A Possible Approach to Proving the Riemann Hypothesis

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Abstract

This paper will start with the derivation of the Euler-Maclaurin formula with singularities, compensate for the series divergence problem of its fitting the zeta function by adding a compensation factor ε, perform analytic continuation on the expanded series, analyze the distribution of its trivial zeros through the laws of Bernoulli numbers, and then construct functions and combine the properties of the gamma function to solve the distribution law of non-trivial zeros.

keywords: Riemann Hypothesis, Analytic Continuation, Euler-Maclaurin Formula, Gamma Function, Zero Analysis.

1 Introduction

The Riemann Hypothesis RH [4][8] is one of the most profound important problems in mathematics, bridging number theory and complex analysis. Proposed by the German mathematician Bernhard Riemann in 1859, in his landmark paper *On the Number of Primes Less Than a Given Magnitude.* Classified as one of the "Seven Millennium Prize Problems" by the Clay Mathematics Institute, with a $1 million reward for its proof or disproof.

Riemann's original work [7]: First extended the zeta function from the real axis to the complex plane, and conjectured that all non-trivial zeros lie on the line $\text{Re}(s) = \frac{1}{2}$; he also noted the connection between this conjecture and the error term of the Prime Number Theorem. 1896 breakthrough: Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the Prime Number Theorem using properties of the zeta function 20th-21st century progress: Numerical calculations have verified the conjecture for the first 10^{13} non-trivial zeros.

The Riemann zeta function is the foundation of the Riemann Hypothesis, defined and extended as follows:

For a complex variable $s = \sigma + it$ (where $\sigma = Re(s)$ is the real part and $t = Im(s)$ is the imaginary part), the zeta function is defined by the infinite series:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \tag{1}$$

The analytic continuation of the Riemann zeta function is a classic problem in complex analysis. Riemann's original work [7] has been elaborated on by [5], which includes an English translation and interpretations from a modern perspective. Based on Riemann's pioneering work , the zeta function can be analytically continued via its functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

Here, $\Gamma(s)$ denotes the gamma function, a generalization of the factorial function to complex numbers.

Trivial zeros:
The zeta function has zeros at all negative even integers $s = -2, -4, -6, \ldots$;
Non-trivial zeros:
All zeros lying in the critical strip $0 < \text{Re}(s) < 1$; the Riemann Hypothesis makes the core assertion:

$$\boxed{\text{All non-trivial zeros of the Riemann zeta function have real part } \frac{1}{2}}$$

2 Preliminaries

Before stating the main theorem, I recall some basic definitions and results.

2.1 Bernoulli Numbers: Recurrence Relation

Bernoulli numbers $\{B_n\}_{n=0}^{\infty}$ are a sequence of rational numbers fundamental in number theory and analysis. They satisfy the following recurrence relation:

- **Initial condition**: $B_0 = 1$.
- For $n \geq 1$, the sum $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$, where $\binom{n+1}{k}$ denotes the binomial coefficient.

This recurrence allows computing Bernoulli numbers iteratively: starting from $B_0 = 1$, one can find $B_1, B_2, \ldots$ successively. For example, solving the equation for $n = 1$ gives $B_1 = -\frac{1}{2}$, and for $n = 2$, we get $B_2 = \frac{1}{6}$, etc, In some contexts, $B_1$is defined as $\frac{1}{2}$, which will be mentioned later.

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \tag{3}$$

2.2 Euler-Maclaurin Formula

The Euler-Maclaurin Formula bridges discrete summation and continuous integration, playing a pivotal role in approximating sums and analyzing series. For a function f that is m-times continuously differentiable on $[a, b]$ (denoted $f \in C^m[a, b]$), the formula is:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] + R_m \tag{4}$$

$$\sum_{k=a}^{b} f(k) \sim \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \tag{5}$$

Here:

- $f^{(2j-1)}$ is the $(2j - 1)$-th derivative of f.
- R_m is the remainder term, which can be bounded using higher derivatives of f and Bernoulli numbers.

The formula quantifies the difference between summing f at integer points and integrating f, with corrections involving Bernoulli numbers and derivatives of f.

2.3 Gamma Function [9][1]

The Gamma function (denoted as $\Gamma(z)$), proposed by Euler, is a tool for extending the factorial function to the complex plane. It generalizes the factorial of positive integers to all complex numbers except non-negative integer poles ($z = 0, -1, -2, \ldots$), serving as a key link in number theory and complex analysis.

- $$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{6}$$

  Significance:For positive integers n, it satisfies $\Gamma(n) = (n - 1)!$

- $$\Gamma(z) = \lim_{n \to +\infty} \frac{n! \cdot n^z}{z(z + 1) \cdots (z + n)} \tag{7}$$

  Significance: It directly reflects the nature of factorial limits. The denominator clearly shows the positions of poles, while the numerator term n^z acts as a normalization factor for convergence.

- $$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{8}$$

  Multiplication Formula (Legendre's Formula)

- $$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{9}$$

  Reflection Formula (Residue Formula)

2.4 The analytic continuation of the Riemann zeta function

In his book *Introduction to Analytic Number Theory* [2].

Apostol used the **Euler-Maclaurin formula with a remainder term** to provide an analytic continuation for the Riemann zeta function. By applying this formula to partial sums and combining the limiting process with remainder analysis, he obtained an explicit formula valid throughout the complex plane (except at $s = 1$):

$$\zeta(s) = -\frac{1}{1-s} + \frac{1}{2} + \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}(s)_{2j-1} + \int_1^\infty \frac{\tilde{B}_{2m}(x)}{(2m)!}\cdot\frac{(s)_{2m}}{x^{s+2m}}dx \tag{10}$$

$(s)_k$ is the Pochhammer symbol, and $\tilde{B}_{2m}(x)$ are the periodic Bernoulli polynomials. This expression achieves the analytic continuation of $\zeta(s)$ and clearly reveals its simple pole at $s = 1$, This formula converges when $\mathrm{Re}(s) > 1-2m$. From this formula, the distribution law of the trivial zeros can be easily observed, but it is rather difficult to derive information about the non-trivial zeros.

3 Main Conclusions and Proofs

Theorem 3.1. *For a singular function $f(x)$, x_0 is a pole of $f(x)$, the Euler-Maclaurin formula can be transformed into:*

$$\sum_{k=a}^{b} f(k) = \sum_{j=0}^{\infty} \frac{\varepsilon B_j}{j!}\int_{a-1}^{b} \frac{d^j}{dx^j} f(x)\,dx \qquad B_1 = \frac{1}{2} \tag{11}$$

$$\sum_{k=a}^{b} f(k) = \sum_{j=0}^{\infty} \frac{\varepsilon B_j}{j!}\int_{a}^{b+1} \frac{d^j}{dx^j} f(x)\,dx \qquad B_1 = -\frac{1}{2} \tag{12}$$

ε is the compensation factor.

Proof of Theorem3.1

Perform the Taylor expansion of $f(x)$ at $x = x_1$.

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(x_1)}{j!}(x-x_1)^j \tag{13}$$

Define it:

$$Q(x,N) = \sum_{j=0}^{N} \frac{f^{(j)}(x_1)}{j!}(x-x_1)^j \tag{14}$$

Both a and b lie within the convergence domain.

$$\begin{aligned}\sum_{k=a}^{b} f(k) &= f(x_1)\sum_{j=a}^{b}(j-x_1)^0 + \frac{f^{(1)}(x_1)}{1!}\sum_{j=a}^{b}(j-x_1)^1 + \frac{f^{(2)}(x_1)}{2!}\sum_{j=a}^{b}(j-x_1)^2 + ...\\ &= f(x_1)\sum_{j=1}^{b-x_1} j^0 + \frac{f^{(1)}(x_1)}{1!}\sum_{j=1}^{b-x_1} j^1 + \frac{f^{(2)}(x_1)}{2!}\sum_{j=1}^{b-x_1} j^2 + ...\\ &- [f(x_1)\sum_{j=1}^{a-x_1-1} j^0 + \frac{f^{(1)}(x_1)}{1!}\sum_{j=1}^{a-x_1-1} j^1 + \frac{f^{(2)}(x_1)}{2!}\sum_{j=1}^{a-x_1-1} j^2 + ...]\\ &= \lim_{N\to\infty}[\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{b-x_1}(j-x_1)^j - \sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{a-x_1-1}(j-x_1)^j]\end{aligned} \tag{15}$$

Based on the formula for calculating the sum of powers:

$$S(N,k) = \sum_{j=1}^{N} j^k = \frac{1}{k+1}\sum_{m=0}^{k}\binom{k+1}{m} B_m N^{k+1-m} \tag{16}$$

Therefore, the left half of Formula (15) is as follows

$$\begin{aligned}&\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{b-x_1}(j-x_1)^j\\ &= \frac{B_0}{0!}[\frac{f^{(1)}(x_1)}{1!}(b-x_1) + \frac{f^{(2)}(x_1)}{2!}\frac{(b-x_1)^2}{2} + \frac{f^{(3)}(x_1)}{3!}\frac{(b-x_1)^3}{3} + ... + \frac{f^{N}(x_1)}{N!}\frac{(b-x_1)^N}{N}]\\ &+ \frac{B_1}{1!}[\frac{f^{(2)}(x_1)}{2!}(b-x_1) + \frac{f^{(3)}(x_1)}{3!}(b-x_1)^2 + ... + \frac{f^{N}(x_1)}{N!}(b-x_1)^{N-1}]\\ &+ \frac{B_2}{2!}[\frac{2f^{(3)}(x_1)}{3!}(b-x_1) + ... + \frac{(N-1)f^{N}(x_1)}{N!}(b-x_1)^{N-2}]\\ &+ ...\end{aligned} \tag{17}$$

Define the deviation function:

$$\varepsilon(x,N)(f(x) - f(x_1)) = \sum_{j=1}^{N}\frac{f^{(j)}(x_1)}{j!}(x-x_1)^j \tag{18}$$

From the Taylor expansion, we know that:

$$\lim_{N\to\infty}\varepsilon(x,N) = 1 \qquad 0 < \varepsilon(x,N) < 1 \tag{19}$$

$$\varepsilon(x,N) \to 0 \qquad N \to 0 \tag{20}$$

Substitute Formula (18) into the above Formula (17).

$$\begin{aligned}&\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{b-x_1}(j-x_1)^j\\ &= \frac{B_0}{0!}\varepsilon(b,N)\int_{x_1}^{b} f(x)dx + \frac{B_1}{1!}\varepsilon(b,N-1)(f(b)-f(x_1)) + \frac{B_2}{2!}\varepsilon(b,N-2)(f^{(1)}(b)-f^{(1)}(x_1)) + ...\end{aligned} \tag{21}$$

Next:

$$\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{b-x_1}(j-x_1)^j-\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{a-x_1-1}(j-x_1)^j$$
$$=\frac{B_0}{0!}\varepsilon_1(b,N)\int_{x_1}^{b}f(x)dx+\frac{B_1}{1!}\varepsilon_1(b,N-1)(f(b)-f(x_1))+\frac{B_2}{2!}\varepsilon_1(b,N-2)(f^{(1)}(b)-f^{(1)}(x_1))+...$$
$$-\frac{B_0}{0!}\varepsilon_2(a-1,N)\int_{x_1}^{b}f(x)dx-\frac{B_1}{1!}\varepsilon_2(a-1,N-1)(f(b)-f(x_1))$$
$$-\frac{B_2}{2!}\varepsilon_2(a-1,N-2)(f^{(1)}(b)-f^{(1)}(x_1))-...$$
$$=\sum_{j=0}^{N-1}\varepsilon_1(b,N-j)\frac{B_j}{j!}\int_{x_1}^{b}\frac{d^j}{dx^j}f(x)dx-\sum_{j=0}^{N-1}\varepsilon_2(a-1,N-j)\frac{B_j}{j!}\int_{x_1}^{a-1}\frac{d^j}{dx^j}f(x)dx \tag{22}$$

so.

$$\sum_{k=a}^{b}f(k)=\mathop{lim}_{N\to\infty}[\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{b-x_1}(j-x_1)^j-\sum_{k=0}^{N}\frac{f^{(k)}(x_1)}{k!}\sum_{j=1}^{a-x_1-1}(j-x_1)^j]$$
$$=\mathop{lim}_{N\to\infty}[\sum_{j=0}^{N-1}\varepsilon_1(b,N-j)\frac{B_j}{j!}\int_{x_1}^{b}\frac{d^j}{dx^j}f(x)dx-\sum_{j=0}^{N-1}\varepsilon_2(a-1,N-j)\frac{B_j}{j!}\int_{x_1}^{a-1}\frac{d^j}{dx^j}f(x)dx] \tag{23}$$
$$=\mathop{lim}_{N\to\infty}[\sum_{j=0}^{N-1}\varepsilon(N-j)\frac{B_j}{j!}\int_{a-1}^{b}\frac{d^j}{dx^j}f(x)dx]$$

It can be expressed in the following form:

$$\sum_{k=a}^{b}f(k)=\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}\int_{a-1}^{b}\frac{d^j}{dx^j}f(x)dx \tag{24}$$

Under the condition that $B_1=-\frac{1}{2}$, it can be obtained through transformation that

$$\sum_{k=a}^{b}f(k)=\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}\int_{a}^{b+1}\frac{d^j}{dx^j}f(x)dx \tag{25}$$

Theorem 3.2. *The formal series of the zeta function after analytic continuation is as follows:*

- $(-s+1)_k^{\downarrow}$ *is a falling factorial.*

$$\zeta(s)=\frac{1}{s-1}\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}(-s+1)_j^{\downarrow} \tag{26}$$

Proof of Theorem3.2

Lemma 3.3. *according to the Riemann zeta function*(1).

$$\zeta(s)=\lim_{M\to\infty}\sum_{k=1}^{M}\frac{1}{k^s} \tag{27}$$

Now we combine the zeta function with formula(25).

$$\zeta(s)=\lim_{M\to\infty}\sum_{k=1}^{M}\frac{1}{k^s}=\lim_{M\to\infty}\lim_{N\to\infty}\sum_{j=0}^{N-1}\varepsilon(N-j)\frac{B_j}{j!}\int_{1}^{M+1}\frac{d^j}{dx^j}\frac{1}{x^s}dx \tag{28}$$

Expanding it, we can obtain:

$$\zeta(s)=\lim_{M\to\infty}\mathop{lim}_{N\to\infty}\left(\frac{B_0}{0!}\frac{\varepsilon(N)}{(-s+1)(M+1)^{s-1}}+\frac{B_1}{1!}\frac{\varepsilon(N-1)}{(M+1)^s}+\frac{B_2}{2!}\frac{\varepsilon(N-2)(-s)}{(M+1)^{s+1}}+...+C(s)\right) \tag{29}$$

$$C(s)=-\lim_{N\to\infty}\left(\frac{\varepsilon(N)B_0}{0!}\frac{1}{-s+1}+\frac{\varepsilon(N-1)B_1}{1!}+\frac{\varepsilon(N-2)B_2}{2!}(-s)+\frac{\varepsilon(N-3)B_3}{3!}(-s)(-s-1)+...\right) \tag{30}$$

$$=\frac{1}{s-1}\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}(-s+1)_j^{\downarrow} \tag{31}$$

$$C(s)=\frac{1}{s-1}\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}(-s+1)_j^{\downarrow} \tag{32}$$

Define $H(s,N)$:

$$H\left(s,M\right)=\frac{1}{-s+1}\lim_{M\to\infty}\sum_{j=0}^{\infty}\frac{\varepsilon B_k}{\left(M+1\right)^{s-1+j}j!}\left(-s+1\right)_j^{\downarrow} \tag{33}$$

$$\zeta\left(s\right)-\lim_{M\to\infty}H\left(s,M\right)=C\left(s\right) \tag{34}$$

When $Re(s)>1$

$$\lim_{M\to\infty}H\left(s,M\right)=0 \tag{35}$$

$Re(s)<1$

$$\lim_{M\to\infty}H\left(s,M\right)=\infty \tag{36}$$

According to the definition of analytic continuation, it can be obtained that:

$$\zeta\left(s\right)=C(s)=\frac{1}{s-1}\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}(-s+1)_j^{\downarrow} \tag{37}$$

4 Analysis of Zero Distribution Law

4.1 Distribution of Trivial Zeros

we define this function below.

$$h\left(s\right)=\sum_{j=0}^{\infty}\frac{\varepsilon B_j}{j!}\left(-s+1\right)_j^{\downarrow}=\lim_{N\to\infty}\sum_{j=0}^{N-1}\frac{\varepsilon\left(N-j\right)B_j}{j!}\left(-s+1\right)_j^{\downarrow} \tag{38}$$

For any sufficiently large T, we have:

$$\lim_{N\to\infty}\sum_{j=0}^{T}\frac{\varepsilon\left(N-j\right)B_j}{j!}\left(-s+1\right)_j^{\downarrow}=\sum_{j=0}^{T}\frac{B_j}{j!}\left(-s+1\right)_j^{\downarrow} \tag{39}$$

$$\zeta\left(s\right)=\lim_{T\to\infty}\lim_{N\to\infty}\sum_{j=0}^{T}\frac{\varepsilon\left(N-j\right)B_j}{j!}\left(-s+1\right)_j^{\downarrow}=\lim_{T\to\infty}\sum_{j=0}^{T}\frac{B_j}{j!}\left(-s+1\right)_j^{\downarrow} \tag{40}$$

The divergence of the formula on the right-hand side is with respect to T, but we do not observe T; therefore the zeros on both sides of the equation are equal.

Lemma 4.1. *According to the Bernoulli Recurrence Formula*(3).

$$\sum_{k=0}^{n}\binom{n+1}{k}B_k=\sum_{k=0}^{n}\frac{(n+1)_k^{\downarrow}}{k!}B_k \tag{41}$$

It is easy to see that $f(s)$ is the case when n tends to infinity in the Bernoulli recurrence formula. Since the Bernoulli numbers of odd terms with $n>1$ are zero, $-2,-4,-6,...$ are the non-trivial zeros of the zeta function.

4.2 Distribution Law of Non-Trivial Zeros

According to Formula (40), in order to study the non-trivial zeros, I need to separate the trivial zeros from the formal series, and here N is taken as the cutoff.

$$H\left(N\right)=\sum_{j=0}^{N}\left(-s+1\right)_{j}\frac{B_{j}}{j!} \tag{42}$$

Next, we separate the trivial zeros.

$$H(4)=\frac{B_4}{4!}((-s-2)(-s-4)(-s-5)(-s+9)) \tag{43}$$

$$H(6)=\frac{B_6}{6!}((-s-2)(-s-4)(-s-6)(-s-7)(a_1+b_1s+c_1s^2)) \tag{44}$$

$$\ldots \tag{45}$$

$$H(N)=\frac{B_N}{N!}((-s-2)(-s-4)(-s-6)\ldots(a+bs+cs^2+ds^3+\ldots)) \tag{46}$$

Define it.

$$g\left(s,N/2\right)=a+bs+cs^2+ds^3+\ldots+zs^N \tag{47}$$

$$g\left(s\right)=\lim_{N\to\infty}g\left(s,N/2\right) \tag{48}$$

For large N:

$$H(N)=\frac{B_N}{N!}((-s-2)(-s-4)(-s-6)\ldots g(s,N/2)) \tag{49}$$

The case where Bernoulli numbers are zero is not considered here, and this does not affect the result.

$$\zeta\left(s\right)=\frac{1}{s-1}\lim_{N\to\infty}H\left(N\right) \tag{50}$$

$$\zeta\left(s\right)=\lim_{N\to\infty}\frac{B_{2N}}{\left(2N\right)!}\frac{\prod_{j=1}^{N}\left(-s-2j\right)}{s-1}g(s,N) \tag{51}$$

$$\zeta\left(1-s\right)=\lim_{N\to\infty}\frac{B_{2N}}{\left(2N\right)!}\frac{\prod_{j=1}^{N}\left(s-2j-1\right)}{-s}g\left(1-s,N\right) \tag{52}$$

Lemma 4.2. *Here, we introduce the defining formula of the zeta function*(2).

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Since this formal series retains the properties of the zeta function, we can utilize this relational expression.

$$\lim_{N\to\infty}\frac{\prod_{j=1}^{N}\left(-s-2j\right)}{s-1}g\left(s,N\right)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(1-s\right)\lim_{N\to\infty}\frac{\prod_{j=1}^{N}\left(s-2j-1\right)}{-s}g\left(1-s,N\right) \tag{53}$$

To study the properties of $g(s,N)$, we isolate it on one side.

$$\lim_{N\to\infty}\frac{g\left(s,N\right)}{g\left(1-s,N\right)}=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma\left(1-s\right)\frac{1-s}{s}\lim_{N\to\infty}\frac{\prod_{j=1}^{N}\left(s-2j-1\right)}{\prod_{j=1}^{N}\left(-s-2j\right)} \tag{54}$$

Lemma 4.3. *According to the Gauss's infinite product form*(7) *of the Gamma function.*

$$\Gamma\left(z\right)=\lim_{n\to\infty}\frac{n!\,n^z}{z\left(z+1\right)\left(z+2\right)\left(z+3\right)\ldots\ldots\left(z+n\right)}=\lim_{n\to\infty}\frac{n^z}{z\left(1+z\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\ldots\ldots\left(1+\frac{z}{n}\right)} \tag{55}$$

$$\lim_{N\to\infty} \frac{\prod_{j=1}^{N}(s-2j-1)}{\prod_{j=1}^{N}(-s-2j)} = \lim_{N\to\infty} \frac{\prod_{j=1}^{N}\left(1-\frac{1-s}{2j}\right)}{\prod_{j=1}^{N}\left(1+\frac{s}{2j}\right)} = \lim_{N\to\infty} \frac{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)}{\frac{1-s}{2}\Gamma\left(\frac{1-s}{2}\right)} N^{\frac{1}{2}-s} \tag{56}$$

$$\lim_{N\to\infty} \frac{g(s,N)}{g(1-s,N)} = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\lim_{N\to\infty} N^{\frac{1}{2}-s} \tag{57}$$

Lemma 4.4. *It can be concluded from the multiplication formula*(8) *and reflection formula*(9) *of the Gamma function that*

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$$

so

$$\begin{aligned}\lim_{N\to\infty} \frac{g(s,N)}{g(1-s,N)} &= 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\lim_{N\to\infty} N^{\frac{1}{2}-s}\\ &= 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\frac{2^{-s}}{\sqrt{\pi}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\lim_{N\to\infty} N^{\frac{1}{2}-s}\\ &= 2^s\pi^{s-1}\sin(\frac{\pi s}{2})\frac{2^{-s}}{\sqrt{\pi}}\frac{\pi}{\sin(\frac{\pi s}{2})}\lim_{N\to\infty} N^{\frac{1}{2}-s}\\ &= \lim_{N\to\infty}(\frac{N}{\pi})^{\frac{1}{2}-s}\end{aligned} \tag{58}$$

4.3 Prove the Riemann Hypothesis.

In this section, I will prove the Riemann Hypothesis. First, let's organize the conditions.

Returning to our definition of $g(s)$.

$$g(s,N/2) = a + bs + cs^2 + ds^3 + ... + zs^N$$

$$g(s) = \lim_{N\to\infty} g(s,N/2)$$

- It can be concluded that For a fixed N, $g(s)$ and $g(1-s)$ have the same highest power.

According to the analytical definition of the Riemann zeta function.

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

- In the region $0 < \sigma < 1$, if s_0 is a non-trivial zero of the zeta function, then $1-s_0$ must also be a non-trivial zero of it, Since $g(s)$ is a function describing non-trivial zeros, the corresponding relationship is that if $g(s_0) = 0$, then it must be that $g(1-s_0) = 0$.

- $$\frac{g(s)}{g(1-s)} = 0 \qquad \left(\sigma > \frac{1}{2}\right) \tag{59}$$

- $$\frac{g(s)}{g(1-s)} = \infty \qquad \left(\sigma < \frac{1}{2}\right) \tag{60}$$

Here we use proof by contradiction: suppose there exists a point with $\sigma \neq \frac{1}{2}$ such that $g(s_0) = 0$.

Case 1: Assume $\sigma_0 < \frac{1}{2}$

Since $g(s_0) = 0$, $g(1-s_0) = 0$, and $1-s_0 = (1-\sigma_0)+it_0$. At this time, $1-\sigma_0 > \frac{1}{2}$; By the definition of zero order, $g(s)$ can be expanded at s_0 as:

$$g(s) = \lim_{N\to\infty}(s-s_0)^N \cdot h(s) \tag{61}$$

$h(s)$ is analytic near s_0 and $h(s_0) \neq 0$;

Similarly, $g(1-s)$ can be expanded at $s = 1 - s_0$ as:

$$g(1-s) = \lim_{N\to\infty} (s-(1-s_0))^N \cdot m(s) \tag{62}$$

the orders are the same; $m(s)$ is analytic near $1-s_0$ and $m(1-s_0) \neq 0$;
we need to analyze the limit of $\frac{g(s)}{g(1-s)}$ near s_0.

$$\frac{g(s)}{g(1-s)} = \lim_{N\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^N \frac{h(s)}{m(s)} \tag{63}$$

$$\lim_{s\to s_0} \frac{g(s)}{g(1-s)} = \lim_{s\to s_0}\lim_{N\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^N \frac{h(s)}{m(s)} = \lim_{N\to\infty} \left(\frac{0}{2s_0-1}\right)^N \frac{h(s)}{m(s)} = 0 \tag{64}$$

According to formula(60), when $\sigma_0 < \frac{1}{2}$, the limit of formula(64) should be ∞, but the actual calculated limit is 0, which is a contradiction. Hence, $\sigma_0 < \frac{1}{2}$ does not hold.

Case 2:Assume $\sigma_0 > \frac{1}{2}$

Since $g(s_0) = 0$, $g(1-s_0) = 0$, and $1-s_0 = (1-\sigma_0)+it_0$. At this time, $1-\sigma_0 < \frac{1}{2}$ (because $\sigma_0 > \frac{1}{2}$); Similarly:

$$\begin{aligned} \frac{g(s)}{g(1-s)} &= \lim_{N\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^N \frac{h(s)}{m(s)} \\ \lim_{s\to 1-s_0} \frac{g(s)}{g(1-s)} &= \lim_{s\to 1-s_0}\lim_{N\to\infty} (\frac{s-s_0}{s-(1-s_0)})^N \frac{h(s)}{m(s)} \\ &= \lim_{N\to\infty} (\frac{1-2s_0}{0})^N \frac{h(s)}{m(s)} \\ &= \infty \end{aligned} \tag{65}$$

According to formula(59), when $\sigma_0 > \frac{1}{2}$, the limit of formula(65) should be 0, but the actual calculated limit is ∞, which is a contradiction. Hence, $\sigma_0 > \frac{1}{2}$ does not hold.

The assumption that $g(s_0) = 0$ and $\sigma_0 \neq \frac{1}{2}$ leads to contradictions. the only possible case is $\sigma_0 = \frac{1}{2}$.

Therefore, all non-trivial zeros of the Riemann zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, and the Riemann Hypothesis is proven.

5 Conclusion

The Riemann Hypothesis is related to more than 1,000 mathematical propositions, and its proof is of great significance. The key of this paper lies in adding a compensation factor to correct the divergent behavior of the Euler-Maclaurin formula when expanding the zeta function, thereby obtaining the power series form of its analytic continuation, and then constructing it into a zero-point form to solve the non-trivial zeros. This method can also be transplanted to the solution of other L-functions, which is of great significance.

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