A Possible Approach to Proving the Riemann Hypothesis

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Abstract

This paper will expand the Riemann zeta function via the Euler-Maclaurin formula, analyze it together with the Riemann zeta function after analytic continuation, and derive a new formula describing the behavior of the zeta function. By analyzing this new formula related to Bernoulli numbers, the distribution law of its trivial zeros will be identified. Furthermore, the infinite product form of the gamma function and the analytic continuation definition of the Riemann zeta function will be used to find the distribution law of non-trivial zeros, ultimately proving the Riemann Hypothesis that $\mathrm{Re}(s) = \frac{1}{2}$.

keywords: Riemann Hypothesis, Analytic Continuation, Euler-Maclaurin Formula, Gamma Function, Zero Analysis.

1 Introduction

The Riemann Hypothesis is one of the most profound important problems in mathematics, bridging number theory and complex analysis.

1.1 Research Background

Proposed by the German mathematician Bernhard Riemann in 1859, in his landmark paper *On the Number of Primes Less Than a Given Magnitude*. Classified as one of the "Seven Millennium Prize Problems" by the Clay Mathematics Institute, with a $1 million reward for its proof or disproof.

1.2 Literature Review of Previous Studies

Riemann's original work: First extended the zeta function from the real axis to the complex plane, and conjectured that all non-trivial zeros lie on the line $\mathrm{Re}(s) = \frac{1}{2}$; he also noted the connection between this conjecture and the error term of the Prime Number Theorem. 1896 breakthrough: Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the Prime Number Theorem using properties of the zeta function 20th-21st century progress: Numerical calculations have verified the conjecture for the first 10^{13} non-trivial zeros.

1.3 Core Concepts: The Riemann Zeta Function[3]

The Riemann zeta function is the foundation of the Riemann Hypothesis, defined and extended as follows:

1.3.1 Definition (Domain Re$(s) > 1$)

For a complex variable $s = \sigma + it$ (where $\sigma = \text{Re}(s)$ is the real part and $t = \text{Im}(s)$ is the imaginary part), the zeta function is defined by the infinite series:

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s} \tag{1}$$

1.3.2 Analytical Continuation

To extend the zeta function beyond $\text{Re}(s) > 1$, the following functional equation is used, which is valid for all complex numbers $s \neq 1$ (where $s = 1$ is a simple pole):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

Here, $\Gamma(s)$ denotes the gamma function, a generalization of the factorial function to complex numbers.

1.4 Key Conclusions and the Riemann Hypothesis

Trivial zeros: The zeta function has zeros at all negative even integers $s = -2, -4, -6, \ldots$; **Non-trivial zeros**: All zeros lying in the "critical strip" $0 < \text{Re}(s) < 1$; the Riemann Hypothesis makes the core assertion:

$$\boxed{\text{All non-trivial zeros of the Riemann zeta function have real part } \frac{1}{2}}$$

2 Preliminaries

Before stating the main theorem, I recall some basic definitions and results.

2.1 Bernoulli Numbers: Recurrence Relation[4]

Bernoulli numbers $\{B_n\}_{n=0}^{\infty}$ are a sequence of rational numbers fundamental in number theory and analysis. They satisfy the following recurrence relation:

- **Initial condition**: $B_0 = 1$.
- For $n \geq 1$, the sum $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$, where $\binom{n+1}{k}$ denotes the binomial coefficient.

This recurrence allows computing Bernoulli numbers iteratively: starting from $B_0 = 1$, one can find $B_1, B_2, \ldots$ successively. For example, solving the equation for $n = 1$ gives $B_1 = -\frac{1}{2}$, and for $n = 2$, we get $B_2 = \frac{1}{6}$, etc, In some contexts, $B_1$is defined as $\frac{1}{2}$, which will be mentioned later.

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \tag{3}$$

2.2 Euler-Maclaurin Formula[2]

The Euler-Maclaurin Formula bridges discrete summation and continuous integration, playing a pivotal role in approximating sums and analyzing series. For a function f that is m-times continuously differentiable on $[a, b]$ (denoted $f \in C^m[a, b]$), the formula is:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right] \tag{4}$$

Here:

- B_{2j} are Bernoulli numbers with even indices (odd-indexed Bernoulli numbers $B_{2j+1} = 0$ for $j \geq 1$, simplifying the formula).
- $f^{(2j-1)}$ is the $(2j-1)$-th derivative of f.
- R_m is the remainder term, which can be bounded using higher derivatives of f and Bernoulli numbers (as shown in related lemmas if needed).

The formula quantifies the difference between summing f at integer points and integrating f, with corrections involving Bernoulli numbers and derivatives of f.

2.3 Gamma Function

The Gamma function (denoted as $\Gamma(z)$), proposed by Euler, is a tool for extending the factorial function to the complex plane. It generalizes the factorial of positive integers to all complex numbers except non-negative integer poles ($z = 0, -1, -2, \ldots$), serving as a key link in number theory and complex analysis.

- $$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{5}$$

  Significance:For positive integers n, it satisfies $\Gamma(n) = (n-1)!$.

- $$\Gamma(z) = \lim_{n \to +\infty} \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)} \tag{6}$$

  Significance: It directly reflects the nature of factorial limits. The denominator clearly shows the positions of poles, while the numerator term n^z acts as a normalization factor for convergence.

- $$\mathit{\Gamma}(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \mathit{\Gamma}(z)\, \mathit{\Gamma}\left(z + \frac{1}{2}\right) \tag{7}$$

  Multiplication Formula (Legendre's Formula).

- $$\mathit{\Gamma}(z)\, \mathit{\Gamma}(1-z) = \frac{\pi}{\sin(\pi z)} \tag{8}$$

  Reflection Formula (Residue Formula).

2.4 Dirichlet Eta Function[1]

The Dirichlet eta function (denoted as $\eta(s)$) is an alternating Dirichlet series, defined for a complex number $s = \sigma + it$ as:

$$\eta(s) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j^s} \tag{9}$$

Its domain of convergence is $\mathrm{Re}(s) > 0$ $(s \neq 1)$, which is broader than that of the original series of the Riemann zeta function (convergent only when $\mathrm{Re}(s) > 1$). For all complex numbers s with $\mathrm{Re}(s) > 0$ $(s \neq 1)$, the following identity holds:

$$\eta(s) = (1 - 2^{1-s})\zeta(s) \tag{10}$$

After rearrangement to $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}$, the Riemann zeta function can be analytically continued to the region $\mathrm{Re}(s) > 0$, covering the "critical strip" $(0 < \mathrm{Re}(s) < 1)$ which is crucial for the study of the Riemann Hypothesis. Within the critical strip $0 < \mathrm{Re}(s) < 1$, $\eta(s) = 0$ is completely equivalent to $\zeta(s) = 0$, meaning their non-trivial zeros (including the order of zeros) are identical.

3 Main Results and Proofs

Theorem 3.1. *Analytic Continuation Expression of the Riemann zeta-Function for $\sigma \neq 1$.*

- *B_k the k-th Bernoulli number, $(-s+1)_k$ is a falling factorial.*

  $$(-s+1)_0 = 1 \qquad (-s+1)_1 = (-s+1) \qquad (-s+1)_2 = (-s+1)(-s) \ \ldots$$

  Then, for any complex number s satisfying $\sigma \neq 1$.

  $$\zeta(s) = \begin{cases} \sum_{j=1}^{\infty} \frac{1}{j^s} & \sigma > 1 \\ \frac{1}{s-1} \sum_{j=0}^{\infty} \frac{B_j}{j!} (-s+1)_j & \sigma < 1 \end{cases}$$

When studying the divergent behavior of the Riemann zeta function for $\sigma < 1$ in the complex plane, I found that its divergence is similar to a spiral as the number of terms increases, and the coordinates of the spiral's center are highly likely to be the value of the Riemann zeta function after analytic continuation. Therefore, I expand the Riemann zeta function via the Euler-Maclaurin formula here.
Proof of Theorem3.1.

Lemma 3.2. *according to the Euler-Maclaurin formula*(4)*.*

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\, dx + \frac{f(a) + f(b)}{2} + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a) \right]$$

I want to write the Euler-Maclaurin formula as a more concise one.

$$\sum_{k=a}^{b} f(k) = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_a^{b+1} \frac{d^j}{dx^j} f(x)\, dx \qquad \left(B_1 = -\frac{1}{2} \right)$$

or

$$\sum_{k=a}^{b} f(k) = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_{a+1}^{b} \frac{d^j}{dx^j} f(x)\, dx \qquad \left(B_1 = \frac{1}{2}\right)$$

To prepare for the following content, replace a with 1 and b with n.

$$\sum_{k=1}^{n} f(k) = \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} f(x)\, dx \tag{11}$$

Lemma 3.3. *according to the Riemann zeta function*(1).

$$\zeta(s) = \sum_{j=1}^{\infty} \frac{1}{j^s}$$

$$\zeta(s) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s} \tag{12}$$

Now we combine the zeta function with formula(11).

$$\zeta(s) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s} = \lim_{n\to\infty} \sum_{j=0}^{\infty} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} \frac{1}{x^s} dx \tag{13}$$

We will expand this summation.

$$\lim_{n\to\infty} \sum_{j=0} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} \frac{1}{x^s} dx = \lim_{n\to\infty} (\frac{B_0}{0!} \frac{1}{(-s+1)(n+1)^{s-1}} + \frac{B_1}{1!} \frac{1}{(n+1)^s} + \frac{B_2}{2!} \frac{-s}{(n+1)^{s+1}} + ... + C(s)) \tag{14}$$

$$C(s) = -(\frac{B_0}{0!} \frac{1}{-s+1} + \frac{B_1}{1!} (-s+1)(-s) + \frac{B_2}{2!} (-s+1)(-s)(-s-1) + ...) = -\frac{1}{-s+1} \sum_{j=0}^{\infty} \frac{B_j}{j!} (-s+1)_j \tag{15}$$

It can be seen from the above formula that the divergence of the Riemann zeta function for $\sigma < 1$ is affected by $\frac{B_0}{0!} \frac{1}{(-s+1)(n+1)^{s-1}}$. Considering the process of analytic continuation of the Riemann zeta function in its integral form, the behavior of the singularity bypassed during the analytic continuation of the zeta function can be equivalent to us removing $\frac{B_0}{0!} \frac{1}{(-s+1)(n+1)^{s-1}}$; in this case, $\zeta(s) = C(s)$.

To verify this idea, I define a function where the independent variable is the number of terms of the zeta function.

$$T(n) = \sum_{j=1}^{n} \frac{1}{j^s}$$

$$T(n) \to \frac{1}{(-s+1)\, n^{1-s}} + C(s)_1 \qquad (n \to \infty) \tag{16}$$

Lemma 3.4. *Here, we introduce the Dirichlet eta function* (9)*, Since* $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}$ *is the analytic continuation of the Riemann zeta function for* $Re(s) > 0$ *.*

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ... \tag{17}$$

We derive the Dirichlet eta function using the Riemann zeta function in the following manner.

$$\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + ... \\
\frac{2}{2^s}\zeta(s) &= \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \frac{2}{8^s} + \frac{2}{10^s} + \frac{2}{12^s} + \frac{2}{14^s} + ... \\
\zeta(s) - \frac{2}{2^s}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + ... \\
&\quad -0 - \frac{2}{2^s} - 0 - \frac{2}{4^s} - 0 - \frac{2}{6^s} - 0 - \frac{2}{8^s} - ... \\
&= \eta(s)
\end{aligned}$$

Define this relational expression.

$$F(n) = \frac{2}{2^s}\sum_{k=1}^{n}\frac{1}{k^s} = 2\sum_{k=1}^{n}\frac{1}{(2k)^s} \tag{18}$$

By Formula (11), similarly to $T(n)$, we can obtain

$$F(n) \to \frac{1}{(-s+1)(2n)^{1-s}} + C(s)_2 \qquad (n \to \infty) \tag{19}$$

$$C(s)_2 = 2^{1-s}C(s)_1 \tag{20}$$

Then we can explain the origin of the Dirichlet eta function.

$$\eta(s) = \lim_{n\to\infty}\left(T(n) - F\left(\frac{n}{2}\right)\right) = C(s)_1 - C(s)_2 = \left(1 - \frac{2}{2^s}\right)C(s)_1 \tag{21}$$

Here, when $\zeta(s)$ is subtracted from $2^{1-s}\zeta(s)$,$\zeta(s)$ needs to have twice as many terms as $2^{1-s}\zeta(s)$, so naturally $F(n)$ needs to be transformed into $(F\left(\frac{n}{2}\right))$.

According to Formula (10).

$$\eta(s) = (1 - 2^{1-s})\zeta(s)$$

$$\eta(s) = \left(1 - \frac{2}{2^s}\right)C(s)_1$$

Here it is proved that $C(s)$ is indeed the analytic continuation form of the Riemann zeta function, and its domain is $\sigma < 1$.

$$\zeta(s) = \begin{cases} \sum_{j=1}^{\infty}\frac{1}{j^s} & \sigma > 1 \\ \frac{1}{s-1}\sum_{j=0}^{\infty}\frac{B_j}{j!}(-s+1)_j & \sigma < 1 \end{cases} \tag{22}$$

4 Analysis of Zero Distribution Law

4.1 Distribution of Trivial Zeros

we define this function below.

$$f(s) = (s-1)\zeta(s) = \sum_{j=0}^{\infty} \frac{B_j}{j!}(-s+1)_j \tag{23}$$

Lemma 4.1. *According to the Bernoulli Recurrence Formula*(3).

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = \sum_{k=0}^{n} \frac{(n+1)_k}{k!} B_k = 0 \tag{24}$$

$$f(s) = \frac{B_0}{0!}\cdot 1 + \frac{B_1}{1!}\cdot(-s+1) + \frac{B_2}{2!}\cdot(-s+1)(-s) + \frac{B_3}{3!}\cdot(-s+1)(-s)(-s-1) + \cdots + \frac{B_j}{j!}\cdot\prod_{k=0}^{j-1}(-s+1-k) + \cdots \tag{25}$$

Since all Bernoulli numbers with odd indices greater than 2 are equal to 0, it can be seen from the expanded form of $f(s)$ that $f(s) = 0$ when $s = -2, -4, -6, \ldots$. This also provides an understanding of the origin of non-trivial zeros from another perspective.

4.2 Distribution Law of Non-Trivial Zeros

The content of the Riemann Hypothesis is that the real parts of all non-trivial zeros of the zeta function are equal to 1/2. To verify this hypothesis, I will perform a special transformation on $f(s)$ and then study the distribution of its non-trivial zeros.

defin $H(n)$

$$H(n) = \sum_{k=0}^{n} (-s+1)_k \frac{B_k}{k!} \tag{26}$$

From the recurrence formula of Bernoulli numbers, the roots of $H(n)$ with respect to s can be expressed as the product of negative even roots and complex roots, define $g(s)$ as a function that affects the complex roots.

$$H(4) = \frac{B_4}{4!}((-s-2)(-s-4)(-s-5)(-s+9))$$

$$H(6) = \frac{B_6}{6!}\left((-s-2)(-s-4)(-s-6)(-s-7)\left(a_1 + b_1 s + c_1 s^2\right)\right)$$

...

$$H(n) = \frac{B_n}{n!}\left((-s-2)(-s-4)(-s-6)\ldots\left(a + bs + cs^2 + ds^3 + \ldots\right)\right)$$

Define it:

$$g(s) = \lim_{n\to\infty}\left(a + bs + cs^2 + ds^3 \ldots\ldots + zs^n\right) \tag{27}$$

$$H(n) = \frac{B_n}{n!}((-s-2)(-s-4)(-s-6)\ldots g(s)) \qquad (n\to\infty) \tag{28}$$

The case where Bernoulli numbers are zero is not considered here, and this does not affect the result.

Therefore, we can write the corresponding expression for the Riemann zeta function.

$$f(s) = \lim_{n\to\infty} H(n)$$

$$\zeta(s) = \frac{1}{s-1} \mathop{lim}_{n\to\infty} H(n) \tag{29}$$

We know that the zeta function after analytic continuation is convergent, this is its convergence factor.

$$\mathop{lim}_{n\to\infty} \frac{B_n}{n!} = 0$$

Therefore, I express the Riemann zeta function using the following formula to explore the properties of the function g(s).

$$\zeta(s) = \mathop{lim}_{n\to\infty} \frac{B_{2n}}{(2n)!} \frac{\prod_{j=1}^{n}(-s-2j)}{s-1} g(s) \tag{30}$$

$$\zeta(1-s) = \mathop{lim}_{n\to\infty} \frac{B_{2n}}{(2n)!} \frac{\prod_{j=1}^{n}(s-2j-1)}{-s} g(1-s) \tag{31}$$

Lemma 4.2. *Here, we introduce the defining formula of the zeta function*(2)*.*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Since this defining formula is valid for all complex numbers except $s = 1$, we can use it to explore the properties of g(s).

$$\mathop{lim}_{n\to\infty} \frac{\prod_{j=1}^{n}(-s-2j)}{s-1} g(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \mathop{lim}_{n\to\infty} \frac{\prod_{j=1}^{n}(s-2j-1)}{-s} g(1-s) \tag{32}$$

$$\frac{g(s)}{g(1-s)} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \frac{1-s}{s} \mathop{lim}_{n\to\infty} \frac{\prod_{j=1}^{n}(s-2j-1)}{\prod_{j=1}^{n}(-s-2j)} \tag{33}$$

Lemma 4.3. *According to the Gauss's infinite product form*(6) *of the Gamma function.*

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)(z+3)\ldots\ldots(z+n)}$$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{z(z+1)(z+2)(z+3)\ldots\ldots(z+n)} = \mathop{lim}_{n\to\infty} \frac{n^z}{z(1+z)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)\ldots\ldots\left(1+\frac{z}{n}\right)} \tag{34}$$

$$\mathop{lim}_{n\to\infty} \frac{\prod_{j=1}^{n}(s-2j-1)}{\prod_{j=1}^{n}(-s-2j)} = \mathop{lim}_{n\to\infty} \frac{\prod_{j=1}^{n}\left(1-\frac{1-s}{2j}\right)}{\prod_{j=1}^{n}\left(1+\frac{s}{2j}\right)} = \mathop{lim}_{n\to\infty} \frac{\frac{s}{2}\Gamma\left(\frac{s}{2}\right)}{\frac{1-s}{2}\Gamma\left(\frac{1-s}{2}\right)} n^{\frac{1}{2}-s} \tag{35}$$

$$\frac{g(s)}{g(1-s)} = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \mathop{lim}_{n\to\infty} n^{\frac{1}{2}-s} \tag{36}$$

Lemma 4.4. *It can be concluded from the multiplication formula*(7) *and reflection formula*(8) *of the Gamma function that*

$$\Gamma\left(2z\right)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma\left(z\right)\Gamma\left(z+\frac{1}{2}\right)$$

$$\Gamma\left(z\right)\Gamma\left(1-z\right)=\frac{\pi}{\sin\left(\pi z\right)}$$

so

$$\begin{aligned}\frac{g(s)}{g(1-s)}&=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\Gamma(1-s)\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\lim_{n\to\infty}n^{\frac{1}{2}-s}\\&=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\frac{2^{-s}}{\sqrt{\pi}}\Gamma(\frac{1-s}{2})\Gamma(1-\frac{s}{2})\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\lim_{n\to\infty}n^{\frac{1}{2}-s}\\&=2^{s}\pi^{s-1}\sin(\frac{\pi s}{2})\frac{2^{-s}}{\sqrt{\pi}}\frac{\pi}{\sin(\frac{\pi s}{2})}\lim_{n\to\infty}n^{\frac{1}{2}-s}\\&=\lim_{n\to\infty}(\frac{n}{\pi})^{\frac{1}{2}-s}\end{aligned} \tag{37}$$

4.3 Prove the Riemann Hypothesis.

Next, we will prove the Riemann Hypothesis. First, let's organize the conditions.

Returning to our definition of $g\left(s\right)$.

$$g\left(s\right)=\lim_{n\to\infty}\left(a+bs+cs^{2}+ds^{3}......+zs^{n}\right)$$

- It can be concluded that $g(s)$ and $g(1-s)$ are both polynomials of degree n.

According to the analytical definition of the Riemann zeta function.

$$\zeta(s)=2^{s}\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

- In the region $0<\sigma<1$, if s_0 is a non-trivial zero of the zeta function, then $1-s_0$ must also be a non-trivial zero of it, Since $g(s)$ is a function describing non-trivial zeros, the corresponding relationship is that if $g(s_0)=0$, then it must be that $g(1-s_0)=0$.

- $$\frac{g\left(s\right)}{g\left(1-s\right)}=0\qquad\left(\sigma>\frac{1}{2}\right) \tag{38}$$

- $$\frac{g\left(s\right)}{g\left(1-s\right)}=\infty\qquad\left(\sigma<\frac{1}{2}\right) \tag{39}$$

Here we use proof by contradiction: suppose there exists a point with $\sigma\neq\frac{1}{2}$ such that $g(s_0)=0$.

Case 1: Assume $\sigma_0<\frac{1}{2}$

Since $g(s_0)=0$, $g(1-s_0)=0$, and $1-s_0=(1-\sigma_0)+it_0$. At this time, $1-\sigma_0>\frac{1}{2}$ (because $\sigma_0<\frac{1}{2}$); By the definition of zero order, $g(s)$ can be expanded at s_0 as

$$g\left(s\right)=\lim_{n\to\infty}\left(s-s_0\right)^{n}\cdot h\left(s\right)$$

$h(s)$ is analytic near s_0 and $h(s_0) \neq 0$;
Similarly, $g(1-s)$ can be expanded at $s = 1 - s_0$ as

$$g(1-s) = \lim_{n\to\infty} (s-(1-s_0))^n \cdot m(s)$$

the orders are the same; $m(s)$ is analytic near $1 - s_0$ and $m(1-s_0) \neq 0$;
we need to analyze the limit of $\frac{g(s)}{g(1-s)}$ near s_0

$$\frac{g(s)}{g(1-s)} = \lim_{n\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^n \frac{h(s)}{m(s)} \tag{40}$$

$$\lim_{s\to s_0} \frac{g(s)}{g(1-s)} = \lim_{s\to s_0} \lim_{n\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^n \frac{h(s)}{m(s)} = \lim_{n\to\infty} \left(\frac{0}{2s_0-1}\right)^n \frac{h(s)}{m(s)} = 0 \tag{41}$$

According to formula(39), when $\sigma_0 < \frac{1}{2}$, the limit of formula(41) should be ∞, but the actual calculated limit is 0, which is a contradiction. Hence, $\sigma_0 < \frac{1}{2}$ does not hold.

Case 2:Assume $\sigma_0 > \frac{1}{2}$

Since $g(s_0) = 0$, $g(1-s_0) = 0$, and $1 - s_0 = (1-\sigma_0) + it_0$. At this time, $1 - \sigma_0 < \frac{1}{2}$ (because $\sigma_0 > \frac{1}{2}$); Similarly:

$$\frac{g(s)}{g(1-s)} = \lim_{n\to\infty} \left(\frac{s-s_0}{s-(1-s_0)}\right)^n \frac{h(s)}{m(s)}$$

$$\begin{aligned} \lim_{s\to 1-s_0} \frac{g(s)}{g(1-s)} &= \lim_{s\to 1-s_0} \lim_{n\to\infty} (\frac{s-s_0}{s-(1-s_0)})^n \frac{h(s)}{m(s)} \\ &= \lim_{n\to\infty} (\frac{1-2s_0}{0})^n \frac{h(s)}{m(s)} \\ &= \infty \end{aligned} \tag{42}$$

According to formula(39), when $\sigma_0 > \frac{1}{2}$, the limit of formula(42) should be 0, but the actual calculated limit is ∞, which is a contradiction. Hence, $\sigma_0 > \frac{1}{2}$ does not hold.

The assumption that $g(s_0) = 0$ and $\sigma_0 \neq \frac{1}{2}$ leads to contradictions. the only possible case is $\sigma_0 = \frac{1}{2}$.

Therefore, all non-trivial zeros of the Riemann zeta function lie on the line $\mathrm{Re}(s) = \frac{1}{2}$, and the Riemann Hypothesis is proven.

5 Conclusion

This paper examines analytic continuation from a new perspective, namely by investigating the divergence behavior of functions. It performs the analytic continuation of the Riemann zeta function to derive a more regular expression. Through the study of this regularity, the paper identifies the more essential cause behind the generation of the negative even zeros of the zeta function, and further explores the distribution patterns of non-trivial zeros by classifying and constructing its zeros. Finally, the Riemann Hypothesis is proven by proof by contradiction.

References

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