A New Approach to the Analytic Continuation of the Riemann Zeta Function and Its Zero Analysis

Chen Zihang

October 26, 2025

Abstract

This paper mainly studies the new analytic continuation method of the Riemann zeta function and the properties of the zeta function after analytic continuation, and explores the laws of its zeros based on these properties. The author conducts the analytic continuation of the Riemann zeta function via the Euler-Maclaurin formula, and finds that it has a close connection with the Bernoulli recurrence formula through analysis. Through special construction and mutual transformation with the gamma function, a special symmetric relationship of the zeta function is discovered, thereby identifying the distribution law of the non-trivial zeros of the zeta function. The new methods and new ideas used in this paper include the new analytic continuation form of the zeta function, the special construction method, and so on.

1 Introduction

The Riemann Hypothesis is one of the most profound important problems in mathematics, bridging number theory and complex analysis.

1.1 Research Background

Proposed by the German mathematician Bernhard Riemann in 1859, in his landmark paper *On the Number of Primes Less Than a Given Magnitude*. Classified as one of the "Seven Millennium Prize Problems" by the Clay Mathematics Institute, with a \$1 million reward for its proof or disproof.

1.2 Literature Review of Previous Studies

Riemann's original work: First extended the zeta function from the real axis to the complex plane, and conjectured that all non-trivial zeros lie on the line $\mathrm{Re}(s) = \frac{1}{2}$; he also noted the connection between this conjecture and the error term of the Prime Number Theorem. 1896 breakthrough: Jacques Hadamard and Charles Jean de la Vallée-Poussin independently proved the Prime Number Theorem using properties of the zeta function 20th-21st century progress: Numerical calculations have verified the conjecture for the first 10^{13} non-trivial zeros.

1.3 Core Concepts: The Riemann Zeta Function[3]

The Riemann zeta function is the foundation of the Riemann Hypothesis, defined and extended as follows:

1.3.1 Definition (Domain Re$(s) > 1$)

For a complex variable $s = \sigma + it$ (where $\sigma = \mathrm{Re}(s)$ is the real part and $t = \mathrm{Im}(s)$ is the imaginary part), the zeta function is defined by the infinite series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

1.3.2 Analytical Continuation

To extend the zeta function beyond $\mathrm{Re}(s) > 1$, the following functional equation is used, which is valid for all complex numbers $s \neq 1$ (where $s = 1$ is a simple pole):

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

Here, $\Gamma(s)$ denotes the gamma function, a generalization of the factorial function to complex numbers.

1.4 Key Conclusions and the Riemann Hypothesis

Trivial zeros: The zeta function has zeros at all negative even integers $s = -2, -4, -6, \ldots$; **Non-trivial zeros**: All zeros lying in the "critical strip" $0 < \mathrm{Re}(s) < 1$; the Riemann Hypothesis makes the core assertion:

$$\boxed{\text{All non-trivial zeros of the Riemann zeta function have real part } \frac{1}{2}}$$

2 Preliminaries

Before stating the main theorem, I recall some basic definitions and results.

2.1 Bernoulli Numbers: Recurrence Relation[4]

Bernoulli numbers $\{B_n\}_{n=0}^{\infty}$ are a sequence of rational numbers fundamental in number theory and analysis. They satisfy the following recurrence relation:

- **Initial condition**: $B_0 = 1$.
- For $n \geq 1$, the sum $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$, where $\binom{n+1}{k}$ denotes the binomial coefficient.

This recurrence allows computing Bernoulli numbers iteratively: starting from $B_0 = 1$, one can find $B_1, B_2, \ldots$ successively. For example, solving the equation for $n = 1$ gives $B_1 = -\frac{1}{2}$, and for $n = 2$, we get $B_2 = \frac{1}{6}$, etc, In some contexts, $B_1$is defined as $\frac{1}{2}$, which will be mentioned later.

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \tag{3}$$

2.2 Euler-Maclaurin Formula[2]

The Euler-Maclaurin Formula bridges discrete summation and continuous integration, playing a pivotal role in approximating sums and analyzing series. For a function f that is m-times continuously differentiable on $[a, b]$ (denoted $f \in C^m[a, b]$), the formula is:

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{j=1} \frac{B_{2j}}{(2j)!}\left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right] \tag{4}$$

Here:

- B_{2j} are Bernoulli numbers with even indices (odd-indexed Bernoulli numbers $B_{2j+1} = 0$ for $j \geq 1$, simplifying the formula).
- $f^{(2j-1)}$ is the $(2j-1)$-th derivative of f.
- R_m is the remainder term, which can be bounded using higher derivatives of f and Bernoulli numbers (as shown in related lemmas if needed).

The formula quantifies the difference between summing f at integer points and integrating f, with corrections involving Bernoulli numbers and derivatives of f.

2.3 Gamma Function

The Gamma function (denoted as $\Gamma(z)$), proposed by Euler, is a tool for extending the factorial function to the complex plane. It generalizes the factorial of positive integers to all complex numbers except non-negative integer poles ($z = 0, -1, -2, \ldots$), serving as a key link in number theory and complex analysis.

- $$\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt \tag{5}$$

  Significance:For positive integers n, it satisfies $\Gamma(n) = (n-1)!$

- $$\Gamma(z) = \lim_{n\to+\infty} \frac{n! \cdot n^z}{z(z+1)\cdots(z+n)} \tag{6}$$

  Significance: It directly reflects the nature of factorial limits. The denominator clearly shows the positions of poles, while the numerator term n^z acts as a normalization factor for convergence.

- $$\mathit{\Gamma}(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \mathit{\Gamma}(z)\, \mathit{\Gamma}\left(z + \frac{1}{2}\right) \tag{7}$$

  Multiplication Formula (Legendre's Formula)

- $$\mathit{\Gamma}(z)\, \mathit{\Gamma}(1-z) = \frac{\pi}{\sin(\pi z)} \tag{8}$$

  Reflection Formula (Residue Formula)

2.4 Dirichlet Eta Function[1]

The Dirichlet eta function (denoted as $\eta(s)$) is an alternating Dirichlet series, defined for a complex number $s = \sigma + it$ as:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \tag{9}$$

Its domain of convergence is $\mathrm{Re}(s) > 0$ ($s \neq 1$), which is broader than that of the original series of the Riemann zeta function (convergent only when $\mathrm{Re}(s) > 1$). For all complex numbers s with $\mathrm{Re}(s) > 0$ ($s \neq 1$), the following identity holds:

$$\eta(s) = (1 - 2^{1-s})\zeta(s) \tag{10}$$

After rearrangement to $\zeta(s) = \frac{\eta(s)}{1-2^{1-s}}$, the Riemann zeta function can be analytically continued to the region $\mathrm{Re}(s) > 0$, covering the "critical strip" ($0 < \mathrm{Re}(s) < 1$) which is crucial for the study of the Riemann Hypothesis. Within the critical strip $0 < \mathrm{Re}(s) < 1$, $\eta(s) = 0$ is completely equivalent to $\zeta(s) = 0$, meaning their non-trivial zeros (including the order of zeros) are identical.

3 Main Results and Proofs

Theorem 3.1. *Analytic Continuation Expression of the Riemann zeta-Function for $\sigma \neq 1$*

- *Let $\zeta(s)$ denote the Riemann zeta-function, $\Gamma(s)$ the Gamma function, and B_k the k-th Bernoulli number, $(-s+1)_k$ is a falling factorial. Then, for any complex number s satisfying $\sigma \neq 1$*

$$(-s+1)_0 = 1 \qquad (-s+1)_1 = (-s+1) \qquad (-s+1)_2 = (-s+1)(-s)$$

......

$$\zeta(s) = \begin{cases} \sum_{n=1} \frac{1}{n^s} & \sigma > 1 \\ \frac{1}{s-1} \sum_{n=0} \frac{B_n}{n!} (-s+1)_n & \sigma < 1 \end{cases}$$

Proof of Theorem3.1

Lemma 3.2. *according to the Euler-Maclaurin formula*(4)*.*

$$\sum_{k=a}^{b} f(k) = \int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2} + \sum_{j=1} \frac{B_{2j}}{(2j)!} \left[f^{(2j-1)}(b) - f^{(2j-1)}(a)\right]$$

Transorming Euler-Maclaurin Formula

$$\sum_{k=a}^{b} f(k) = \sum_{j=0} \frac{B_j}{j!} \int_a^{b+1} \frac{d^j}{dx^j} f(x)\,dx \qquad \left(B_1 = -\frac{1}{2}\right)$$

or

$$\sum_{k=a}^{b} f(k) = \sum_{j=0} \frac{B_j}{j!} \int_{a+1}^{b} \frac{d^j}{dx^j} f(x)\,dx \qquad \left(B_1 = \frac{1}{2}\right)$$

These two formulas are equivalent to the Euler-Maclaurin formula. The purpose of this approach is to make them appear more symmetric and concise, laying the groundwork for the proof in the following text.
Parameter substitution

$$\sum_{k=1}^{n} f(k) = \sum_{j=0} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} f(x)\, dx \tag{11}$$

Lemma 3.3. *according to the Riemann zeta function*(1).

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$\zeta(s) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s} \tag{12}$$

Now we combine the zeta function with formula(9).

$$\zeta(s) = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{j^s} = \lim_{n\to\infty} \sum_{j=0} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} \frac{1}{x^s} dx \tag{13}$$

Here, we discuss the part where $\sigma < 1$. From the above formula, it can be seen that as n tends to infinity, the zeta function can be expressed as a main function plus a constant independent of n.

$$\lim_{n\to\infty} \sum_{j=0} \frac{B_j}{j!} \int_{1}^{n+1} \frac{d^j}{dx^j} \frac{1}{x^s} dx = \lim_{n\to\infty} \Big(\frac{B_0}{0!} \frac{1}{(-s+1)(n+1)^{s-1}} + \frac{B_1}{1!} \frac{1}{(n+1)^s} + \frac{B_2}{2!} \frac{-s}{(n+1)^{s+1}} + ... + C\Big) \tag{14}$$

$$C = -\Big(\frac{B_0}{0!} \frac{1}{-s+1} + \frac{B_1}{1!}(-s+1)(-s) + \frac{B_2}{2!}(-s+1)(-s)(-s-1) + ...\Big) = -\frac{1}{-s+1} \sum_{k=0} \frac{B_k}{k!} (-s+1)_k \tag{15}$$

Since the part of the function concerning n is divergent, it is conjectured that the zeros of the analytic continuation of the zeta function are related to the constant C, $(-s+1)_k$ is a falling factorial.
Define this relational expression.

$$T(n) = \sum_{k=1}^{n} \frac{1}{k^s}$$

$$T(n) \to \frac{1}{(-s+1)\, n^{1-s}} + C_1 \qquad (n \to \infty) \tag{16}$$

Lemma 3.4. *Here, we introduce the Dirichlet eta function* (9)*, as its zeros are consistent with those of the Riemann zeta function.*

$$\eta(s) = \sum_{k=1} (-1)^{k-1} \frac{1}{k^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + ... \tag{17}$$

We observe the process of deriving the Dirichlet eta function from the Riemann zeta function.

$$\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + ... \\
\frac{2}{2^s}\zeta(s) &= \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \frac{2}{8^s} + \frac{2}{10^s} + \frac{2}{12^s} + \frac{2}{14^s} + ... \\
\zeta(s) - \frac{2}{2^s}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s} + ... \\
&\quad -0 - \frac{2}{2^s} - 0 - \frac{2}{4^s} - 0 - \frac{2}{6^s} - 0 - \frac{2}{8^s} - ...... \\
&\left(1 - \frac{2}{2^s}\right)\zeta(s) = \eta(s)
\end{aligned}$$

Define this relational expression.

$$F(n) = \frac{2}{2^s} \sum_{k=1}^{n} \frac{1}{k^s} = 2 \sum_{k=1}^{n} \frac{1}{(2k)^s} \tag{18}$$

By the same reasoning, we can obtain

$$F(n) \to \frac{1}{(-s+1)(2n)^{1-s}} + C_2 \qquad (n \to \infty) \tag{19}$$

$$C_2 = 2^{1-s} C_1 \tag{20}$$

Then we can explain the origin of the Dirichlet eta function.

$$\eta(s) = \lim_{n \to \infty} \left(T(n) - F\left(\frac{n}{2}\right)\right) = C_1 - C_2 = \left(1 - \frac{2}{2^s}\right) C_1 \tag{21}$$

Because when the $\zeta(s)$ is subtracted by $(\frac{2}{2^s}\zeta(s))$, the latter function undergoes a double shift, so naturally $F(n)$ needs to be transformed into $(F\left(\frac{n}{2}\right))$, From the above relationship, it can be seen that this constant C is indeed the equivalent function of the Riemann zeta function after analytic continuation for $\sigma < 1$. Since C is independent of n and is a function of s, Viewed from another perspective, the definition of the Riemann zeta function obtained via analytic continuation also bypasses the divergent points through integration. It can also be seen from the above formula that the derivation of C essentially involves the elimination of the divergent function. We can conclude that

$$\zeta(s) = \begin{cases} \sum_{n=1} \frac{1}{n^s} & \sigma > 1 \\ \frac{1}{s-1} \sum_{n=0} \frac{B_n}{n!} (-s+1)_n & \sigma < 1 \end{cases} \tag{22}$$

4 Analysis of Zero Distribution Law

4.1 Distribution of Trivial Zeros

we define this function below.

$$f\left(s\right)=\sum_{k=0}\left(-s+1\right)_{k}\frac{B_{k}}{k!}=\left(s-1\right)\zeta\left(s\right) \tag{23}$$

Lemma 4.1. *According to the Bernoulli Recurrence Formula*(3).

$$\sum_{k=0}^{n}\left(\begin{array}{c}n+1\\k\end{array}\right)B_{k}=\sum_{k=0}^{n}\frac{\left(n+1\right)_{k}}{k!}B_{k} \tag{24}$$

It can be seen that the first n terms of $f(s)$ correspond to the Bernoulli recurrence formula, It follows from the generation rule of Bernoulli numbers that $f(s)=0$ when $s=-2,-4,-6,\ldots$,This also explains, from another perspective, why negative even numbers are the zeros of the zeta function.

4.2 Distribution Law of Non-Trivial Zeros

The content of the Riemann Hypothesis is that the real parts of all non-trivial zeros of the zeta function are equal to 1/2. To verify this hypothesis, I will perform a special transformation on $f(s)$ and then study the distribution of its non-trivial zeros.

defin again

$$H\left(n\right)=\sum_{k=0}^{n}\left(-s+1\right)_{k}\frac{B_{k}}{k!} \tag{25}$$

From the recurrence formula of Bernoulli numbers, the roots of $H(n)$ with respect to s can be expressed as the product of negative even roots and complex roots, define $g(s)$ as a function that affects the complex roots.

$$H\left(4\right)=\frac{B_{4}}{4!}\left(\left(-s-2\right)\left(-s-4\right)\left(-s-5\right)\left(-s+9\right)\right)$$

$$H\left(6\right)=\frac{B_{6}}{6!}\left(\left(-s-2\right)\left(-s-4\right)\left(-s-6\right)\left(-s-7\right)\left(a_{1}+b_{1}s+c_{1}s^{2}\right)\right)$$

......

$$H\left(n\right)=\frac{B_{n}}{n!}\left(\left(-s-2\right)\left(-s-4\right)\left(-s-6\right)......\left(a+bs+cs^{2}+ds^{3}+......\right)\right)$$

$$H\left(n\right)=\frac{B_{n}}{n!}\left(\left(-s-2\right)\left(-s-4\right)\left(-s-6\right)......g\left(s\right)\right)\qquad\left(n\rightarrow\infty\right) \tag{26}$$

The case where Bernoulli numbers are zero is not considered here, and this does not affect the result.

$$f\left(s\right)=\lim_{n\rightarrow\infty}H\left(n\right)$$

$$\zeta\left(s\right)=\frac{1}{s-1}\mathop{lim}_{n\rightarrow\infty}H\left(n\right) \tag{27}$$

We know that the zeta function after analytic continuation is convergent.

$$\lim_{n\to\infty}\frac{B_n}{n!}=0$$

$$s\in C \qquad \lim\left(-s-2\right)\left(-s-4\right)\left(-s-6\right)......=\infty \qquad \lim g\left(s\right)=\infty$$

$$\zeta\left(s\right)=\lim_{n\to\infty}\frac{B_{2n}}{\left(2n\right)!}\frac{\prod_{j=1}^{n}\left(-s-2j\right)}{s-1}g\left(s\right) \tag{28}$$

$$\zeta\left(1-s\right)=\lim_{n\to\infty}\frac{B_{2n}}{\left(2n\right)!}\frac{\prod_{j=1}^{n}\left(s-2j-1\right)}{-s}g\left(1-s\right) \tag{29}$$

Lemma 4.2. *Here, we introduce the defining formula of the zeta function*(2)*.*

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Since this defining formula is valid for all complex numbers except $s=1$, we can

$$\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\left(-s-2j\right)}{s-1}g\left(s\right)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\mathit{\Gamma}\left(1-s\right)\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\left(s-2j-1\right)}{-s}g\left(1-s\right) \tag{30}$$

To study the properties of $g(s)$, we isolate it on one side.

$$\frac{g\left(s\right)}{g\left(1-s\right)}=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\mathit{\Gamma}\left(1-s\right)\frac{1-s}{s}\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\left(s-2j-1\right)}{\prod_{j=1}^{n}\left(-s-2j\right)} \tag{31}$$

Lemma 4.3. *According to the Gauss's infinite product form*(6) *of the Gamma function.*

$$\mathit{\Gamma}\left(z\right)=\lim_{n\to\infty}\frac{n!\,n^z}{z\left(z+1\right)\left(z+2\right)\left(z+3\right)......\left(z+n\right)}$$

$$\mathit{\Gamma}\left(z\right)=\lim_{n\to\infty}\frac{n!\,n^z}{z\left(z+1\right)\left(z+2\right)\left(z+3\right)......\left(z+n\right)}=\lim_{n\to\infty}\frac{n^z}{z\left(1+z\right)\left(1+\frac{z}{2}\right)\left(1+\frac{z}{3}\right)......\left(1+\frac{z}{n}\right)} \tag{32}$$

$$\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\left(s-2j-1\right)}{\prod_{j=1}^{n}\left(-s-2j\right)}=\lim_{n\to\infty}\frac{\prod_{j=1}^{n}\left(1-\frac{1-s}{2j}\right)}{\prod_{j=1}^{n}\left(1+\frac{s}{2j}\right)}=\lim_{n\to\infty}\frac{\frac{s}{2}\mathit{\Gamma}\left(\frac{s}{2}\right)}{\frac{1-s}{2}\mathit{\Gamma}\left(\frac{1-s}{2}\right)}n^{\frac{1}{2}-s} \tag{33}$$

$$\frac{g\left(s\right)}{g\left(1-s\right)}=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\mathit{\Gamma}\left(1-s\right)\frac{\mathit{\Gamma}\left(\frac{s}{2}\right)}{\mathit{\Gamma}\left(\frac{1-s}{2}\right)}\lim_{n\to\infty}n^{\frac{1}{2}-s} \tag{34}$$

Lemma 4.4. *It can be concluded from the multiplication formula*(7) *and reflection formula*(8) *of the Gamma function that*

$$\mathit{\Gamma}\left(2z\right)=\frac{2^{2z-1}}{\sqrt{\pi}}\mathit{\Gamma}\left(z\right)\mathit{\Gamma}\left(z+\frac{1}{2}\right)$$

$$\mathit{\Gamma}\left(z\right)\mathit{\Gamma}\left(1-z\right)=\frac{\pi}{\sin\left(\pi z\right)}$$

so

$$\frac{g\left(s\right)}{g\left(1-s\right)}=\lim_{n\to\infty}\left(\frac{n}{\pi}\right)^{\frac{1}{2}-s} \tag{35}$$

4.3 To understand Formula(35)

Returning to our definition of $g\left(s\right)$

$$g\left(s\right)=\lim_{n\to\infty}\left(a+bs+cs^{2}+ds^{3}......+zs^{n}\right) \tag{36}$$

$g(s)$ is analytic on the complex plane $\mathbb{C}$ except for finitely many isolated poles, and has no poles in the critical strip $0<\sigma<1$, For any fixed t.

$$\frac{g\left(s\right)}{g\left(1-s\right)}=0 \qquad \left(\sigma>\frac{1}{2}\right) \tag{37}$$

$$\frac{g\left(s\right)}{g\left(1-s\right)}=\infty \qquad \left(\sigma<\frac{1}{2}\right) \tag{38}$$

Suppose there exists a non-trivial zero $s_0=\sigma_0+it_0$ of $g(s)$ such that $\sigma_0\neq\frac{1}{2}$. Since s_0 is a zero of $g(s)$, $1-s_0=(1-\sigma_0)+it_0$ is also a non-trivial zero of $g(s)$.

- If $\sigma_0>\frac{1}{2}$, then $1-\sigma_0<\frac{1}{2}$. For $s\to 1-s_0$ (where $\sigma\to 1-\sigma_0<\frac{1}{2}$), the ratio $\frac{g(s)}{g(1-s)}$ to necessarily tend to ∞. However, as $g(s)\to g(1-s_0)=0$ and $g(1-s)\to g(s_0)=0$, the ratio becomes a $0/0$ indeterminate form (it cannot be guaranteed to tend to ∞), which contradicts the "limit exists and is uniquely ∞".
- If $\sigma_0<\frac{1}{2}$, similarly, $1-\sigma_0>\frac{1}{2}$. The ratio is required to necessarily tend to 0, but it still reduces to a $0/0$ indeterminate form (it cannot be guaranteed to tend to 0), contradicting the "limit exists and is uniquely 0".

The counterassumption is invalid. Therefore, all non-trivial zeros of $g(s)$ must satisfy $\mathrm{Re}(s)=\frac{1}{2}$.

5 Conclusion

This paper examines analytic continuation from a new perspective, namely by investigating the divergence behavior of functions. It performs the analytic continuation of the Riemann zeta function to derive a more regular expression. Through the study of this regularity, the paper identifies the more essential cause behind the generation of the negative even zeros of the zeta function, and further explores the patterns of non-trivial zeros by classifying and constructing its zeros. This paper holds significant value for understanding analytic continuation. Unlike Riemann, who conducted the analytic continuation of the zeta function using integral methods, the method proposed in this paper can directly calculate the values of the zeta function for $s<1$ without relying on symmetric relationships.

References

References

[1] Chinnaraji Annamalai. Computation of analog theorem for dirichlet eta function and riemann zeta function. 2024.

[2] Carlo Bellingeri, Peter K Friz, and Sylvie Paycha. Generalized euler–maclaurin formula and signatures. *International Mathematics Research Notices*, 2025(8), April 2025.

[3] Walter Dittrich. On riemann's paper," on the number of primes less than a given magnitude". *arXiv preprint arXiv:1609.02301*, 2016.

[4] Masanobu Kaneko. A recurrence formula for the bernoulli numbers. 1995.