Different Perspectives on Power of a Point

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Abstract

This paper is devoted to an in-depth study of the concept of the power of a point and its applications to the solution of olympiad-level geometry problems. The discussion encompasses the classical definitions of the power of a point, the radical axis, and the radical center, as well as their various generalizations — including the interpretation of a point as a circle of zero radius, the notion of coaxial circles, and the linearity property of power differences. Detailed examples drawn from both national and international mathematical olympiads are presented to showcase the effectiveness of these methods in addressing both classical and modern geometric problems. Furthermore, the paper considers potential extensions and applications within a broader framework of elementary geometry, with particular emphasis on their value as practical tools for olympiad training and problem solving.

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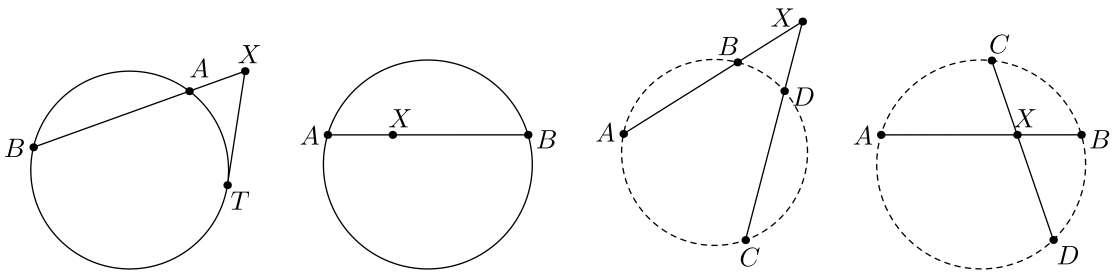
Part I
Power of a Point in Euclidean Geometry

§1 Introduction



Definition 1.1. Let some line passing through a point X intersect a circle ω at points A and B. The *power of a point* X with respect to a circle ω is the expression $d^2 - R^2$, where d is the distance from X to the center of the circle ω, and R is the radius of this circle, or $XA \cdot XB$. Note that this is true when X lies inside the circle ω. When X lies outside the circle ω, its power is taken with the minus sign, precisely $R^2 - d^2$, or $-XA \cdot XB$. Finally, it is equal to the square of the tangent, that is, XT^2. Henceforth, the power of a point X with respect to a circle ω will be denoted by $\operatorname{Pow}(X, \omega)$.

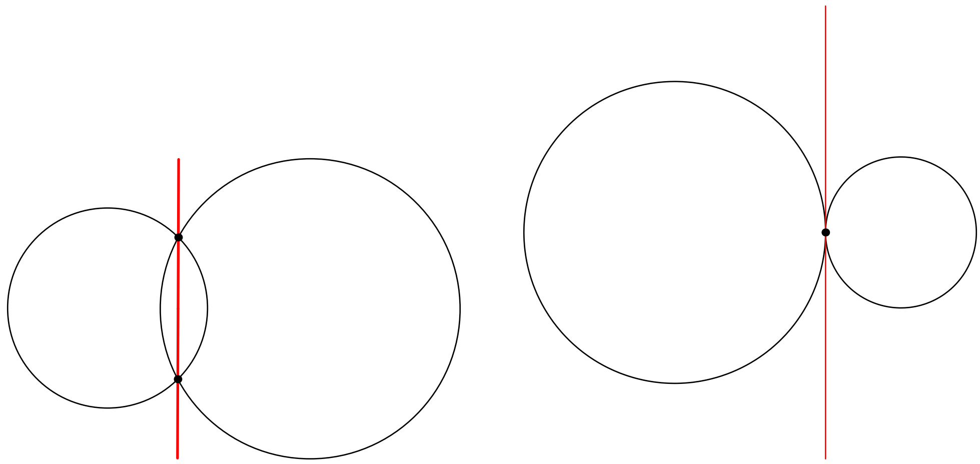
The definition directly implies the criterion of cyclic quadrilateral:

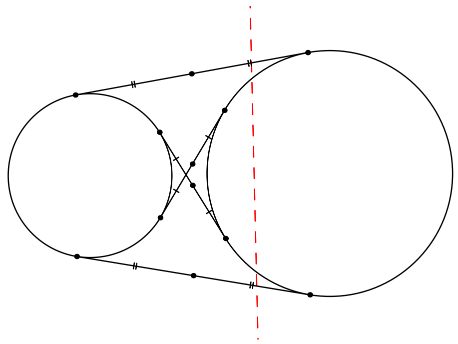
Theorem 1.2

Lines AB and CD intersect at point X. Then points A, B, C and D are concyclic (i.e. lie on the same circle) if and only if $XA \cdot XB = XC \cdot XD$ holds true.

Definition 1.3. The *radical axis* of two circles ω_1 and ω_2 is the locus of points P whose power is the same with respect to ω_1 and ω_2, that is,

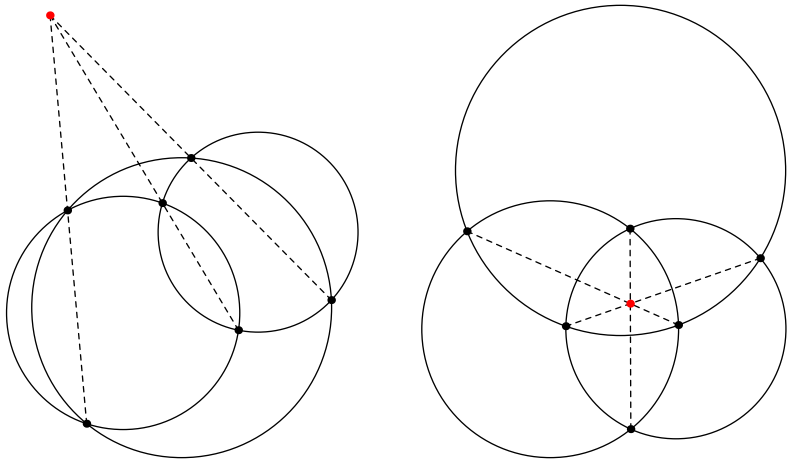
$$\operatorname{Pow}(P, \omega_1) = \operatorname{Pow}(P, \omega_2)$$





Fact 1.4. The radical axis is a line perpendicular to the line, connecting the centers of two circles (interested readers can prove this using the definition of the power of a point and a little knowledge of Cartesian coordinates).

Definition 1.5. The *radical center* of three circles is the point of intersection of the three radical axes of the pairs of circles.



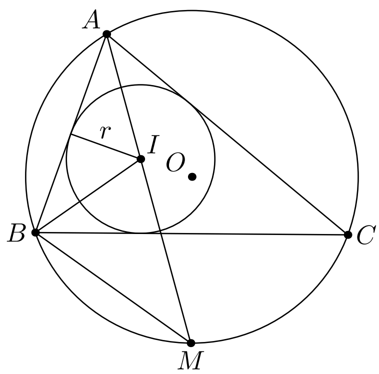
Let's take a look at a few examples.

Example 1.6 (Euler's theorem)

Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then

$$OI^2 = R(R - 2r).$$

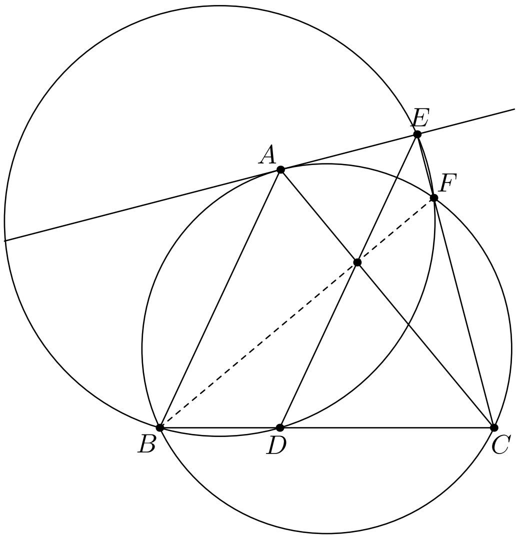
In particular, $R \geq 2r$.

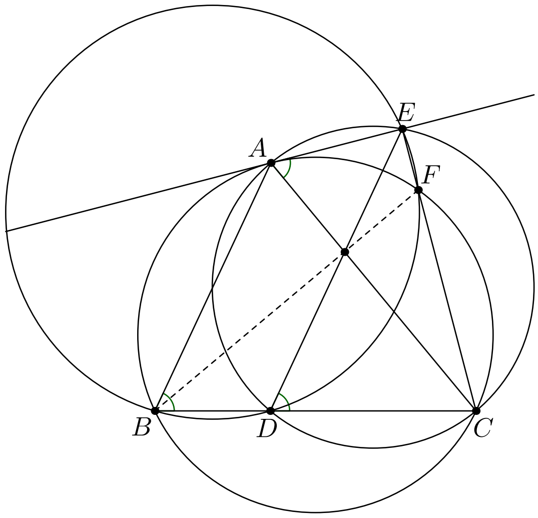


Proof. Let the line AI intersect the circumcircle of the triangle a second time at point M. Since $MC = MB$, $\angle BIC = 180° - \frac{\angle B + \angle C}{2}$, and $\angle BMC = 180° - \angle A$, then M is the circumcenter triangle BIC (essentially, we proved Lemma B.1 here). Therefore, $BM = MC = 2Rsin\angle\frac{A}{2}$. On the other hand, it's easy to check that $AI = \frac{r}{sin\frac{\angle A}{2}}$. Therefore, $R^2 - OI^2 = AI \cdot IM = 2Rr$. ■

Example 1.7 (APMO 2020)

Let Γ be the circumcircle of $\triangle ABC$. Let D be a point on the side BC. The tangent to Γ at A intersects the parallel line to BA through D at point E. The segment CE intersects Γ again at F. Suppose B, D, F, E are concyclic. Prove that AC, BF, DE are concurrent.





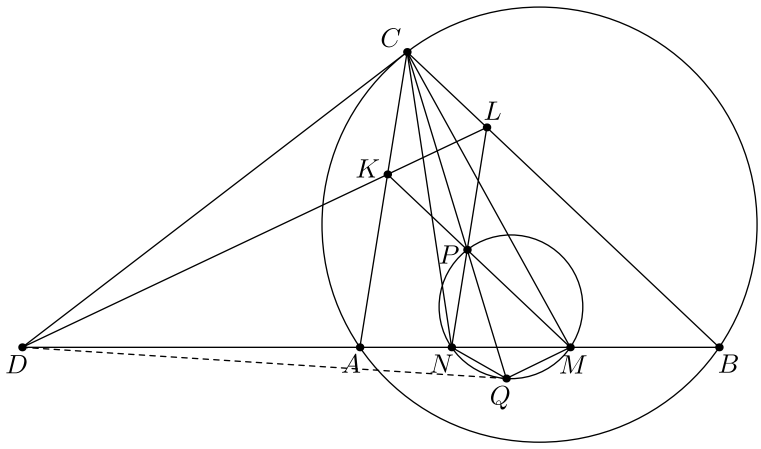
Proof. Note that $\angle EDC = \angle ABC = \angle EAC$, where the first equality follows from the parallelism of lines AB and ED, and the second from the fact that line AE is tangent to the circumcircle of triangle ABC. Thus, $DAEC$ is cyclic. But then the lines BF, DE, and AC are the three radical axes of the three circles and intersect at one point. ■

Remark 1.8. Both conditions are equivalent to the triangle ABC being isosceles with base AB.

Next up, let's tackle a challenging problem proposed on International Zhautykov Olympiad in 2023.

Example 1.9 (IZhO 2023)

The tangent at C to Ω, the circumcircle of scalene triangle ABC intersects AB at D. Through point D, a line is drawn that intersects segments AC and BC at K and L respectively. On the segment AB points M and N are marked such that $AC \parallel NL$ and $BC \parallel KM$. Lines NL and KM intersect at point P lying inside the triangle ABC. Let ω be the circumcircle of MNP. Suppose CP intersects ω again at Q. Show that DQ is tangent to ω.



Solution. First, note that according to Thales' theorem,

$$\frac{DA}{DN} = \frac{DK}{DL} = \frac{DM}{DB},$$

from which, together with the fact that DC is tangent to (ABC), we have $DC^2 = DA \cdot DB = DN \cdot DM$. Hence, DC is also tangent to (MCN), or

$$\angle DCN = \angle CMN$$

$$\angle DCA + \angle ACN = \angle BCM + \angle CBM,$$

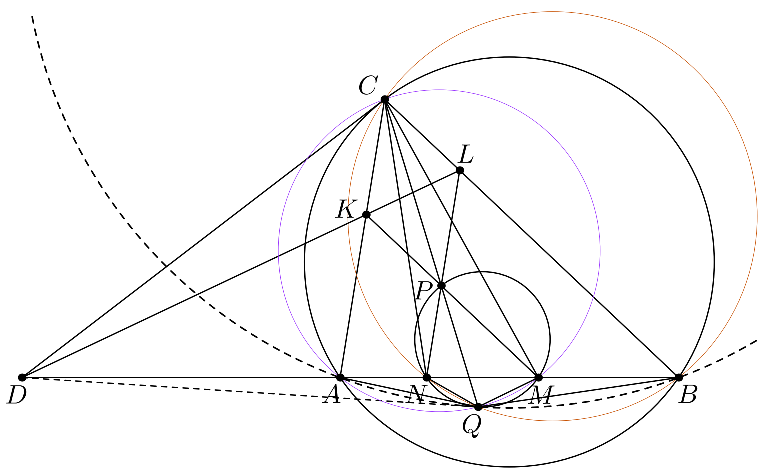
so $\angle ACN = \angle BCM$. Now, $\angle CAM = \angle CAB = \angle LNB = \angle PNM = \angle PQM = \angle CQM$, therefore, $ACMQ$ is cyclic. Similarly, $BCNQ$ is cyclic. Using this, we see that

$$\angle CQA = \angle CMA$$

$$\angle CQN + \angle AQN = \angle CMP + \angle PMA$$

$$\angle AQN = \angle BCM.$$

Similarly, we conclude that $\angle AQN = \angle BQM$. By Lemma B.2, circles (MQN) and (AQB) are tangent. Therefore, their radical axis is their common tangent at point Q. Earlier, we proved that $DA \cdot DB = DN \cdot DM$, which is equivalent to saying that the power of point D with respect to these two circles is the same. Therefore, D lies on this radical axis—the common tangent to the two circles. ■



§1.1 Practice Problems

1.1 Given two non-concentric circles of different radii, and one of them lies inside the other. Construct the radical axis of these circles using only a compass and a ruler.

1.2 (USAJMO 2012) Given a triangle ABC, let P and Q be points on segments AB and AC, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic (in other words, these four points lie on a circle).

1.3 On the sides AB, BC, CA of triangle ABC, points C_1, C_2, A_1, A_2, B_1, B_2 are taken in pairs, respectively. It is known that $A_2A_1C_2C_1$, $C_2C_1B_2B_1$, and $B_2B_1A_2A_1$ are all cyclic. Prove that these six points lie on the same circle.

1.4 (IMO 2008) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2.

Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

1.5 (Italy TST 2001) The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at point M. The bisector of $\angle ACD$ meets the ray BA at K. Given that $MA \cdot MC + MA \cdot CD = MB \cdot MD$, prove that $\angle BKC = \angle CDB$.

1.6 (IGO 2015) In triangle ABC, M, N, K are midpoints of sides BC, AC, AB, respectively. Construct two semicircles with diameter AB, AC outside of triangle ABC. MK, MN intersect with semicircles in X, Y. The tangents to semicircles at X, Y intersect at point Z. Prove that $AZ \perp BC$.

1.7 Triangle ABC has a circumcenter O. The altitudes AA_1, BB_1, CC_1 intersect at point H. Lines A_1B_1 and AB intersect at point C_2. Define points A_2 and B_2 similarly. Prove that A_2, B_2, C_2 lie on a single line perpendicular to OH.

1.8 Prove that the bases of the external bisectors of a triangle lie on a single line perpendicular to the line connecting the circumcenter and incenter.

1.9 Points X and Y are taken on sides AB and BC of acute-angled triangle ABC, respectively. Circles ω_1 and ω_2 are constructed on segments CX and AY as diameters. Prove that the radical axis of ω_1 and ω_2 passes through the orthocenter of triangle ABC.

1.10 (KazMO 2008) Suppose that B_1 is the midpoint of the arc AC, containing B, in the circumcircle of $\triangle ABC$, and let I_b be the B-excircle's center. Assume that the external angle bisector of $\angle ABC$ intersects AC at B_2. Prove that B_2I is perpendicular to B_1I_B, where I is the incenter of $\triangle ABC$.

1.11 On the sides BC, CA, AB of the triangle ABC points A_1, A_2, B_1, B_2, C_1, C_2 are taken in pairs respectively so that $AA_1 = AA_2 = BB_1 = BB_2 = CC_1 = CC_2$. Prove that midpoints of these segments are concyclic.

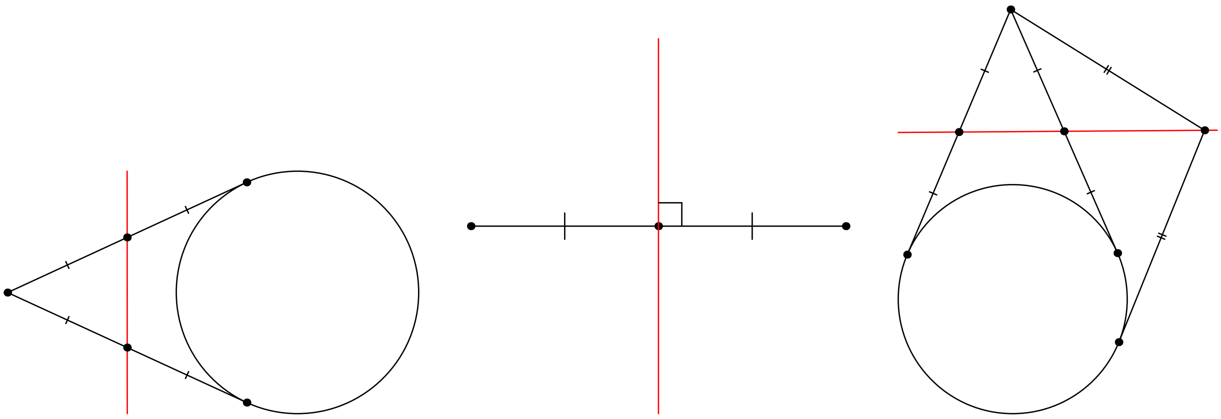
1.12 (Regional 2020) In triangle ABC, circle ω passes through points A and B and intersects segments BC and AC at points D and E, respectively. The angle bisector of BAD intersects ω a second time at point M, and lines BD and ME intersect at point K. Let the perpendicular dropped from point K onto line AM intersect line AC at point N. Prove that $\angle BNK = \angle DNK$.

1.13 (IGO 2023) Let ABC be a triangle and P be the midpoint of arc BAC of circumcircle of triangle ABC with orthocenter H. Let Q, S be points such that $HAPQ$ and $SACQ$ are parallelograms. Let T be the midpoint of AQ, and R be the intersection point of the lines SQ and PB. Prove that AB, SH and TR are concurrent.

§2 Degenerate Circles

In the previous sections, we considered the concept of the power of a point with respect to "ordinary" circles, i.e. circles whose radii are non-zero. But what if we treat a point as a circle with radius 0? Then the power of one point with respect to another is the square of the distance between them.

So if we look at two points as circles, their radical axis is the perpendicular bisector of the segment between them.

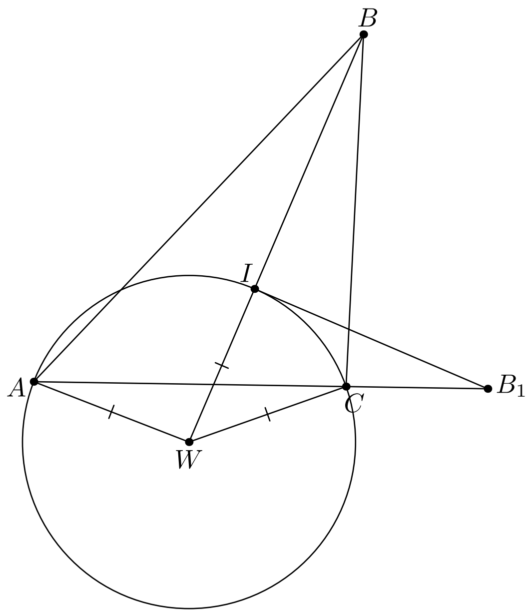


Degenerate Cases of Radical Axis

Let's now move on to some examples.

Example 2.1

I is the incenter of triangle ABC. The line passing through point I perpendicular to line BI intersects line AC at point B_1. Points A_1, C_1 are defined similarly. Prove that points A_1, B_1, C_1 lie on the same line.



Solution. Sometimes it is possible to prove that three points lie on the same line by showing that each of them has the same power with respect to two circles. However, in our case we have only one circle — the incircle. And how do we even compute the powers of the points A_1, B_1, C_1 with respect to the incircle?

Each of these points is constructed symmetrically with respect to the incenter I. Therefore, we take the point I itself as one of the "circles". Then, for example, the power of the point B_1 with respect to the "circle" I equals B_1I^2.

Which circle should we take as the second one? The incircle does not fit, since B_1I^2 is greater than the power of B_1 with respect to it. The circumcircle, however, does. By Lemma B.1, the center of the circumcircle of triangle AIC lies on the bisector BI. Hence, the line B_1I is tangent to the circumcircle of triangle AIC. Thus,

$$B_1I^2 = B_1C \cdot B_1A.$$

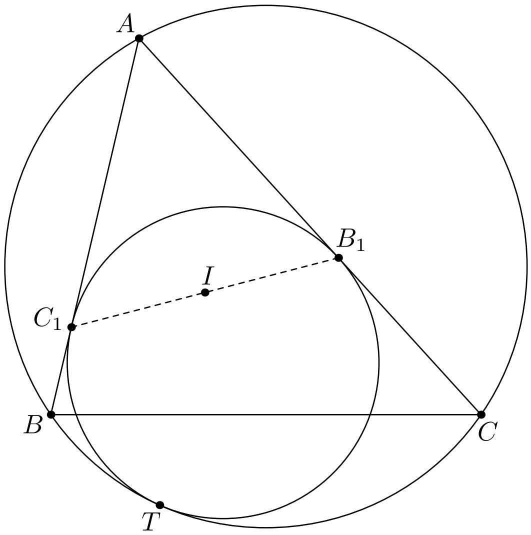
But the product $B_1C \cdot B_1A$ is exactly the power of the point I with respect to the circumcircle of triangle ABC.

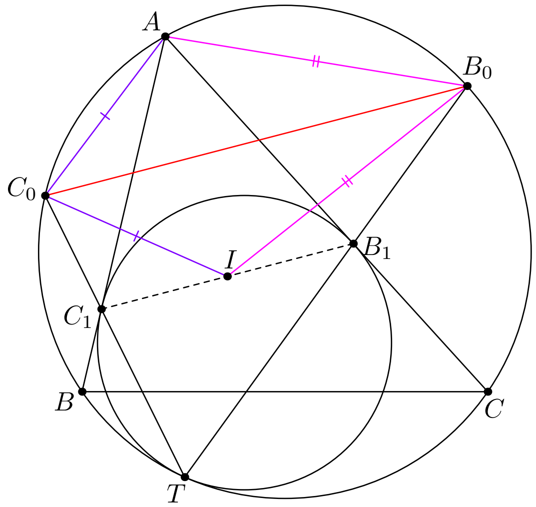
Therefore, we have shown that the powers of the point B_1 with respect to the "circle" I and the circumcircle of triangle ABC are equal — that is, B_1 lies on their radical axis. Similarly, the points A_1 and C_1 also lie on the same radical axis. Hence, the points A_1, B_1, C_1 are collinear.

■

Example 2.2 (Mixtilinear Incircle)

The circle ω is tangent to the sides AB and AC of triangle ABC at points C_1 and B_1, respectively, and is internally tangent to the circumcircle at point T. Prove that the incenter I of triangle ABC lies on the line A_1B_1. (ω is called the A-*mixtilinear incircle* of triangle ABC.)

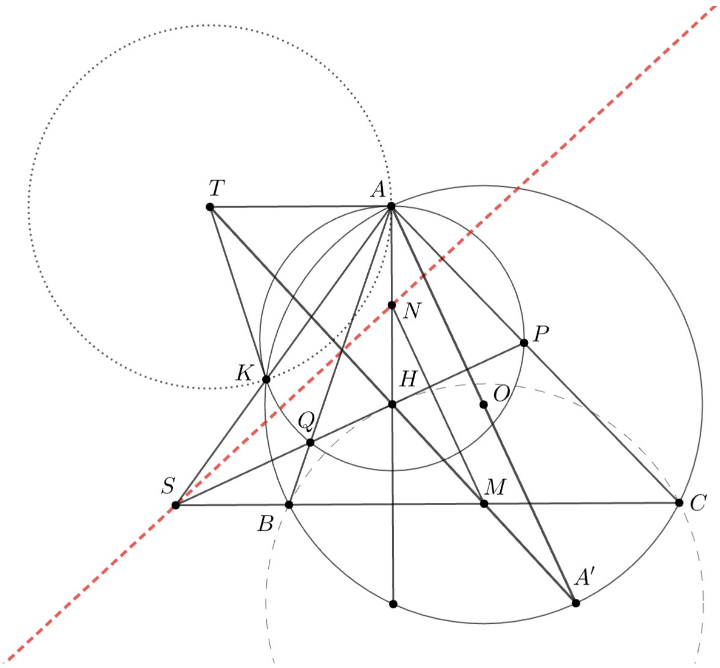




Solution. Note that by Shooting Lemma, the line TB_1 passes through the midpoint of the arc AC of the circumcircle not containing the point B. Similarly, the line TC_1 passes through the midpoint of the arc AB not containing the point C. Denote these by B_0 and C_0, respectively. It also follows from Shooting Lemma that $B_0A^2 = B_0B_1 \cdot B_0T$. Therefore, the power of point B_0 is the same with respect to circle ω and point A. The same is true for point C_0. We obtain that the line B_0C_0 is the radical axis of the point A and the circle ω. Therefore, line B_0C_0 passes through the midpoints of the segments AB_1 and AC_1. This means that the line B_0C_0 contains the median of the triangle C_1AB_1. Consequently, the reflection of point A about the line B_0C_0 lies on B_1C_1. On the other hand, by Lemma B.1, $AC_0 = C_0I$ and $AB_0 = B_0I$. Therefore, the reflection of point B about line B_0C_0 is I and thus we are done. ■

Example 2.3 (KazMO 2023)

The altitudes of an acute triangle ABC intersect at H. The tangent line at H to the circumcircle of triangle BHC intersects the lines AB and AC at points Q and P respectively. The circumcircles of triangles ABC and APQ intersect at point K ($K \neq A$). The tangent lines at the points A and K to the circumcircle of triangle APQ intersect at T. Prove that TH passes through the midpoint of segment BC.



Solution. Let N be the midpoint of AH. We will prove that the lines HT and HM are perpendicular to the line SN. Denote by ω the circle of zero radius centered at point H, and by Ω the circle centered at T with radius TA. Let $S = AK \cap PQ \cap BC$.

From the similarity of triangles ABC and APQ, and from $\angle BAH = \angle CAA'$, it follows that $AO \perp PQ$. Since MN is the midline of triangle HAA', we also have $PQ \perp MN$. Therefore, point H is the orthocenter of triangle SNM, which implies $HM \perp SN$.

The point N lies on the circle Ω, since $AT \perp AH$, and moreover,

$$NA^2 = NH^2,$$

which means that N lies on the radical axis of Ω and ω. Furthermore,

$$SH^2 = SB \cdot SC = SK \cdot SA,$$

so point S also lies on the same radical axis. Hence, the line SN is the common radical axis of Ω and ω, and therefore $HT \perp SN$.

■

§2.1 Practice Problems

2.1. The incircle of triangle ABC with center I touches sides AB, BC, AC at points C_0, A_0, B_0. Line BI intersects A_0C_0 at point K. Prove that the circumcenter of triangle BKB_0 lies on line AC.

2.2. (MMO 2024) In an acute triangle ABC, altitude AH is drawn. Points M and N are the midpoints of segments BH and CH. Prove that the intersection point of the perpendiculars dropped from points M and N to lines AB and AC, respectively, is equidistant from points B and C.

2.3. (Tuymaada 2024) Extension of angle bisector BL of the triangle ABC (where $AB < BC$) meets its circumcircle at N. Let M be the midpoint of BL. Isosceles triangle BDC with base BC and $\angle BDC = \angle ABC$ is constructed outside the triangle ABC. Prove that $CM \perp DN$.

2.4. (Iran TST 2011) In acute triangle ABC angle B is greater than angle C. Let M is midpoint of BC. D and E are the feet of the altitude from C and B respectively. K and L are midpoint of ME and MD respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.

2.5. (Polish MO 2018) An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle escribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.

2.6. (BMO 2015) Let $\triangle ABC$ be a scalene triangle with incentre I and circumcircle ω. Lines AI, BI, CI intersect ω for the second time at points D, E, F, respectively. The parallel lines from I to the sides BC, AC, AB intersect EF, DF, DE at points K, L, M, respectively. Prove that the points K, L, M are collinear.

2.7. A circle is inscribed in triangle ABC and touches sides BC, CA, and AB at points X, Y, and Z, respectively. A point K is marked on the plane. The perpendicular bisectors of segments KX, KY, and KZ intersect lines BC, CA, and AB at points X_1, Y_1, and Z_1, respectively. Prove that points X_1, Y_1, and Z_1 are collinear.

2.8. (IMO Shortlist 2009) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms. Prove that $GR = GS$.

2.9. (CGMO 2015) Let Γ_1 and Γ_2 be two non-overlapping circles. A, C are on Γ_1 and B, D are on Γ_2 such that AB is an external common tangent to the two circles, and CD is an internal common tangent to the two circles. AC and BD meet at E. F is a point on Γ_1, the tangent line to Γ_1 at F meets the perpendicular bisector of EF at M. MG is a line tangent to Γ_2 at G. Prove that $MF = MG$.

2.10. (All-Russian 2011) Perimeter of triangle ABC is 4. Point X is marked on the ray AB and point Y is marked on the ray AC so that $AX = AY = 1$. Line segments BC and XY intersect at point M. Prove that perimeter of one of triangles ABM or ACM is 2.

2.11. (USEMO 2021) Let ABC be a triangle with circumcircle ω, and let X be the reflection of A in B. Line CX meets ω again at D. Lines BD and AC meet at E, and lines AD and BC meet at F. Let M and N denote the midpoints of AB and AC. Can line EF share a point with the circumcircle of triangle AMN?

2.12. (Iran MO 2017) Let ABC be an acute-angle triangle. Suppose that M be the midpoint of BC and H be the orthocenter of ABC. Let $F \equiv BH \cap AC$ and $E \equiv CH \cap AB$. Suppose that X is a point on EF such that $\angle XMH = \angle HAM$ and A, X are on different sides of MH. Prove that AH bisects MX.

Hint: (M), (AHM), (AEF).

2.13. (ELMO 2022) Let $ABCDE$ be a convex pentagon such that $\triangle ABE$, $\triangle BEC$, and $\triangle EDB$ are similar (with vertices in order). Lines BE and CD intersect at point T. Prove that line AT is tangent to the circumcircle of $\triangle ACD$.

Hint: $\triangle ABD \sim \triangle AEC$, (A).

§3 Forgotten Coaxiality Lemma and Pencils of Circles

Claim 3.1

Let a circle ω be given by the equation $f(x, y) = 0$, where

$$f(x, y) = (x - a)^2 + (y - b)^2 - R^2.$$

Then $f(x_0, y_0)$ equals the power of the point (x_0, y_0) with respect to ω.

Proof. Indeed, for $P(x_0, y_0)$ we have

$$f(x_0, y_0) = (x_0 - a)^2 + (y_0 - b)^2 - R^2 = PO^2 - R^2 = \text{Pow}(P, \omega).$$

∎

Let two circles be given with functions f_1 and f_2. Then, for $k \neq 1$, the equation

$$f_1(P) - k\, f_2(P) = 0$$

defines either a circle (possibly a single point) or an empty set. If it's a circle, it shares the same radical axis with the two original circles (i.e., it's coaxial with them). Moreover, every circle coaxial with the two originals can be obtained in this way.

Lemma 3.2

Let ω_1 and ω_2 be two circles, and let P and Q be two points. If

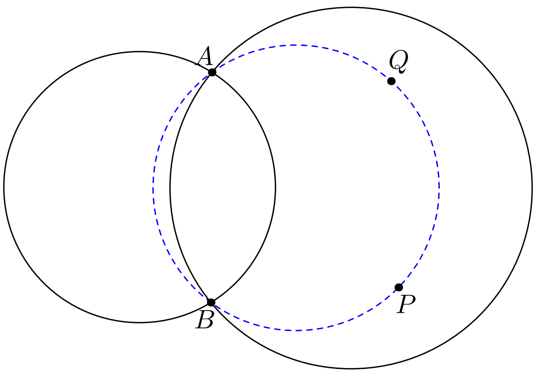
$$\frac{\text{Pow}(P, \omega_1)}{\text{Pow}(P, \omega_2)} = \frac{\text{Pow}(Q, \omega_1)}{\text{Pow}(Q, \omega_2)},$$

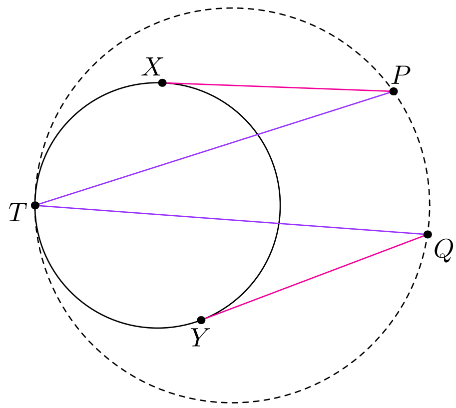
then P and Q lie on a circle coaxial with ω_1 and ω_2. In particular, if ω_1 and ω_2 intersect at A and B, then the new circle also passes through A and B. Conversely, if P and Q lie on a circle coaxial with ω_1 and ω_2, then the above equality holds.

Definition 3.3. A *pencil* of circles is the family of all circles coaxial with two fixed ones; equivalently, the set of circles given by equations

$$\lambda\, f(x, y) + \mu\, g(x, y) = 0,$$

where $f(x, y)$ and $g(x, y)$ are fixed polynomials of the form $x^2 + y^2 + Ax + By + C$, for all $\lambda, \mu \in \mathbb{R}$.





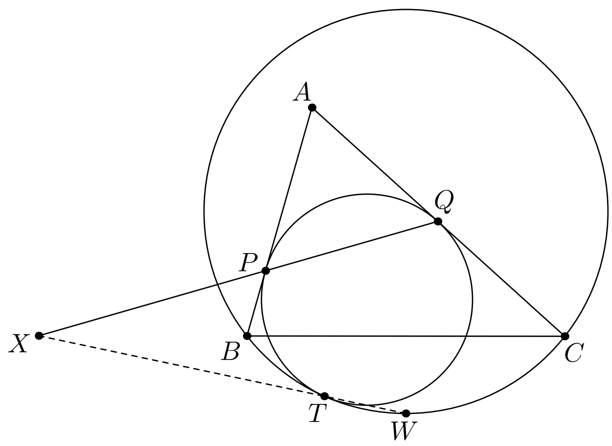
In light of the above, it is natural to ask what happens if one of the ”circles” degenerates to a point. Surprisingly, this yields a very convenient tangency test: suppose two circles are claimed to be tangent at a point T, and that on one of them two points P and Q are already marked. Then it is enough to verify that

$$\frac{PT}{QT} = \frac{PX^2}{QY^2}.$$

Equivalently, the ratio of distances $PT : QT$ equals the ratio of the tangent lengths from P and Q to the second circle.

Example 3.4

A circle Ω passes through vertices B and C of a non-isosceles triangle ABC and contains A in its interior. A circle ω is tangent to AB and AC at P and Q, and is internally tangent to Ω at T. Let $X = BC \cap PQ$. Prove that the line TX passes through the midpoint of the arc BTC of Ω.



Solution. The line TX passes through the midpoint of the arc BTC of Ω if and only if TX is the external angle bisector of $\angle BTC$. By the Angle Bisector Theorem it suffices to check that

$$\frac{TB}{TC} = \frac{XB}{XC}.$$

By Menelaus' Theorem applied to $\triangle ABC$ and the line PQ, the latter ratio equals

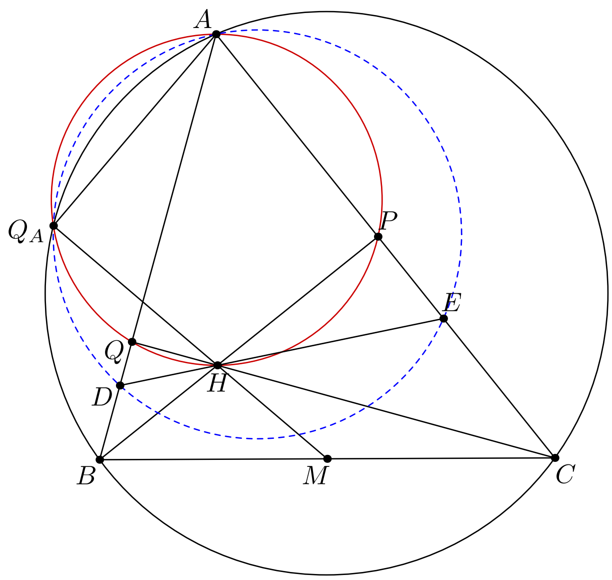
$$\frac{BP}{PA} \cdot \frac{AQ}{QC}.$$

Since AP and AQ are tangent segments from A to ω, they are equal; thus the product reduces to $\dfrac{BP}{QC}$, whose square is the ratio of the powers of B and C with respect to ω. On the other hand, $\left(\frac{TB}{TC}\right)^2$ is the ratio of the powers of B and C with respect to the point T.

Finally, observe that ω, Ω, and the point T are coaxial, and invoke the lemma on coaxiality: the equality of the corresponding power ratios implies that TX indeed passes through the midpoint of the arc BTC. ■

Example 3.5 (IMO Shortlist 2005)

Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.



Solution. In fact, it suffices to prove that (ABC), (AH), (ADE) are circles (AH) and (ADE) are coaxial. Let BP and CQ be the altitudes of $\triangle ABC$. Then

$$\angle QHD \;=\; 90^\circ - \angle ADE \;=\; 90^\circ - \angle AEH \;=\; \angle PHE \;=\; \angle DHB.$$

Thus DE is the internal angle bisector of $\angle QHB$.

By our lemma, we want

$$\frac{\text{Pow}(D,(AH))}{\text{Pow}(D,(ABC))} = \frac{\text{Pow}(E,(AH))}{\text{Pow}(E,(ABC))} \iff \frac{DQ\cdot DA}{DB\cdot DA} = \frac{EP\cdot EA}{EC\cdot EA} \iff \frac{DQ}{DB} = \frac{EP}{EC}.$$

But this is immediate, since

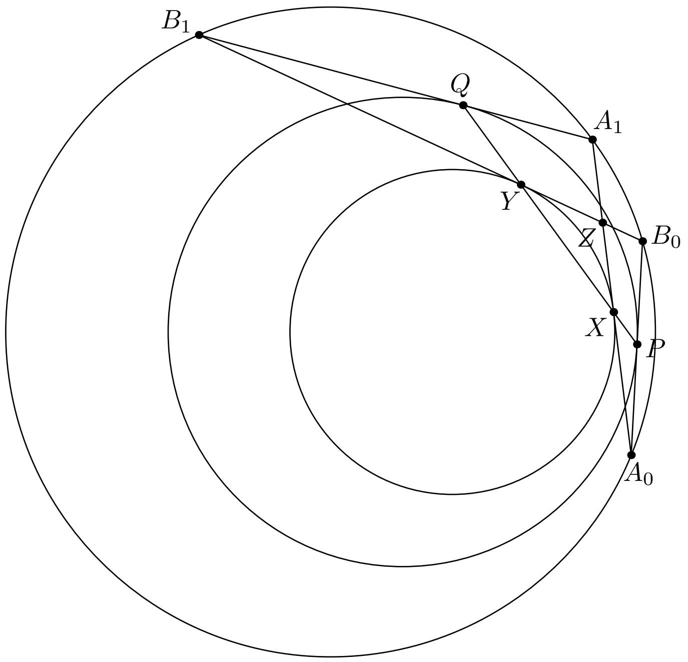
$$\frac{DQ}{DB} = \frac{HQ}{HB} = \frac{HP}{HC} = \frac{EP}{EC}$$

by the Angle Bisector Theorem. Therefore, by our lemma, the circles (ABC), (AH), and (ADE) are coaxial; let their common chord be AQ_A. Hence, by a well-known fact, $HM \perp AQ_A$ (to prove it, just see that the reflection of H about M is the antipode of A in (ABC)).

■

Example 3.6 (Poncelet's Porism)

Let $\omega_1, \omega_2, \ldots$ be circles belonging to the same pencil, and let A_0 be a point lying on the circle ω_0. The tangent to ω_1 drawn from A_0 intersects ω_0 again at A_1, the tangent to ω_2 drawn from A_1 intersects ω_0 again at A_2, and so on. The tangent to ω_{i+1} drawn from A_i intersects ω_0 again at A_{i+1}. Suppose that for some n, the point A_n coincides with A_0. Then for any point B_0 lying on ω_1, the point B_n constructed in the same manner will coincide with B_0.



Note that it is not always possible to construct such a point B_n. For instance, if B_0 lies inside ω_1, the construction fails. In what follows we assume that such a point B_n has indeed been obtained.

Proof. It suffices to show that each segment A_iB_i is tangent to a fixed circle from our pencil. Then, if A_0 happens to coincide with A_n, the tangent from A_0 must coincide with the tangent from A_n to that circle (we always take tangents on the same side). Hence their intersections with ω_0 must coincide as well; these intersection points are B_0 and B_n.

Let the lines A_0A_1 and B_0B_1 be tangent to ω_1 at X and Y, respectively, and let $Z = A_0A_1 \cap B_0B_1$. Then $\triangle XZY$ is isosceles (since X and Y are the tangent points from Z to ω_1), so the angles at X and Y are equal. Moreover, $\angle B_1A_1A_0 = \angle B_1B_0A_0$. Therefore the triangles XQA_1 and YPB_0 are similar, where $P = XY \cap A_0B_0$ and $Q = XY \cap A_1B_1$. Consequently, $\angle PQA_1 = \angle QPB_0$, and hence there exists a circle ω' tangent to the segments A_0B_0 and A_1B_1 at P and Q, respectively.

We now show that ω' belongs to our pencil. It is enough to prove that ω_0 lies in the pencil generated by ω_1 and ω'. To that end, we will show that the ratios of the powers of the points A_0, A_1, B_0, B_1 with respect to ω_1 and ω' are equal. It is easy to see that these ratios are, respectively,

$$\frac{A_0X^2}{A_0P^2}, \qquad \frac{A_1X^2}{A_1Q^2}, \qquad \frac{B_0Y^2}{B_0P^2}, \qquad \frac{B_1Y^2}{B_1Q^2}.$$

By the similarity of $\triangle A_0XP$ and $\triangle B_1YQ$ we have

$$\frac{A_0X^2}{A_0P^2} = \frac{B_1Y^2}{B_1Q^2}.$$

Similarly, by the similarity of $\triangle B_0YP$ and $\triangle A_1XQ$,

$$\frac{A_1X^2}{A_1Q^2} = \frac{B_0Y^2}{B_0P^2}.$$

It remains to verify that

$$\frac{A_0X}{A_0P} = \frac{B_0Y}{B_0P}.$$

By the law of sines,

$$\frac{A_0X}{A_0P} = \frac{\sin\angle A_0PX}{\sin\angle A_0XP} = \frac{\sin\angle B_0PY}{\sin\angle B_0YP} = \frac{B_0Y}{B_0P}.$$

In the same way one shows that the segments A_iB_i and $A_{i+1}B_{i+1}$ are tangent to one and the same circle of our pencil. It is also clear that the segments A_iB_i can be tangent to at most one such circle. Therefore this circle is the same for all i, and it is precisely ω'. ■

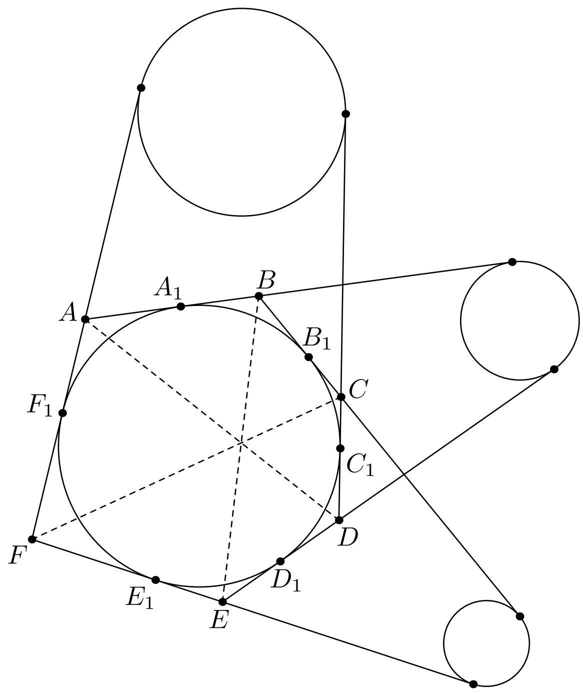
Poncelet's Porism implies, in particular, that if a polygon is inscribed in one circle and circumscribed about another, then it can be "rotated" between those two circles. During this motion, each diagonal of the polygon is tangent to a circle coaxial with the inscribed and circumscribed circles.

Beyond Poncelet's Porism, properties of radical axes also yield Brianchon's Theorem, now in full generality. We first state it.

Example 3.7 (Brianchon's Theorem)

Let lines l_i, $i = 1, \ldots, 6$, be tangent to a single circle, and let $A_{ij} = l_i \cap l_j$. Then the lines $A_{12}A_{45}$, $A_{23}A_{56}$, and $A_{34}A_{61}$ are concurrent.

Proof. So let us be given a circumcribed hexagon $ABCDEF$. We must prove that AD, BE and CF intersect at one point.



Consider the circles $\omega_1, \omega_2, \omega_3$, each tangent to the pairs of lines AB and DE, BC and EF, CD and FA, respectively, such that the points of tangency are at a distance a from the corresponding points of tangency of the sides of the hexagon with its incircle (denoted $A_1, B_1, \ldots, F_1$).

Then, the power of point A with respect to the circles ω_1 and ω_3 equals $-(AA_1 + a)^2$. Hence, A lies on the radical axis of these two circles. The same holds for point D. Therefore, AD is the radical axis of the circles ω_1 and ω_3.

Similarly, one can show that BE is the radical axis of the circles ω_1 and ω_2, and CF is the radical axis of the circles ω_2 and ω_3. Consequently, the lines AD, BE, and CF concur at a single point — namely, the radical center of the circles ω_1, ω_2, and ω_3.

■

§3.1 Practice Problems

3.1. In an acute triangle ABC, altitudes AD, BE, and CF are drawn. Point M is the midpoint of side BC. Prove that the circumcircles of triangles ABC, AEF, and ADM are coaxial.

3.2. In a scalene triangle ABC, points D, E, and F are the points of tangency of the incircle with sides BC, CA, and AB, respectively. Let P be the foot of the perpendicular from point D to EF. Lines BP and CP intersect sides AC and AB at points Y and X, respectively. Prove that the circumcircles of triangles ABC, AEF, and AXY have a second common point of intersection distinct from A.

3.3. The mixtilinear incircle touches the sides AB and AC of triangle ABC at points P and Q, respectively, and touches the circumcircle of triangle ABC at point T. Segments AT and PQ intersect at point S. Prove that $\angle ABS = \angle ACS$.

3.4. Quadrilateral $ABCD$ is inscribed in circle Ω. Circle ω is tangent to lines AB and CD at points X and Y and intersects arc AD of circle Ω at points K and L. Line XY intersects AC and BD at points Z and T. Prove that points K, L, Z, and T lie on the same circle.

3.5. (Iran MO 2nd Round 2021) Circle ω is inscribed in quadrilateral $ABCD$ and is tangent to segments BC, AD at E, F , respectively.DE intersects ω for the second time at X. if the circumcircle of triangle DFX is tangent to lines AB and CD , prove that quadrilateral $AFXC$ is cyclic.

3.6. (CGMO 2017) Let the $ABCD$ be a cyclic quadrilateral with circumcircle ω_1. Lines AC and BD intersect at point E,and lines AD, BC intersect at point F. Circle ω_2 is tangent to segments EB, EC at points M, N respectively,and intersects with circle ω_1 at points Q, R.Lines BC, AD intersect line MN at S, T respectively.Show that Q, R, S, T are concyclic.

3.7. (Romania TST 2010) Let ℓ be a line, and let γ and γ' be two circles. The line ℓ meets γ at points A and B, and γ' at points A' and B'. The tangents to γ at A and B meet at point C, and the tangents to γ' at A' and B' meet at point C'. The lines ℓ and CC' meet at point P. Let λ be a variable line through P and let X be one of the points where λ meets γ, and X' be one of the points where λ meets γ'. Prove that the point of intersection of the lines CX and $C'X'$ lies on a fixed circle.

3.8. (IMO Shortlist 2012) Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

3.9. (IMO 2015) Let ABC be an acute triangle with $AB > AC$. Let Γ be its circumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90^\circ$ and let K be the point on Γ such that $\angle HKQ = 90^\circ$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

Hint: The problem is equivalent to showing that (KQH), (FKM), and the circle centered at K with radius 0 are coaxial.

3.10. (USA TST 2022) Let ABC be an acute triangle. Let M be the midpoint of side BC, and let E and F be the feet of the altitudes from B and C, respectively. Suppose that the common external tangents to the circumcircles of triangles BME and CMF intersect at a point K, and that K lies on the circumcircle of ABC. Prove that line AK is perpendicular to line BC.

Hint: Reflect the orthocenter

§4 Linearity of Power of a Point

A function f on the plane is called *linear* (affine) if either of the following equivalent conditions holds:

- For any points A, B, C and real numbers λ, μ such that C divides AB in the ratio $\frac{\overline{AC}}{\overline{CB}} = \frac{\lambda}{\mu}$ (here, lenghts are directed), we have

$$f(C) = \frac{\lambda}{\lambda+\mu} f(A) + \frac{\mu}{\lambda+\mu} f(B),$$

equivalently,

$$f(C) = \frac{\overline{BC}}{\overline{AB}} f(A) + \frac{\overline{AC}}{\overline{AB}} f(B).$$

- There exist real numbers a, b, c such that for every point $A = (x, y)$,

$$f(A) = ax + by + c.$$

The power of a point $P(x, y)$ with respect to a circle is the function

$$f(P) = (x-a)^2 + (y-b)^2 - R^2,$$

where (a, b) is the center and R is the radius. The circle itself is given by the equation $f(P) = 0$, while the radical axis of two circles is the locus $f_1(P) - f_2(P) = 0$, where f_1 and f_2 are the corresponding functions.

More generally, let n circles be given with functions $f_1, f_2, \ldots, f_n$, and consider

$$F(P) = a_1 f_1(P) + a_2 f_2(P) + \cdots + a_n f_n(P),$$

where $a_1, a_2, \ldots, a_n$ are real numbers. The equation $F(P) = c$ (with constant c) can be rewritten as

$$A(x^2+y^2) + Bx + Cy + D = c, \qquad \text{where } A = a_1 + a_2 + \cdots + a_n.$$

There are two possible cases:

- If $A \neq 0$, then the equation can be brought to the form of a (scaled) circle equation. It may define a circle; or, if the resulting squared radius is non-positive, a single point or the empty set.
- If $A = 0$, then the equation is linear. It may define a line (if $B \neq 0$ or $C \neq 0$), the entire plane, or the empty set.

When $F(P)$ is linear (i.e., when $a_1 + a_2 + \cdots + a_n = 0$), its value at a point can be computed as a linear combination of its values at other points. For example, if S lies on segment PQ, then

$$F(S) = \frac{\overline{SQ}}{\overline{PQ}} F(P) + \frac{\overline{SP}}{\overline{PQ}} F(Q).$$

Key Lemma.

Lemma 4.1 (Linearity of PoP)

Denote by $\operatorname{Pow}(X, \omega)$ the power of a point X with respect to a circle ω. Then the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(X) = \operatorname{Pow}(X, \omega_1) - \operatorname{Pow}(X, \omega_2)$$

is **linear**.

Proof. Let $O_1 = (x_1, y_1)$ and $O_2 = (x_2, y_2)$ be the centers of ω_1, ω_2, and let their radii be r_1, r_2. For $X = (x, y)$ we have

$$\begin{aligned}\operatorname{Pow}(X, \omega_1) - \operatorname{Pow}(X, \omega_2) &= \big((x - x_1)^2 + (y - y_1)^2 - r_1^2\big) - \big((x - x_2)^2 + (y - y_2)^2 - r_2^2\big) \\ &= 2(x_2 - x_1)\, x + 2(y_2 - y_1)\, y + \big(x_1^2 - x_2^2 + y_1^2 - y_2^2 + r_2^2 - r_1^2\big),\end{aligned}$$

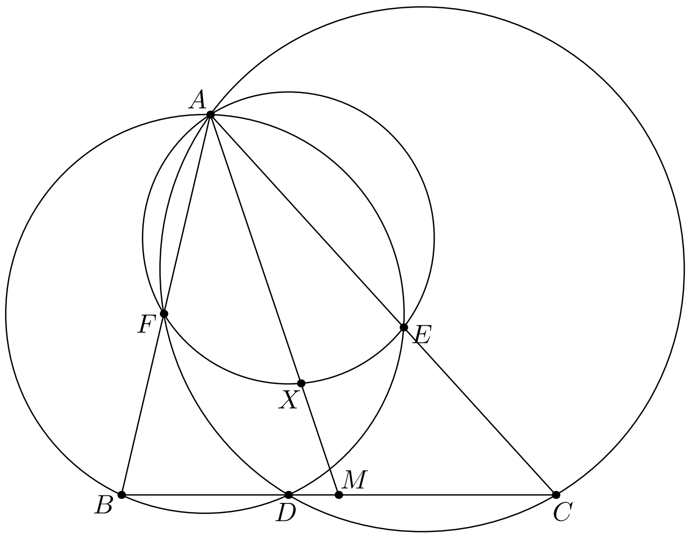
which is a linear function of x and y. ■

The main idea is to introduce the function given by the difference of powers with respect to two "nice" circles, and then length-bash via this function.

We begin with a classical problem illustrating this method.

Example 4.2 (ELMO Shortlist 2013)

In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.



Solution. Define $f(P) = \operatorname{Pow}(P, \Gamma_1) - \operatorname{Pow}(P, \Gamma_2)$.

Let ω and ω_1 be the circumcircles of triangles ABC and AEF, respectively. It suffices to show that if $(AEF) \cap AM = X$, then $\operatorname{Pow}(M, \omega_1) = MX \cdot MA$ is a constant. We have

$$\begin{aligned}f(M) &= \frac{1}{2}(f(B) + f(C)) \\ &= \frac{1}{2}(BE \cdot BA - 0 + CF \cdot CA - 0) \\ &= \frac{1}{2}(BD \cdot BC + CD \cdot BC) \\ &= \frac{BC^2}{2}.\end{aligned}$$

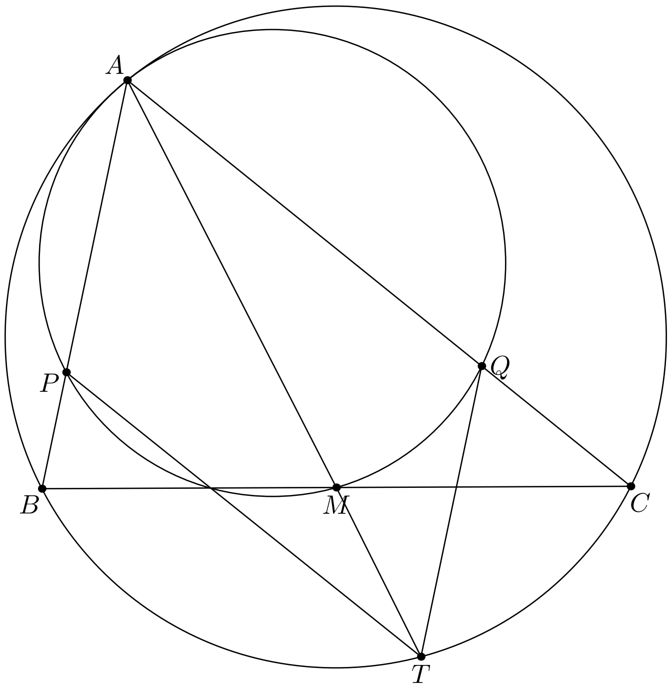
Thus,

$$\frac{BC^2}{4} - \operatorname{Pow}(M, \omega_1) = f(M) = \frac{BC^2}{2}.$$

Therefore, $\operatorname{Pow}(M, \omega_1)$ is a constant, as desired. ■

Example 4.3 (IMO Shortlist 2015)

Let ABC be an acute triangle and let M be the midpoint of BC. A circle ω passing through A and M meets the sides AB and AC at points P and Q respectively. Let T be the point such that $APTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{AT}{AM}$.



Solution. Let (A) be a circle with center A and radius 0. Denote by $\text{Pow}_\omega(P)$ the power of the point P with respect to the circle ω.

Define $f(X) = \text{Pow}(X, (A)) - \text{Pow}(X, (ABC))$.

By linearity, we have $f(A) + f(T) = f(P) + f(Q)$. Thus,

$$AT^2 = PA^2 - (-PA \cdot PB) + QA^2 - (-QA \cdot QB) = PA \cdot AB + QA \cdot AC.$$

Now let $R \neq M$ be the second intersection of ω with BC, then

$$PB \cdot AB + QC \cdot AC = BM \cdot BR + CM \cdot CR = \frac{BC^2}{2}.$$

It follows that $AT^2 + \frac{BC^2}{2} = AB^2 + AC^2 \implies AT^2 = AB^2 + AC^2 - \frac{BC^2}{2}$. By Stewart's theorem, $\frac{BC}{2}(AB^2 + AC^2) = BC \cdot (AM^2 + \frac{BC^2}{4}) \implies 2AM^2 = AB^2 + AC^2 - \frac{BC^2}{2} = AT^2$. Therefore, $\frac{AT}{AM} = \sqrt{2}$. ■

Example 4.4 (AoPS)

Given triangle ABC inscribed in (O). Let M be the midpoint of BC, D be the projection of A onto BC. OH meets AM at P. Prove that P lies on the radical axis of (BOC) and the 9-point circle of triangle ABC.

Solution. Let H be the orthocenter of triangle ABC, and let AD intersect ω a second time at E (known to be the midpoint of AH). For any point X, define the function $f : \mathbb{R}^2 \to \mathbb{R}$ as $f(X) = \text{Pow}(X, (BOC)) - \text{Pow}(X, \omega)$, where ω is the nine-point circle.

Note that it suffices to show that

$$0 = f(P) = \frac{OP}{OD} f(D) + \frac{DP}{OH} f(O) \iff \frac{OP}{DP} = -\frac{f(O)}{f(D)}.$$

From the Euler line, we know that O and H are symmetric about the center of ω, so we have

$$f(O) = -\text{Pow}(O, \omega) = -\text{Pow}(H, \omega) = EH \cdot HD = (R\cos A)(2R\cos B\cos C).$$

$$f(D) = \text{Pow}(D, (BOC)) = -BD \cdot DC = -(AB\cos B)(AC\cos C),$$

so

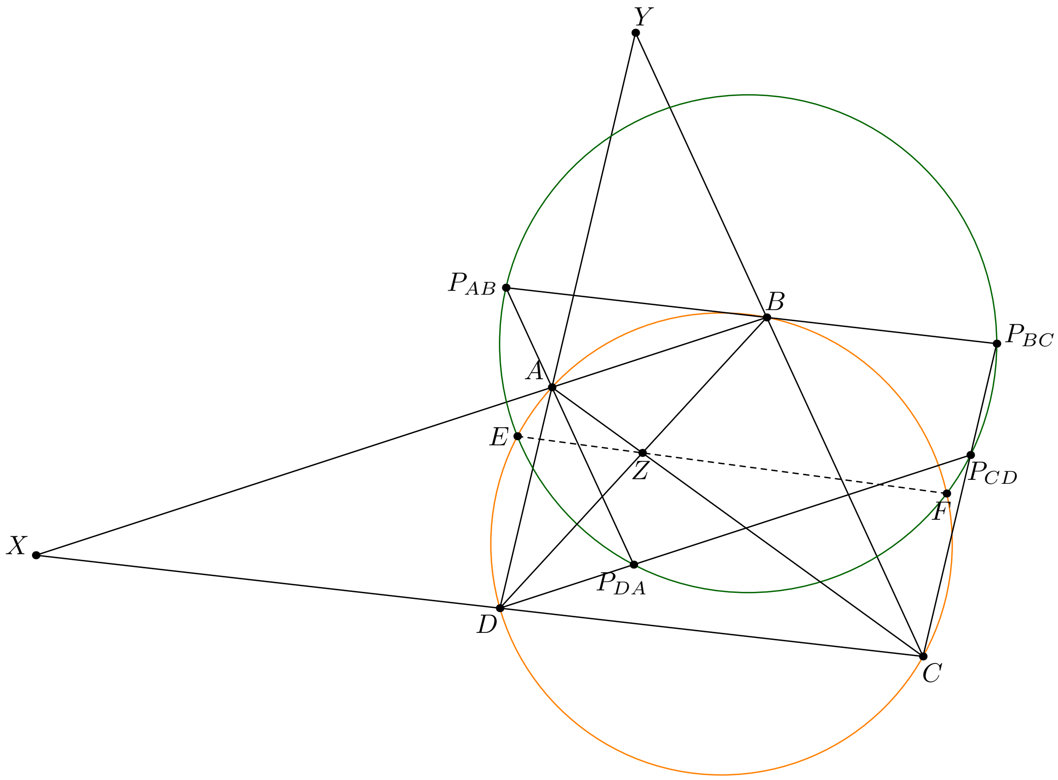
$$-\frac{f(O)}{f(D)} = \frac{2R^2\cos A}{AB \cdot AC}.$$

We also know that $OM||AD$, hence

$$\triangle OPM \sim \triangle DPA \implies \frac{OP}{DP} = \frac{OM}{DA} = \frac{R\cos A}{AB\sin\angle B} = \frac{2R^2\cos A}{AB \cdot AC},$$

where the last equality follows from the law of sines, as desired. ■

Example 4.5 (All-Russian 2024)
A quadrilateral $ABCD$ without parallel sides is inscribed in a circle ω. We draw a line $\ell_a \parallel BC$ through the point A, a line $\ell_b \parallel CD$ through the point B, a line $\ell_c \parallel DA$ through the point C, and a line $\ell_d \parallel AB$ through the point D. Suppose that the quadrilateral whose successive sides lie on these four straight lines is inscribed in a circle γ and that ω and γ intersect in points E and F. Show that the lines AC, BD and EF intersect in one point.



Solution. Let $P_{AB} = l_a \cap l_b$. Define P_{BC}, P_{CD}, and P_{DA} similarly. Let $X = AB \cap CD$, $Y = AD \cap BC$, and $Z = AC \cap BD$. Define

$$f(P) = \text{Pow}(P, \gamma) - \text{Pow}(P, w)$$

for every point P in the plane. By linearity of power of a point, f is a linear function. Notice that $f(A) = AP_{AB} \cdot AP_{DA}$. Using similar triangles ADP_{DA} and YAB and similar triangles ABP_{AB} and BXC yields that $f(A) = \frac{YB\cdot AD}{YA} \cdot \frac{BC\cdot BA}{XB}$. Similarly,

$$f(A) = \frac{YB \cdot (AD \cdot BC \cdot BA)}{YA \cdot XB}$$

$$f(C) = \frac{YD \cdot (CB \cdot DA \cdot DC)}{YC \cdot XD}$$

Using the linearity of f along AC we get that

$$f(Z) = \frac{ZC \cdot f(A) + AZ \cdot f(C)}{ZC + AZ}$$

Applying Menelaus' Theorem to triangle XAC and line BZD gives that

$$\frac{AZ}{ZC} \cdot \frac{XB}{BA} \cdot \frac{CD}{DX} = 1$$

We also have that $YA \cdot YD = YB \cdot YC$ which is sufficient to conclude that $f(Z) = 0$. Thus Z must lie on the radical axes of w and γ. ■

§4.1 Practice Problems

4.1. (USAMO 2013) In triangle ABC, points P, Q, R lie on sides BC, CA, AB respectively. Let ω_A, ω_B, ω_C denote the circumcircles of triangles AQR, BRP, CPQ, respectively. Given the fact that segment AP intersects ω_A, ω_B, ω_C again at X, Y, Z, respectively, prove that

$$YX/XZ = BP/PC.$$

4.2. (IMO Shortlist 2011, reformulated) Let $ABCD$ be a non-cyclic quadrilateral. Prove that

$$\frac{1}{\text{Pow}(A,(BCD))} + \frac{1}{\text{Pow}(B,(ACD))} + \frac{1}{\text{Pow}(C,(ABD))} + \frac{1}{\text{Pow}(D,(ABC))} = 0.$$

4.3. Point P on the inscribed circle of triangle ABC is such that a right triangle can be formed from the tangent segments from point P to the three excircles of triangle ABC. Prove that P lies on the midline of the triangle.

Hint: Length of tangent segment squared is power of a point. Then, construct a function that is zero for points on a midline.

4.4. (MODS Geometry Contest 2024) Let $ABCD$ be a parallelogram. Let line ℓ externally bisect $\angle DCA$ and let ℓ' be the line passing through D which is parallel to line AC. Suppose that ℓ' meets line AB at point E and ℓ at point F, and that ℓ meets the internal bisector of $\angle BAC$ at point X. Further let circle EXF meet line BX at point $Y \neq X$ and the internal bisector of $\angle DCA$ meet circle AXC at point $Z \neq C$. Prove that points D, X, Y, and Z are concyclic.

4.5. In a trapezoid $ABCD$ with bases $AB \parallel CD$, points P and Q are chosen so that $\angle BPC = \angle BQC = 180° - \angle DPA = 180° - \angle DQA$. If $U = AC \cap BD$ and $V = BC \cap DA$, prove that PQ passes through the projection of point U onto the line passing through V parallel to AB.

4.6. (Taiwan TST 2016) Let O be the circumcenter of triangle ABC, and ω be the circumcircle of triangle BOC. Line AO intersects with circle ω again at the point G. Let M be the midpoint of side BC, and the perpendicular bisector of BC meets circle ω at the points O and N.

Prove that the midpoint of the segment AN lies on the radical axis of the circumcircle of triangle OMG, and the circle whose diameter is AO.

4.7. (RMM 2019) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let E be the midpoint of AC. Denote by ω and Ω the circumcircles of the triangles ABE and CDE, respectively. Let P be the crossing point of the tangent to ω at A with the tangent to Ω at D. Prove that PE is tangent to Ω.

4.8. (Sharygin Geometry Olympiad 2023) The incircle ω of a triangle ABC centered at I touches BC at point D. Let P be the projection of the orthocenter of ABC to the median from A. Prove that the circles (AIP) and ω cut off equal chords on AD.

4.9. (RMM Shortlist 2017) Let $ABCD$ be a convex quadrilateral and let P and Q be variable points inside this quadrilateral so that $\angle APB = \angle CPD = \angle AQB = \angle CQD$. Prove that the lines PQ obtained in this way all pass through a fixed point, or are all parallel.

Hint: $E = AC \cap BD$ $f(X) = \text{Pow}(X, (CDPQ)) - \text{Pow}(X, (ABPQ))$. We need to find constants u, v, w (not all equal to zero) such that $uf(E) + vf(A) + wf(B) = 0$.

Part II
Beyond: Inversion, Conics, and Projective Geometry

§5 Inversion and Power of a Point

In this section we explain how *inversion in a circle* can be viewed purely through the lens of power of a point. We work in the Euclidean plane.

§5.1 Definition via power of a point

Definition 5.1 (Inversion with respect to a circle)**.** Let ω be a fixed circle with center O and radius R. The *inversion* with respect to ω is the transformation that sends to each point $P \neq O$ a point P' on the ray OP such that

$$OP \cdot OP' = R^2.$$

We say that P' is the *inverse* of P with respect to ω, and we write $P' = \mathcal{I}(P)$. Points on ω are fixed by inversion (since $OP = R$ implies $OP' = R$). The center O has no finite image under inversion, but in the extended plane it is swapped with the point at infinity.

Note that

$$OP \cdot OP' = R^2 \quad \Longleftrightarrow \quad OP' = \frac{R^2}{OP}.$$

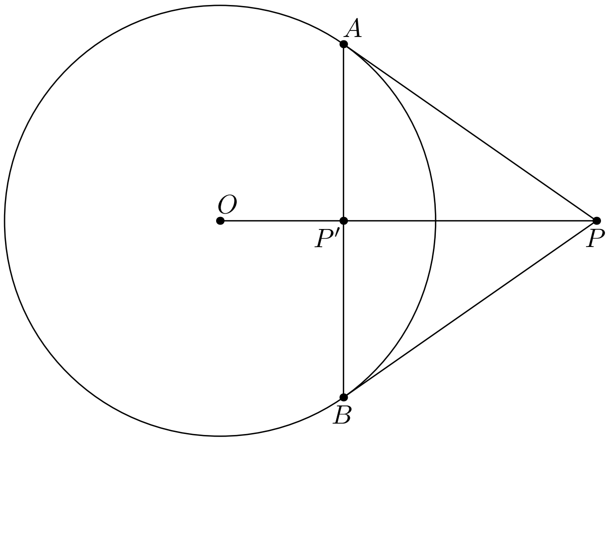
If we set $d = OP$, then $OP' = \dfrac{R^2}{d}$ and

$$\operatorname{Pow}(P, \omega) = d^2 - R^2,$$

$$\operatorname{Pow}(P', \omega) = OP'^2 - R^2 = \frac{R^4}{d^2} - R^2 = -\frac{R^2}{d^2}\left(d^2 - R^2\right) = -\frac{R^2}{OP^2} \operatorname{Pow}(P, \omega).$$

So inversion sends a point P to a point P' whose power with respect to ω is scaled by the factor $-\dfrac{R^2}{OP^2}$. In particular, interior points (with $\operatorname{Pow}(P, \omega) < 0$) go to exterior points ($\operatorname{Pow}(P', \omega) > 0$), and vice versa. Inversion is nothing more than enforcing a *constant power product* $OP \cdot OP' = R^2$ along each ray from O.

How do we construct images of points under inversion, though? Let's be given a point P and the circle of inversion ω centered at O.

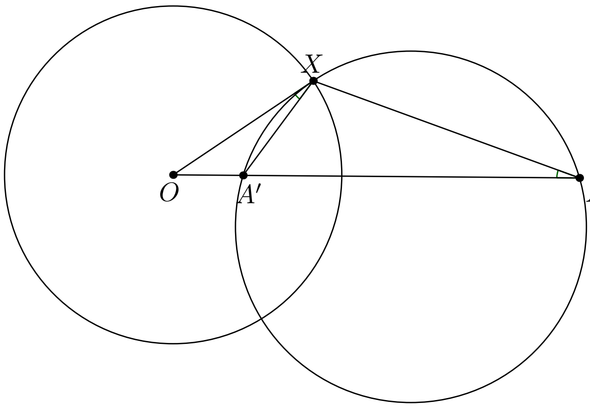


- First, let P lie outside ω, construct two lines touching ω at A and B. Then we claim that the intersection point P' of lines OP and AB. Indeed, using similarity we get $R^2 = OA^2 = OP \cdot OP'$.
- Now let P lie inside ω. We do the opposite. Construct a line perpendicular to OP and let it cut ω at A and B. Then the intersection point P' of tangents at A and B is the desired image of P.

§5.2 Orthogonal circles

Another classical concept that has a clean PoP interpretation is the following.

Definition 5.2. Two circles are said to be *orthogonal* if tangents at each intersection point are perpendicular (or at each intersection point, their radii are perpendicular).



Let us be given a circle $\omega(O, R)$ and a point $A \neq O$. Consider a circle γ that passes through A and is orthogonal to ω. Define X as an intersection point of ω and γ, and $A' \neq A$ as the intersection point of ray OA and γ. Then

$$OX^2 = OA \cdot OA'.$$

We know that $OX = R$, and OA is fixed (since O and A are fixed), so we infer that OA', and hence the point A', is fixed too. Thus, all such circles γ pass through a fixed point A' on the ray OA.

But notice that $OX^2 = OA \cdot OA'$ is precisely the power of O with respect to γ: since γ is orthogonal to ω, we have

$$\text{Pow}(O, \gamma) = R^2 = OX^2 = OA \cdot OA'.$$

In other words, A' is chosen so that the power of O to γ is the constant R^2.

This motivates the following definition.

Definition 5.3 (Inversion via orthogonal circles)**.** Let $\omega(O, R)$ be fixed and let $A \neq O$. Choose any circle γ passing through A and orthogonal to ω. Let $A' \neq A$ be the second intersection point of γ with the ray OA.

By the discussion above, the point A' is independent of the choice of γ (all circles through A orthogonal to ω pass through the same A'). We call A' the *inverse* of A with respect to ω.

> **Remark 5.4.** Equivalently, for each $A \neq O$ there is a unique circle γ orthogonal to ω and passing through A; then A' is the second intersection of γ with the ray OA. The condition $OA \cdot OA' = R^2$ says exactly that the power of O with respect to every such circle γ is the constant R^2.

This induces the following fact.

Theorem 5.5 (Orthogonal circles are fixed under inversion)

Let ω be the circle of inversion, and let γ be a circle orthogonal to ω. Then γ is invariant under inversion in ω (as a set).

§5.3 Line–circle duality via power of a point

One of the basic facts about inversion is that it sends lines and circles to each other. To sum up this section, we present not-hard-to-prove facts which can be used or stated on contests without a proof:

Theorem 5.6 (Lines and circles under inversion)
The following are true:

- A point on the circle of inversion is sent to itself.
- A line passing through the center is sent to itself.
- A line not passing through the center is sent to a circle through the center.
- A circle passing through the center is sent to a line.
- A circle not passing through the center is sent to a circle.
- A circle orthogonal to the circle of inversion is sent to itself.

§6 Pole and Polar with Respect to a Circle

We now connect the notion of power of a point to the classical theory of poles and polars with respect to a circle. Throughout, let ω be a fixed circle with center O and radius R.

Definition 6.1. Let P and P' be inverses with respect to a circle ω centered at O. The line $\pi(P)$ through P' perpendicular to OP is called the *polar* of P with respect to ω. The point P is then called the *pole* of the line $\pi(P)$.

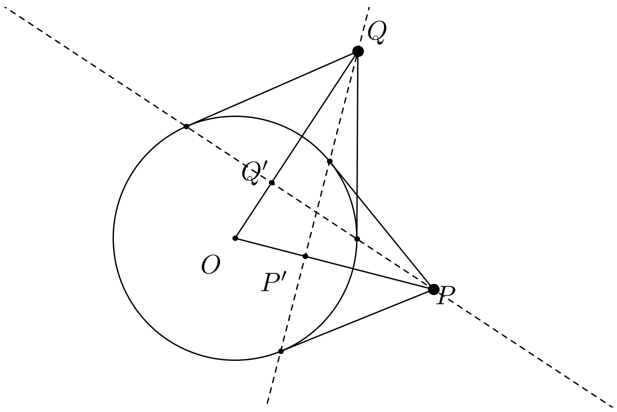
Poles and polars can be constructed in a way similar to the construction of inverse points.

§6.1 La Hire's Theorem (circle version)

One of the fundamental facts about poles and polars is the following reciprocity.

Theorem 6.2 (La Hire)
With respect to a circle ω, the following are equivalent for two points P, Q:

1. Q lies on the polar $\pi(P)$ of P.
2. P lies on the polar $\pi(Q)$ of Q.



Proof. Let P' and Q' be the inverses of P and Q respectively. Then

$$OP \cdot OP' = R^2 = OQ \cdot OQ',$$

i.e. P, Q, Q', P' are concyclic. Thus,

$$P \in \pi(Q) \iff \angle PQ'Q = 90^\circ \iff \angle QP'P = 90^\circ \iff Q \in \pi(P).$$

■

This duality is a metric version (for a circle) of the more general projective relationship between points and lines with respect to a conic. In our context, it is entirely governed by PoP and inversion.

Proposition 6.3 (Pole–polar duality)
With respect to a fixed circle ω, the map $P \mapsto \pi(P)$ extends to a bijection between points and lines in the projective plane, satisfying:

- $Q \in \pi(P)$ if and only if $P \in \pi(Q)$ (La Hire).
- Three points P_1, P_2, P_3 are collinear if and only if their polars $t(P_1), t(P_2), t(P_3)$ are concurrent.

- Dually, three lines ℓ_1, ℓ_2, ℓ_3 are concurrent if and only if their poles are collinear.

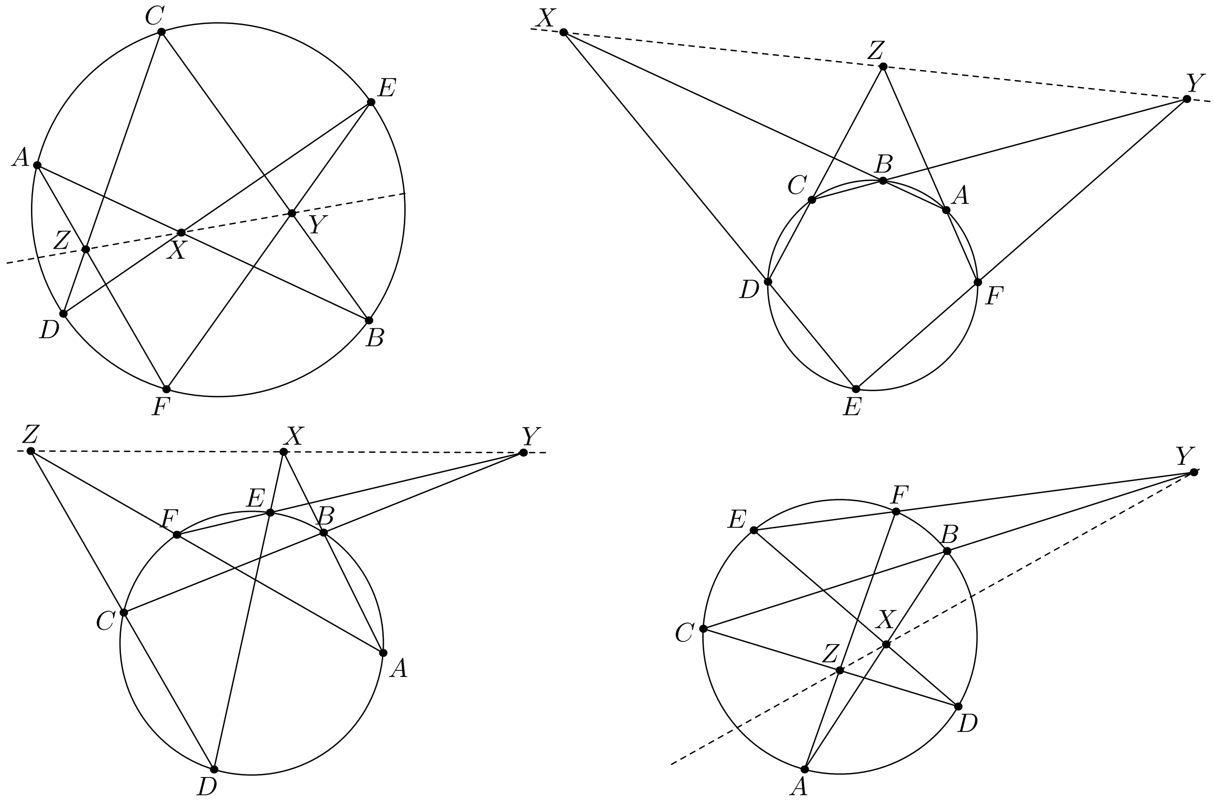
In this sense, the circle ω defines a *polarity*, or *projective duality*, between points and lines.

A striking consequence is that many theorems about points on a conic have *dual* versions about lines tangent to the conic. The most famous pair is Pascal's and Brianchon's theorems.

Theorem 6.4 (Pascal's)
Let $ABCDEF$ be a hexagon inscribed in a circle (nondegenerate conic) $\mathcal{C}$. Let

$$X = AB \cap DE, \quad Y = BC \cap EF, \quad Z = CD \cap FA.$$

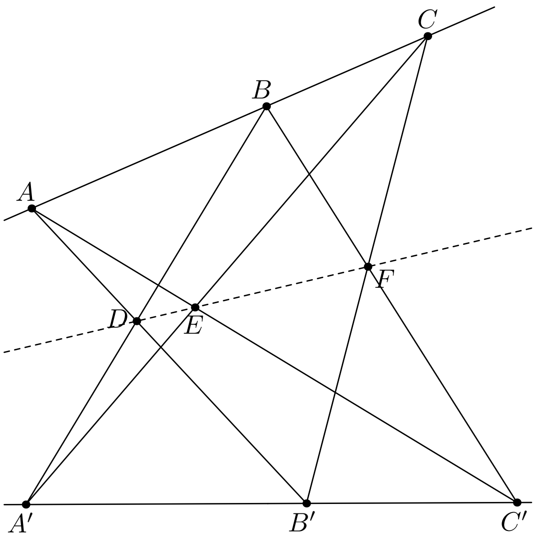
Then the points X, Y, Z are collinear.

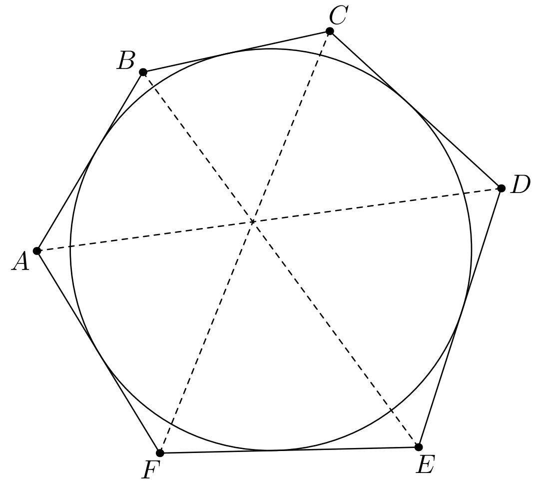


Pascal's can look very different

Theorem 6.5 (Pappus')
Let A, B, C be three points on one line, and A', B', C' be three points on another line. Let three lines AB', BC', CA' intersect three lines $A'B$, $B'C$, $C'A$ at points D, E, F, respectively. Then the points D, E, F are collinear.

Theorem 6.6 (Brianchon's)
Given a hexagon $ABCDEF$ circumscribed around a circle (nondegenerate conic). Then its main diagonals AD, BE, CF intersect at one point.





Remark 6.7. Pole–polar duality with respect to a fixed conic allows us to transform any statement about points and lines into a ”dual” statement: we interchange ”point” with ”line” and replace incidence by its dual (collinearity $\leftrightarrow$ concurrency).

Under this procedure, Pascal's theorem and Brianchon's theorem are dual to each other. In Pascal's theorem, we start with a hexagon inscribed in a conic, and conclude that three intersection points of pairs of opposite sides are collinear. If we interchange points and lines, an inscribed hexagon becomes a circumscribed hexagon, collinearity of the three Pascal points becomes concurrency of three lines joining opposite vertices, and we recover Brianchon's theorem.

In this sense Pascal and Brianchon form a dual pair of theorems. There are also theorems which are *self-dual*, such as Desargues' theorem.

Remark 6.8. Pascal's theorem, just like any other projective theorem, remains true when the hexagon is degenerate. For example, if we take the cyclic hexagon $AABBCC$, we can deduce that the intersection points of the tangents at the triangle's vertices and the sides opposite them lie on a straight line. Sometimes it's also useful to apply Pascal's theorem to multiple six-tuples of points (for example, $AABCCD$ and $BBCDDA$), obtaining an auxiliary fourth point on the same line.

Remark 6.9. Pappus' theorem is just a special case of Pascal's theorem, i.e. when the conic degenerates into a pair of lines.

§7 Projective Upgrade: From Circles to Conics (Optional)

Projective geometry is all geometry.

—Arthur Cayley (1821-1895)

In the main body of this paper we have mostly worked with circles and power of a point in the Euclidean setting. In this final (optional) section, we briefly indicate how some circle results can be "upgraded" to conics using projective geometry. We will keep the discussion informal and only quote the necessary facts.

§7.1 Basics

Definition 7.1. Let O be an arbitrary point in the plane. A *pencil of lines* $\mathcal{L}_O$ centered at O is the set of all possible Euclidean lines passing through O.

Consider a pencil of lines $\mathcal{L}_O$ and an arbitrary line ℓ that does not pass through the point O. Then all lines of the pencil $\mathcal{L}_O$, except one, intersect the line ℓ. Thus, we obtain a map between the points of the line ℓ and the lines of the pencil $\mathcal{L}_O$. This observation motivates the following

Definition 7.2. A *projective line* is an arbitrary pencil of lines $\mathcal{L}_O$. A line from this pencil is called a *projective point*. If ℓ is an arbitrary Euclidean line not passing through O, then the projective point $\mathcal{J} \in \mathcal{L}_O$ corresponding to a line parallel to ℓ is called the *point at infinity* of ℓ. Allowing for a free speech, we will say that the Euclidean line ℓ is extended by the point at infinity $\mathcal{J}$.

§7.2 Circles as special conics

A (nondegenerate) *conic* in the projective plane can be thought of as the projective image of a circle. A projective transformation sends lines to lines and conics to conics, and preserves incidence relations such as collinearity, concurrence, and intersection.

In particular, we may regard a circle as a special type of conic. Conversely, any conic is projectively equivalent to a circle, meaning there exists a projective transformation that sends that conic to a circle.

Thus any theorem about circles that uses only:

- incidence (who lies on which line or conic),
- concurrency,
- and cross-ratio (which we will not exploit explicitly here),

has an analogue for general conics: one applies a projective transformation sending the conic to a circle, proves the statement there, and transforms back.

§7.3 Concurrence of lines from three conics

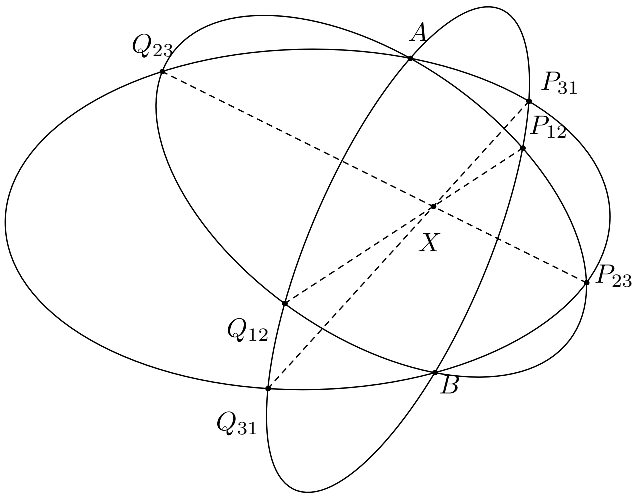
We now state and prove the conic concurrency fact mentioned in the introduction.

Theorem 7.3 (Three conics theorem)

Let three nondegenerate conics C_1, C_2, C_3 in the plane all pass through two distinct points A, B. For each pair (C_i, C_j), let the four intersection points be

$$C_i \cap C_j = \{A, B, P_{ij}, Q_{ij}\}.$$

Let ℓ_{ij} be the line through $P_{ij}Q_{ij}$. Then the three lines $\ell_{12}, \ell_{23}, \ell_{31}$ are concurrent.



Projective proof via circles and radical center. Consider a projective transformation Φ that sends A and B to two fixed points I, J on the line at infinity, often called the *circular points*. In this new projective coordinate system, each conic $\Phi(C_k)$ passes through I and J, hence is a circle in the Euclidean chart.

For each pair (C_i, C_j), the image of their intersection set is

$$\Phi(C_i) \cap \Phi(C_j) = \{I, J, \Phi(P_{ij}), \Phi(Q_{ij})\}.$$

The line $\Phi(\ell_{ij})$ is the line through $\Phi(P_{ij})$ and $\Phi(Q_{ij})$, i.e. the line of the *finite* intersection points of two circles that also intersect at the two circular points. This line is exactly the *radical axis* of the circles $\Phi(C_i)$ and $\Phi(C_j)$.

We thus obtain three circles $\Phi(C_1), \Phi(C_2), \Phi(C_3)$ and their three pairwise radical axes $\Phi(\ell_{12}), \Phi(\ell_{23}), \Phi(\ell_{31})$. By the classical radical center theorem for circles, these three radical axes are concurrent at a point R' (the radical center).

Projective transformations preserve incidence, so the preimages $\ell_{12}, \ell_{23}, \ell_{31}$ under Φ^{-1} are also concurrent at a point $R = \Phi^{-1}(R')$. This is exactly the desired result. ■

Conceptually, this theorem is nothing more than the radical center theorem for three circles, recast in the language of conics via a projective change of coordinates.

§7.4 A Conic Analogue of the Forgotten Coaxiality Lemma

Let C_1 and C_2 be two nondegenerate conics in the affine plane, given by

$$C_1 : \ F_1(x, y) = 0, \qquad C_2 : \ F_2(x, y) = 0,$$

where F_1 and F_2 are quadratic polynomials. For a point $P = (x_0, y_0)$ we define the *generalized power* of P with respect to C_i to be simply the value $F_i(P)$.

Lemma 7.4 (Conic Coaxiality / Pencil Lemma)
Let P and Q be two points not lying on either C_1 or C_2. The following are equivalent:

1. $$\frac{F_1(P)}{F_2(P)} = \frac{F_1(Q)}{F_2(Q)}.$$

2. The points P and Q lie on a common conic from the pencil generated by C_1 and C_2, i.e. on a conic of the form
$$C_{\lambda,\mu} : \ \lambda F_1(x, y) + \mu F_2(x, y) = 0 \qquad (\lambda, \mu) \neq (0, 0).$$

Proof. Suppose

$$\frac{F_1(P)}{F_2(P)} = \frac{F_1(Q)}{F_2(Q)} = t.$$

Then $F_1(P) = tF_2(P)$ and $F_1(Q) = tF_2(Q)$, so both P and Q satisfy $F_1 - tF_2 = 0$, i.e. they lie on the conic $C_{1,-t}$ of the pencil.

Conversely, if P and Q lie on a conic $\lambda F_1 + \mu F_2 = 0$ with $\lambda \neq 0$, then

$$F_1(P) = -\frac{\mu}{\lambda} F_2(P), \qquad F_1(Q) = -\frac{\mu}{\lambda} F_2(Q),$$

and therefore the two ratios coincide. ■

Remark 7.5. When C_1 and C_2 are circles, $F_i(P)$ equals the usual power of P with respect to ω_i, and the pencil $\lambda F_1 + \mu F_2 = 0$ is precisely the coaxial family of circles. Thus the lemma reduces to the classical forgotten coaxiality lemma.

§7.5 Remark on Poncelet's Porism

Many configurations involving power of a point for circles have analogues for general conics. A famous example is Poncelet's Porism:

Theorem 7.6 (Poncelet's Porism, informal)

Let C and D be two nondegenerate conics. Suppose there exists an n-gon that is inscribed in C and circumscribed about D. Then, starting from any point of C, one can construct such an n-gon; in fact there is a whole one-parameter family of them.

In this paper we proved a special case of this porism when both C and D are circles, using only PoP and circle geometry. The general conic–conic case is significantly deeper and uses projective and analytic methods (e.g. elliptic curves). We mention it here only to indicate that PoP ideas for circles sit naturally inside a much richer projective theory of conics.

Further Reading

The literature below gives a broader perspective on the methods appearing in this paper. Some items offer classical background (projective duality, involutions), others showcase modern olympiad geometry applications (moving points, cool ratio lemma), and some connect circle geometry to higher-degree curves.

1. Ratio Lemma
2. A walk through the Projective Plane
3. Point moving
4. On the Desargues' Involution Theorem
5. Clawson Conjugates
6. Isoptic cubics (cubics of foci) and more circulal cubics
7. Poncelet Invariants in the light of Cool ratio lemma

§A Appendix A. Minimal Projective Background

In this appendix we collect the projective notions used in the paper. We state everything over the real plane for intuition, but later we briefly pass to the complex projective plane when needed.

Throughout, we freely use the language "point", "line", "conic" in the projective plane; in an affine chart these correspond to ordinary points, lines, and (nondegenerate) conic sections.

§A.1 Projective line and cross-ratio

Definition A.1 (Projective line)**.** Fix a point O in the Euclidean plane. The set of all lines through O is called the *projective line* with center O; its elements are called *projective points*. If ℓ is any affine line not passing through O, then every projective point except one meets ℓ in exactly one point. The unique line through O parallel to ℓ is called the *point at infinity* of ℓ.

Thus we can think of a projective line as an ordinary line completed by one extra point at infinity.

Definition A.2 (Cross-ratio on a line)**.** Let A, B, C, D be four distinct collinear points. Their *cross-ratio* is

$$[A, B; C, D] = \frac{\overline{AC}}{\overline{BC}} : \frac{\overline{AD}}{\overline{BD}},$$

where lengths are directed. The ordered quadruple $(A, B; C, D)$ is called *harmonic* if $[A, B; C, D] = -1$.

One may equivalently define cross-ratio on a bundle of lines through a vertex O by

$$[\ell_1, \ell_2; \ell_3, \ell_4] = \frac{\sin\angle(\ell_1, \ell_3)}{\sin\angle(\ell_2, \ell_3)} : \frac{\sin\angle(\ell_1, \ell_4)}{\sin\angle(\ell_2, \ell_4)},$$

and check that intersecting the bundle with a fixed line recovers the point-version above.

A key property is that if

$$[A, B; C, D_1] = [A, B; C, D_2],$$

then $D_1 = D_2$. In this sense the cross-ratio plays the role of a projectively natural "distance" along a line, measured relative to three base points.

Definition A.3 (Projective map of a line)**.** A map f from one projective line to another is called *projective* if it preserves cross-ratios:

$$[A, B; C, D] = [f(A), f(B); f(C), f(D)]$$

for all quadruples of distinct points.

Theorem A.4 (Uniqueness on a line)

Any projective self-map of a line is uniquely determined by the images of three distinct points. Equivalently, given two ordered triples of distinct points (A, B, C) and (A', B', C') there exists a unique projective transformation sending $A \mapsto A'$, $B \mapsto B'$, $C \mapsto C'$.

§A.2 Projective plane, homogeneous coordinates, and duality

One can define the projective plane similarly as a bundle of lines in 3-dimensional space.

Definition A.5 (Projective plane)**.** Fix a point O in $\mathbb{R}^3$. The set of all lines through O is the *projective plane*. Each such line is a *projective point*, and each affine plane through O is a *projective line*. If we intersect this bundle with a fixed affine plane $z = 1$, the resulting chart identifies projective points with pairs (x, y) in the usual plane, plus a line at infinity $z = 0$.

In coordinates, a projective point is represented by a nonzero triple $(x : y : z)$, defined up to multiplication by a nonzero scalar; these are the *homogeneous coordinates* of the point. Likewise, a projective line has an equation $ax + by + cz = 0$, again defined up to scalar, so we may represent it by $(a : b : c)$. A point $(x : y : z)$ lies on a line $(a : b : c)$ if and only if $ax + by + cz = 0$.

Note the symmetry between the two triples $(x : y : z)$ and $(a : b : c)$. This is formalized as follows.

Theorem A.6 (Projective duality)
Any statement in projective plane geometry remains valid if one interchanges the words "point" and "line" throughout (and adjusts incidence correspondingly).

This principle lets us systematically "dualize" theorems, for instance the pole–polar duality for conics and the Pascal–Brianchon pair.

Definition A.7. A *projective transformation of the projective plane* is a map that sends projective lines to projective lines.

Projective transformations can be conveniently described in homogeneous coordinates. Consider a linear transformation of the three–dimensional space $\mathbb{R}^3$ given by

$$(x, y, z) \longmapsto \big(a_{11}x + a_{12}y + a_{13}z,\ a_{21}x + a_{22}y + a_{23}z,\ a_{31}x + a_{32}y + a_{33}z\big),$$

where a_{ij} are real numbers. This linear map induces a projective transformation of the projective plane $\mathbb{P}^2$, whose homogeneous coordinates transform as

$$(x : y : z) \longmapsto \big(a_{11}x + a_{12}y + a_{13}z : a_{21}x + a_{22}y + a_{23}z : a_{31}x + a_{32}y + a_{33}z\big).$$

From the viewpoint of Euclidean coordinates on the affine chart $\{z = 1\}$, a projective transformation is given by the usual fractional–linear formulas:

$$(x, y) \longmapsto \left(\frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}\right).$$

There is also a fundamental theorem that allows one to recover a unique projective transformation from the images of a suitable set of points. Namely: knowing the images of four points in general position uniquely determines a projective transformation.

Theorem A.8 (Uniqueness on the plane)
Given two quadruples of points in general position (A, B, C, D) and (A', B', C', D'), there exists a unique projective transformation carrying A, B, C, D to A', B', C', D' respectively.

§A.3 Conics and quadratic forms

We now recall the analytic definition of a conic that works well in the projective setting.

Definition A.9 (Conic in the affine and projective plane)**.** A (real) *conic* in the affine plane is the zero set of a quadratic polynomial

$$F(x, y) \in \mathbb{R}[x, y], \qquad \deg F = 2.$$

It is *nondegenerate* if F is irreducible over $\mathbb{R}$.

To view the same conic in the projective plane, we homogenize F to an homogeneous quadratic form

$$\widetilde{F}(x, y, z) = z^2 F\Big(\frac{x}{z}, \frac{y}{z}\Big)$$

and consider the zero set

$$\mathcal{C} = \{(x : y : z) \neq (0 : 0 : 0) \mid \widetilde{F}(x, y, z) = 0\}.$$

We do not distinguish notationally between the affine conic and its projective closure.

A central fact is that, projectively, all nondegenerate conics look alike:

Theorem A.10 (Every conic is a projective image of a circle)
The projective image of a circle is a conic. Conversely, any nondegenerate conic in the projective plane is the image of some circle under a suitable projective transformation.

Thus any projective statement about circles that only uses incidence and cross-ratio has a direct analogue for general conics.

We also extend cross-ratio to quadruples of points on a conic.

Definition A.11 (Cross-ratio on a conic)**.** Let A, B, C, D be four distinct points on a nondegenerate conic $\mathcal{C}$. Choose any point $X \in \mathcal{C}$ different from those four, and consider the four lines XA, XB, XC, XD. Their cross-ratio is

$$[A, B; C, D]_{\mathcal{C}} := [XA, XB; XC, XD].$$

This value is independent of the choice of X, and agrees with the usual cross-ratio after sending $\mathcal{C}$ to a circle or line by a projective transformation.

With this definition, a conic is projectively isomorphic to the projective line: projective maps of a conic are precisely those that preserve cross-ratio.

§A.4 Poles and polars with respect to a conic

Let $\mathcal{C}$ be a nondegenerate conic given in homogeneous coordinates by a quadratic form $q(v) = 0$, where v is a column vector and Q is a symmetric 3×3 matrix. Define a symmetric bilinear form

$$Q(u, v) = \frac{q(u + v) - q(u) - q(v)}{2}.$$

Definition A.12 (Polar line and pole)**.** Two vectors u, v are called *conjugate* with respect to $\mathcal{C}$ if $Q(u, v) = 0$. For a fixed nonzero vector u, the set of all v with $Q(u, v) = 0$ is a projective line; this is the *polar* of the point represented by u. Conversely, given a projective line ℓ, any two distinct vectors on ℓ have polars that meet in a single point whose vector u is determined up to scale by $Q(u, v_1) = Q(u, v_2) = 0$; this point is called the *pole* of ℓ.

The correspondence "point $\leftrightarrow$ polar line" defined by $\mathcal{C}$ is known as the *polarity* induced by the conic.

Theorem A.13 (Principle of duality for a conic)
Let $\mathcal{C}$ be a nondegenerate conic. Let A, B be two points with polars a, b. Then

$$A \in b \iff B \in a.$$

Equivalently, three points are collinear if and only if their polars are concurrent, and dually, three lines are concurrent if and only if their poles are collinear.

In the Euclidean case where $\mathcal{C}$ is a circle, this recovers the usual notions of poles and polars defined via tangents and power of a point; the algebraic description above allows us to extend these ideas to arbitrary conics in the projective plane.

§A.5 Complex projective plane and circular points

For some global arguments it is convenient to work over the complex projective plane $\mathbb{CP}^2$. All definitions above extend verbatim to this setting. An important role is played by the two *circular points at infinity*,

$$I = (1 : i : 0), \quad J = (1 : -i : 0),$$

which lie on the line at infinity $z = 0$.

A (complex) conic is a (possibly complex) circle in some affine chart if and only if it passes through both I and J. This leads to the following reduction theorem.

Theorem A.14 (Reduction of conics to circles) (a) Over $\mathbb{CP}^2$, any two nondegenerate conics can be simultaneously carried to two circles by a complex projective transformation.

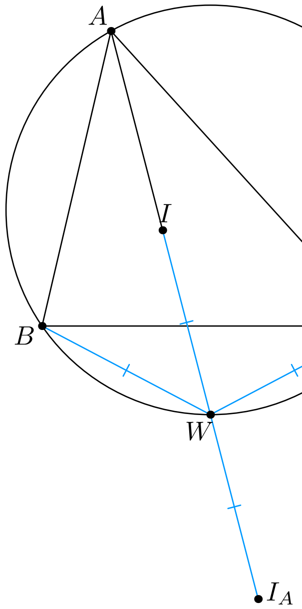
(b) Over $\mathbb{RP}^2$, if two real conics have at most two common points (counted with multiplicity), then there exists a real projective transformation sending both of them to circles.

In this paper we use this theorem in one direction only: certain concurrency statements for three conics through two points are reduced, via a projective transformation sending those points to I and J, to the radical-center theorem for three circles.

§B Appendix B. Common Facts

Lemma B.1 (Fact 5)

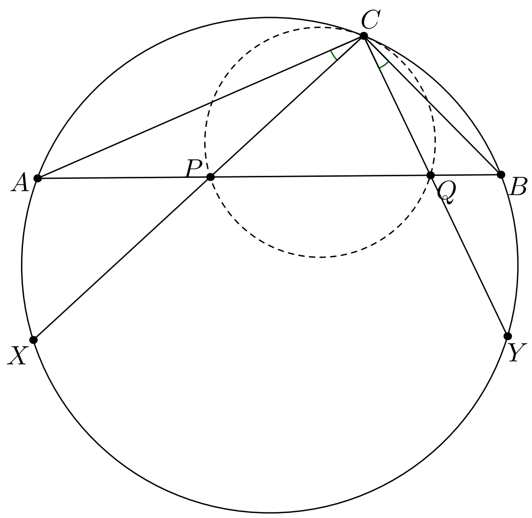
Let I be the incenter of triangle ABC, I_A be the excenter of A, and W be the midpoint of arc BC that does not contain point A. Then $WB = WI = WC = WI_A$.



Proof. It suffices to verify via angle chasing that $WB = WI$ (and similarly $WC = WI$). To prove that WI_A is equal to the other three segments, one needs to observe that W is the center of the circle $(BICI_A)$. ■

Lemma B.2 (Extended Shooting Lemma)

On the circumcircle of triangle ABC, points X and Y are taken so that $\angle ACX = \angle BCY$. Lines CX and CY intersect the segment AB at points P and Q. Then the circumcircles of triangles CPQ and ABC are tangent.



Proof. Let ℓ be tangent to the circle (ABC) at the point C. We'll show that ℓ also touches the circle (CPQ). Note that $\angle(\ell, CB) = \angle CAB$. But $\angle CPQ = \angle ACP + \angle CAP = \angle BCY + \angle(\ell, CB) = \angle(\ell, CQ)$, whence ℓ touches (CPQ). ■

Remark B.3. Shooting Lemma (ordinary) is the case when $X \equiv Y \equiv M$ and $P \equiv Q \equiv L$. Similar triangles show that

$$MA^2 = MB^2 = ML \cdot MC.$$

Theorem B.4 (Stewart's Theorem)

In triangle ABC, AD is an arbitrary cevian. If $AB = c$, $BC = a$, $CA = b$, $AD = d$, $BD = m$, $CD = n$, then

$$b^2m + c^2n = a(d^2 + mn).$$

Proof. Compute the cosine of angle B in triangles ABD and ABC and equate the resulting expressions. ■

§A Appendix C. Abbreviations

- MO — Mathematical Olympiad
- Regional — Kazakhstan Regional Olympiad
- KazMO — Kazakhstan National Olympiad
- All-Russian — All-Russian Olympiad
- IGO — Iranian Geometry Olympiad
- TST — Team Selection Test
- TSTST — Team Selection for the Team Selection Test
- USA(J)MO — USA (Junior) Math Olympiad
- USEMO — US Ersatz Math Olympiad
- ELMO — ELMO
- IZhO — International Zhautykov Olympiad
- APMO — Asia-Pacific Math Olympiad
- MMO — Moscow Math Olympiad
- BMO — Balkan Math Olympiad
- CGMO — Chinese Girls' Math Olympiad
- RMM — Romanian Master of Mathematics
- IMO — International Math Olympiad

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