} dx_{AD} = AD, \tag{42}$$

the magnitudes of the infinitesimal segments of each line in relation to each other must be written as

$$dx_{BD} > dx_{CD} > dx_{AD}. \tag{43}$$

Thus, the segments of line BD have a magnitude greater than that of CD, and similarly for AD. □

Figure 5 is a visual aid for understanding the previous equations. If, by the property of congruence, we can lay the lines BD, CD, and AD next to each other and they are of unequal length, let us then imagine that we can use the vertical dividing lines to denote the infinitesimal segments within each line. Torricelli's example is presented to show that the magnitudes of the segments within BD, CD, and AD are all the same (intrinsically flat) within their respective lines. However, the magnitudes of the segments within BD must not be the same as CD, nor AD. Again, this is what Torricelli meant when he said that points are indistinguishable, whereas segments can differ by their magnitude. They are intrinsically curved relative to each other. We can then also understand that the cardinality within AD must be the same as BD, as well as for CD, which was chosen when we opted to move the perpendicular lines "point by point". Thus, for example, line BD would be written as

$$S_{BD} = \{\{dx_1\}_1, \{dx_1\}_2, \{dx_1\}_n\} \tag{44}$$

and comparing the sum of BD versus AD would give

$$n_{BD} = n_{AD}$$

$$(dx_1)_{BD} > (dx_1)_{AD}$$

$$\sum_{dx_1 \in S_{BD}}^{n_{BD}} (dx_1)_{BD} > \sum_{dx_1 \in S_{AD}}^{n_{AD}} (dx_1)_{AD}. \tag{45}$$

Intrinsically flat lineal lines with relative flatness

Proposition 5 *Imagine now that we have again taken lines BD, CD, and AD and laid them next to each other. We again traverse the segments on the line one for one, but this time we do so in equal magnitude increments, as shown in Figure 6. Line BD is still longer, but only because the relative cardinality (the number of segments) is the greatest, as seen in Figure 6.*

[8] We could even argue that this should be the definition for set theory.

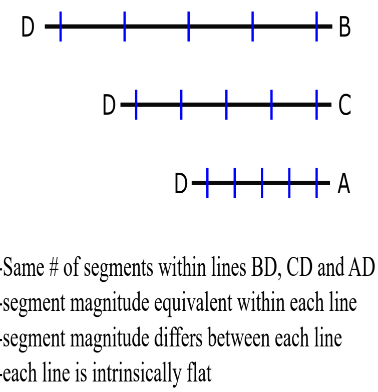

Fig. 5: Intrinsically flat lineal lines with equal cardinality and relative curvature.

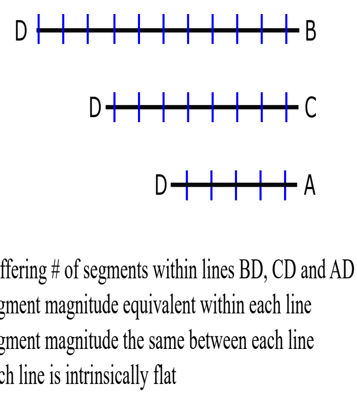

Fig. 6: Intrinsically flat lines with differing cardinality and relative flatness.

Proof We can then write

$$n_{BD} > n_{CD} > n_{AD} \tag{46}$$
$$dx_{BD}^{n} - dx_{BD}^{n-1} = 0 \tag{47}$$
$$dx_{CD}^{n} - dx_{CD}^{n-1} = 0 \tag{48}$$
$$dx_{AD}^{n} - dx_{AD}^{n-1} = 0 \tag{49}$$

$$dx_{BD} = dx_{CD} = dx_{AD} \tag{50}$$

□

Relative curvature and flatness via the ANMC

Remark 1 *From these two simple examples, we can observe the axiom in that we have the choice to make the inequalities*

$$n_{AD} * dx_{AD} < n_{CD} * dx_{CD} < n_{BD} * dx_{BD} \tag{51}$$

true by either making n *all equal and varying* dx *or vice versa. The following represents the same meaning using summation notation (and dropping the "1" directional subscript for the respective lines):*

$$\sum_{dx_{AD} \in S_{AD}}^{n_{AD}} dx_{AD} < \sum_{dx_{CD} \in S_{CD}}^{n_{CD}} dx_{CD} < \sum_{dx_{BD} \in S_{BD}}^{n_{BD}} dx_{BD}. \tag{52}$$

Defining scale factors: Sum of cardinality vs sums of HIs

Notation provides economy of thought, but that notation can have low value if it mischaracterizes the underlying geometry it represents. Let us see what that means by examining sums of infinitesimals and defining scale factors.

Sums of infinitesimals

If line BD is twice as long as AD, then from Figure 5, we could view line BD as the summing of two lines AD, where the magnitudes of each element in one AD are summed with the magnitude of a corresponding element in the other AD ($dx_{AD} + dx_{AD} = dx_{BD}$, element by element). From Figure 6, we could view line BD as the summing of two lines AD, where the cardinality of one line AD is summed with the cardinality of the other line AD, creating a longer line, BD, with larger cardinality ($n_{AD} + n_{AD} = n_{BD}$).

Simply put, this means we have two choices here for the summing of line lengths. We can either sum their elements so that the cardinality stays the same but each element is bigger, or keep the same magnitude for the elements and double their number.

Euclidean scale factor

Now let us consider the common geometric conception that if $A = 4$ is written as a representation of the length of a line, then a scale factor $k = 3$ would give

$$kA = 12. \tag{53}$$

One could say that if we have another line $B = 12$, then k could be defined as

$$k = \frac{B}{A}, \tag{54}$$

as can be seen in Figure 7. However, this conceals something geometrical that can be fleshed out with the

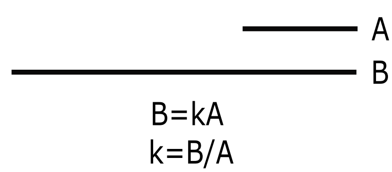

Fig. 7: The common understanding of the scale factor for a simple one-dimensional line.

ANMC.

Euclidean scale factor defined as a quotient of relative cardinalities

Proposition 6 *Let us take Equation 29 and relabel it to compare the length of a line to a "reference" line,*

$$n_a dx_a \leq n_{ref} dx_{ref}. \tag{55}$$

Let

$$n_a dx_a = 4 = A, \tag{56}$$

$$n_{ref} dx_{ref} = 12 = B \tag{57}$$

and

$$dx_a = dx_{ref}. \tag{58}$$

Definition 10 We then define a *relative cardinality* scale factor k as

$$k_{RC} = \frac{n_{ref}}{n_a} = 3. \tag{59}$$

Proof Thus, writing

$$(k_{RC} n_a) dx_a = 12 = B \tag{60}$$

means that line A has $\frac{1}{3}$ the cardinal number of elements of magnitude as the reference line B, as shown in Figure 8. Line B is scaled to three times the length of line A. □

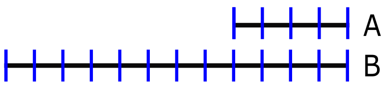

Fig. 8: The scaling of the relative cardinality.

The scale factor defined as a quotient of relative magnitudes

Proposition 7 *The opposing case allowed by the ANMC here is a scaling of the magnitude dx. Again, if I have Equation 29 and relabel it so that I am comparing the length of a line to a "reference" line,*

$$n_a dx_a \leq n_{ref} dx_{ref}. \tag{61}$$

If

$$n_a dx_a = 4 = A \tag{62}$$

and

$$n_{ref} dx_{ref} = 12 = B, \tag{63}$$

but instead this time

$$n_a = n_{ref}. \tag{64}$$

Definition 11 Let us define a *relative magnitude* scale factor j_{RM} as

$$j_{RM} = \frac{dx_{ref}}{dx_a} = 3. \tag{65}$$

Proof Thus, writing

$$n_a (j_{RM} dx_a) = 12 = B \tag{66}$$

means that line A has infinitesimal elements with $\frac{1}{3}$ of the magnitude of those of the reference line B, as shown in Figure 9. Again, line B is scaled to three times the length of line A. □

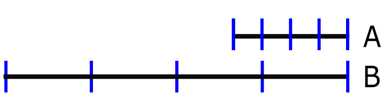

Fig. 9: The scaling of the relative cardinality.

ANMC and area

Before we analyze Torricelli's parallelogram further, let us understand the ramifications of scale factors on area. As with Figures 5 and 6, it should be obvious that we can sum together numbers of elements of length to create a longer line or sum together the magnitudes of elements of length to create a longer line.

In the same vein, there are two methods for summing together two areas, either by their cardinality of elements or by the magnitude of the elements (or some combination of both), as shown in Figure 10 (note that the figures are not exact in terms of cardinality; this is an inherent flaw in using graphical proofs for homogeneous infinitesimals). For the upper sum of area, this is a sum of cardinalities, and we could write

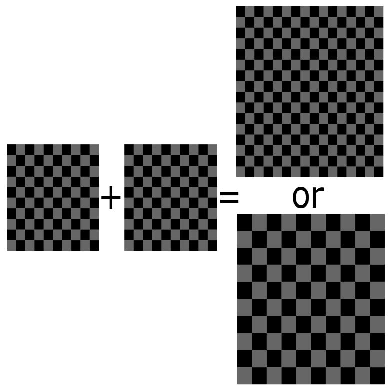

Fig. 10: The summation of areas either via cardinality or magnitude.

$$n_a(dx_1dx_2) + n_b(dx_1dx_2) = (n_a + n_b)(dx_1dx_2). \tag{67}$$

The lower sum of area is a sum of magnitudes and can be written as

$$n(dx_{a1}dx_{a2}) + n(dx_{b1}dx_{b2}) = n(dx_{a1}dx_{a2} + dx_{b1}dx_{b2}). \tag{68}$$

Remark 2 *Note that summation notation*

$$\sum_{dx_1,dx_2\in S}^{n} dx_1, dx_2 \tag{69}$$

does not indicate whether we are summing the magnitudes of elements of area or their cardinality, so we will stick with the previous form to emphasize a difference with Leibniz's notation later.

As with elements of length, we could view this as scaling either the cardinality of elements of area or their magnitudes, such that we can write

$$kn(dx_1dx_2) = (n_1 + n_2 + n_k)(dx_1dx_2), \tag{70}$$

using k to scale the cardinal number of the elements or j in

$$n(j(dx_{a1}dx_{a2})) = n((dx_{a1}dx_{a2})_1 + (dx_{b1}dx_{b2})_2 + (dx_{b1}dx_{b2})_j) \tag{71}$$

to scale the magnitudes of each of the elements of area.

Analysis of the area of Torricelli's parallelogram using ANMC

Now that we have an understanding of area, let us take Torricelli's parallelogram, remove some of the notation, and split up the opposing triangles, as in Figure 11. We have left point E so that we still have some visual reference to the diagonal line in the parallelogram. Without proof, assume that $Area\ 1 = Area\ 2$ and that $Area\ 3 = Area\ 4$.

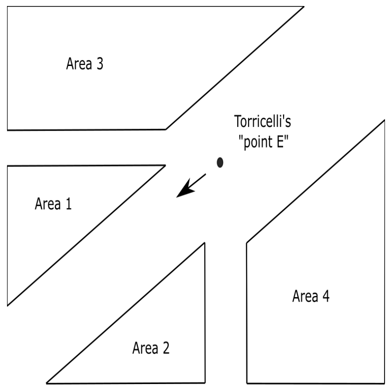

Fig. 11: Torricelli's parallelogram divided up with labeled areas.

We might mentally understand that as E "moves" toward the top right, the area in $Area\ 3$ and $Area\ 4$ decreases while the area in $Area\ 1$ and $Area\ 2$ increases. We may also understand that the previous equalities still hold, in that $Area\ 1 = Area\ 2$ and $Area\ 3 = Area\ 4$. Let us add a concept from ANMC to our graphic to help us understand why this is.

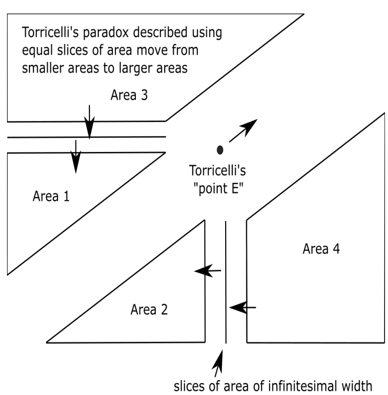

Fig. 12: Torricelli's parallelogram with area being removed from $Area\ 3$ to $Area\ 1$ and from $Area\ 4$ to $Area\ 2$ via slices of area.

Proposition 8 *Using Figure 12 and the definition of areal lines, we can extract the concept that infinitesimal slices of area (areal lines) are being moved from Area 3 to Area 1 and from Area 4 to Area 2. Since the magnitude relationships between the top and bottom areas are constant, let us claim the slices of area in the top and bottom, as well as their cardinality on a one-to-one basis, must also be equal.*

Proof Let us now rotate the top areas as a triangle so that the slices of area for both are vertical in our graphic, as in Figure 13. If we view these slices of area instead as columns of elements of area, we can draw

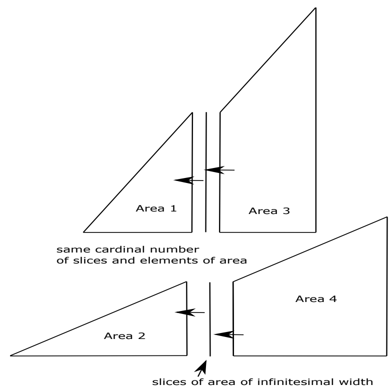

Fig. 13: Torricelli's parallelogram with area being removed from *Area* 3 to *Area* 1 and from *Area* 4 to *Area* 2 via vertical slices of area.

Figure 14.

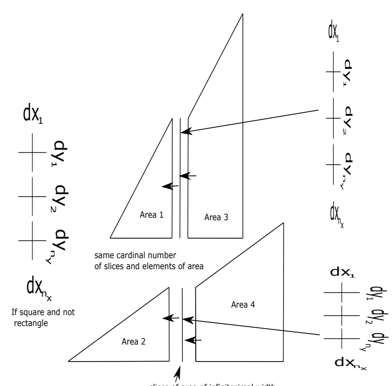

Fig. 14: Torricelli's parallelogram with area being moved from *Area* 3 to *Area* 1 and from *Area* 4 to *Area* 2 via columns of vertical infinitesimal elements of equal area.

In the graphic, note that there are three elements of area shown in the column (and not six elements of length) to conceptually represent the cardinality. We can also see that the elements of area in the top triangle have the same "height" as the "width" of the elements in the bottom triangle and vice versa. We can also see that the top and the bottom have the same cardinal number of slices (and we can assume, number of elements). If we are not averse to counting the number of elements along the bottom of each triangle, we can see that they, too, have the same cardinal number. □

Example 3 *It will be helpful to represent these columns of area with Figure 15, where we have chosen real number values to represent the lengths of the sides and bottoms of the top and bottom triangles. Note that* *we can write that each column of area in the top is equal to each column of area in the bottom,*

$$(n_{ytop}dy_{top})dx_{top} = (n_{ybot}dy_{bot})dx_{bot}. \tag{72}$$

The height of the labeled top column is

$$n_{ytop}dy_{top} = 1 \tag{73}$$

and the height of the labeled bottom column is

$$n_{ybot}dy_{bot} = \frac{1}{2}. \tag{74}$$

The bottom column is twice as "thick" as the top column, and the bottom column has half the height of the top column since they possess equal infinitesimal areas. It is important to note that we are measuring the height of a column of elements of area and not the length of a one-dimensional line! It is also important to note that each element is intrinsically curved within itself and not flat (the dx and the dy are not of equal magnitude).

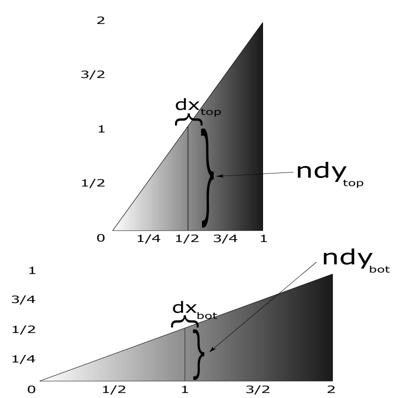

Fig. 15: Torricelli's parallelogram composed of columns of elements of area.

Comparison of the Torricelli ANMC solution with the Leibniz area solution

We can now finally turn to Leibniz's transcendental law of homogeneity. His example is essentially the same as Torricelli's if we think of both as asking, "What is a logically and notationally correct method to find an infinitesimal change in area?" Along with Torricelli's example (Figure 16), we can use a common illustration of the product rule [2] (Figure 17). Each is fundamentally concerned with two strips of area. These are of infinitesimal but non-zero thickness, one along the top and one along the right.

Proposition 9 *Essentially, for Leibniz's example, let*

$$XY = area, \tag{75}$$

so that

$$d(XY) = infinitesimal\ change\ in\ area. \tag{76}$$

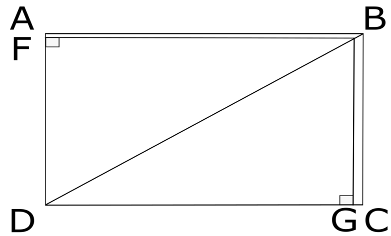

Fig. 16: Torricelli's parallelogram setup to examine the top and right slices of area.

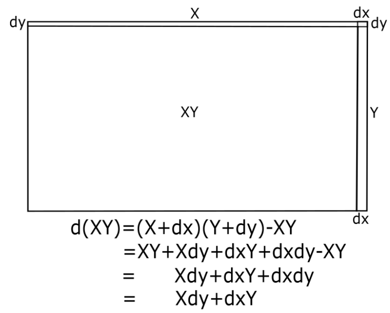

Fig. 17: Leibniz's transcendental law of homogeneity from the product rule. The setup is the change in area of a rectangle to examine the top and right slices of area.

Leibniz adds infinitesimals dx and dy to this notation for X and Y so that

$$d(XY) = (X + dx)(Y + dy) - XY \tag{77}$$

denotes a change in the area of XY. From treating his terms algebraically, he ends up with

$$d(XY) = XY + Xdy + dxY + dxdy - XY, \tag{78}$$

which gives

$$d(XY) = Xdy + dxY + dxdy. \tag{79}$$

Leibniz drops the last term ($dxdy$) to give the equation

$$d(XY) = Xdy + dxY = Xdy + Ydx. \tag{80}$$

One author's paraphrasing of Leibniz's justification [4] is

> A quantity which is infinitely small with respect to another quantity can be neglected if compared with that quantity. Thus all terms in an equation except those of the highest order of infinity, or the lowest order of infinite smallness, can be discarded.

In other words, for our example, $dxdy$ is of a "lower order" than Xdy and Ydx. This is an important conceptual distinction from CPNAHI in that Leibniz does not seem to view Xdy as a row of elements of area, nor does he view the extraneous $dxdy$ as an element of area.

Proof With an eye to not making this introductory paper longer than needed to achieve its goals, let us take Equation 72 and equate these to the top and right slices of area in Leibniz's example,

$$\begin{aligned} n_{ytop}dy_{top} &= X, \\ n_{ybot}dy_{bot} &= Y, \\ dx_{top} &= dx, \\ dy_{bot} &= dy. \end{aligned} \tag{81}$$

Let us note here for now that there is no extraneous $dxdy$ for our example. □

Conclusion

Using the new concepts of Calculus, Philosophy, and Notation of Axiomatic Homogeneous Infinitesimals (CPNAHI) and the axiom of N–M choice (ANMC), I have demonstrated that a relative number of elements and their relative magnitudes are required to quantitatively describe finite measures using infinitesimals and that

$$ndx \leq 1 \tag{82}$$

is a flawed interpretation and simplification of

$$\sum_{dx_m \in S}^{n} dx_m. \tag{83}$$

Without these two sides of the same coin of measurement, different sizes of both infinity and infinitesimals, real finite measures may remain paradoxical.

In future work, I intend to use the previous equation to examine the distinction between a quotient comparing the magnitudes of two lineal infinitesimals,

$$\frac{dx_1}{dx_2}, \tag{84}$$

and the difference in cardinality between to two areal lines,

$$\frac{\Delta n_1 dx_1}{dx_2}. \tag{85}$$

Primitive notion and postulates

CPNAHI primitive notion

Let a homogeneous infinitesimal (HI) be a primitive notion that possesses direction and infinitesimal magnitude.

CPNAHI postulates

1. Postulate of homogeneity: Homogeneous infinitesimals (HIs) can have the property of direction with magnitude, which gives length for one direction, area for two, volume for three, etc. Only HIs of length can sum to create lines, only HIs of area can sum to create area, and only HIs of volume can sum to create volume, etc.[9].
2. HIs conform to the boundaries of any shape.
3. HIs can be adjacent or non-adjacent to other HIs.

[9] This is also in accordance with Eudoxus' theory of proportions, which I view as equivalent to it not being possible to sum heterogeneous infinitesimals. In simpler words, "stacked" two-dimensional planes cannot integrate into a volume.

4. A set of HIs can be a closed set.
5. A lineal line is defined as a closed set of adjacent HIs (paths) with the property of length. These HIs have one direction.
6. An areal line is defined as a closed set of adjacent HIs (paths) with the property of area. These HIs possess two orthogonal directions.
7. A voluminal line is defined as a closed set of adjacent HIs (paths) with the property of volume. These HIs possess three orthogonal directions.
8. Higher directional lines possess higher orthogonal directions.
9. The cardinality of these sets is infinite.
10. The cardinality of these sets can be relatively less than, equal to, or greater than the cardinality of another set and is called *relative cardinality* (n or RC).
11. Postulate of HI proportionality: RC, HI magnitude, and sums each follow Eudoxus' theory of proportion.
12. The magnitudes of a HI can be relatively less than, equal to, or the same as those of another HI.
13. The magnitude of a HI can be null.
14. If the HI within a line is of the same magnitude as the corresponding adjacent HI, then that HI is intrinsically flat relative to the corresponding HI.
15. If the HI within a line is of a magnitude other than equal to or null as the corresponding adjacent HI, then that HI is intrinsically curved relative to the corresponding HI.
16. An HI that is of null magnitude in the same direction as a path is defined as a point. A lineal HI point has zero dimensions. An areal HI point has the property of length orthogonal to the path. A voluminal HI point has the property of area orthogonal to the path.
17. Adjacent points within adjacent areal lines are said to create an arc (i.e., the circumference of a circle).
18. Adjacent points within adjacent voluminal lines are said to create a surface (i.e., the surface of a sphere).

Competing interests

No competing interest is declared.

Author contributions statement

J.P. Baugher conceived of and wrote the manuscript.

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