

# Long time dynamics of the Cauchy problem for the predator-prey model with cross-diffusion

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## Abstract

This paper is concerned with a predator-prey model in  $N$ -dimensional spaces ( $N = 1, 2, 3$ ), given by

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \frac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u), \end{cases}$$

which describes random movement of both predator and prey species, as well as the spatial dynamics involving predators pursuing prey and prey attempting to evade predators. It is shown that any global strong solutions of the corresponding Cauchy problem converge to zero in the sense of  $L^p$ -norm for any  $1 < p \leq \infty$ , and also converge to the heat kernel with respect to  $L^p$ -norm for any  $1 \leq p \leq \infty$ . In particular, the decay rate thereof is optimal in the sense that it is consistent with that of the heat equation in  $\mathbb{R}^N$  ( $N = 2, 3$ ). Undoubtedly, the global existence of solutions appears to be among the most challenging topic in the analysis of this model. Indeed even in the one-dimensional setting, only global weak solutions in a bounded domain have been successfully constructed by far. Nevertheless, to provide a comprehensive understanding of the main results, we append the conclusion on the global existence and asymptotic behavior of strong solutions, although certain smallness conditions on the initial data are required.

**Keywords:** Pursuit-evasion, cross diffusion, decay rate, heat kernel.

**Mathematics Subject Classification:** 35K51, 35B45, 92C17.

## 1 Introduction

The interaction between different species generates a complex ecosystem, and the predator-prey relation is one of the fundamental binary interactions, in which predators pursue preys. Since Lotka-Volterra-type prey-predator system was introduced, various mathematical models have been proposed

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to explore a diverse range of complex ecological phenomena, including Allee effects [38], group defense [42] and so on, and especially predator-prey model invoking some mechanism of spatial interaction is derived to describe complex space-time patterns which reflect nonlocal spatial interactions between the predators and preys (see [36, 34, 28] for example). Therefore the investigation of the diffusive predator-prey systems becomes an important step through which one can get a rather deeper understanding of the effects of the spatial dispersal of the predators and preys on the entire ecosystem. In the spatial predator-prey relationship, besides the kinetic dynamics and the random movement of the predators and preys, the predators and preys may also partially orient their movement towards or away from higher concentrations of the other species, that is predators move toward the gradient direction of prey distribution (called prey-taxis), and/or preys move opposite to the gradient of predator distribution (called predator-taxis). To capture this pursuit-escape phenomenon, Tsyganov et.al [39] proposed the cross-diffusion system of the form

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + T_1(u, v), \\ \frac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) + T_2(u, v), \end{cases} \quad (1.1)$$

where the unknown functions  $u(t, x)$  and  $v(t, x)$  correspond to the predator and prey population densities respectively at position  $x$  and time  $t > 0$ ;  $-\chi \nabla \cdot (u \nabla v)$  and  $+\xi \nabla \cdot (v \nabla u)$  stand for the predators are attracted by the preys with coefficient  $\chi$  and the preys escape from predators with coefficient  $\xi$ , respectively. The functions  $T_1(u, v)$  and  $T_2(u, v)$  represent the inter-specific interaction of  $u$  and  $v$ .

As pointed in [40], (1.1) can be regarded as a distant relative of the classic Shigesada-Kawasaki-Teramoto model ([37])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta[(a_1 + a_2 u + a_3 v)u] + T_1(u, v), \\ \frac{\partial v}{\partial t} = \Delta[(b_1 + b_2 u + b_3 v)v] + T_2(u, v), \end{cases}$$

abbreviated as the SKT model. The SKT model and its simplified version have been widely studied in the past few decades, see for example [8, 20, 26, 24, 25] or the references therein. However up to now, it seems that the global well-posedness theory of solutions thereof has not been fully established yet. Unlike the SKT model, the cross-diffusion in (1.1) originates from a specific mechanism which characters the attractive and repulsive taxis between the two populations simultaneously. As a remark feature, (1.1) possesses a quite colorful wave-like solution, for example, quasi-soliton pattern is identified in [39]. In recent years, the global well-posedness theory of (1.1) has received a considerable attention from researchers. However, the achievement on system (1.1) seems at a rather rudimentary stage with rather limited results available for the references [10, 40, 41, 45, 46]. For instance, for the case  $T_1 = T_2 = 0$  or  $T_1$  and  $T_2$  being Lotka-Volterra response terms, Fuest investigated the existence of global classical solutions of homogeneous Neumann boundary value problem of (1.1) on a smooth bounded domain when the initial values are sufficiently close to the constant equilibrium points, as well as the asymptotic behavior thereof in [10]. Nevertheless, due to the double tactic mechanism in (1.1), there is significant challenge in the rigorous analysis of (1.1) with large initial data. Indeed, to the best of our knowledge, the global solvability for the initial boundary value problem for (1.1) with large initial data is limited in one-dimensional spatial situation yet, see [40, 41, 45] for example.

In contrast to the sparse results on model (1.1) by far, extensive studies have been carried out on the simplified variants thereof. As an important particular case of (1.1) in which the predator-taxis is

neglected, the diffusive predator-prey system only with prey-taxis reads as

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) + T_1(u, v), \\ \frac{\partial v}{\partial t} = \Delta v + T_2(u, v). \end{cases} \quad (1.2)$$

Here the most type of ecological inter-specific interaction can be encapsulated by the choice of  $T_1(u, v) = \gamma u F(v) - uh(u)$ ,  $T_2(u, v) = -uF(v) + k(v)$ ,  $F(v)$  is the predation rate, the parameter  $\gamma > 0$  denotes the conversion rate from prey to predator,  $uh(u)$  and  $k(v)$  are referred to the intra-specific interaction of the predators and preys, respectively. Up to date there are a few results available to (1.2) in the bounded domain (see [6, 13, 16, 47, 43, 50] for example and references cited therein). For instance, the global existence and stability of corresponding solutions to (1.2) with no-flux boundary conditions was established in a two-dimensional bounded domain under some specific condition on  $F, h$  and  $k$  ([16]), while the existence of global bounded solutions was obtained in higher dimensions for small  $\chi > 0$  and in three-dimensional setting for suitably small initial value  $\|v_0\|_{L^\infty(\Omega)}$ , respectively (see [22, 50]). In particular, Winkler ([43]) showed the global existence of weak solutions over a bounded domain with space dimension  $N \leq 5$  and their eventual smoothness under some weak conditions which includes  $k(v) = 0$  and  $h(u) = 0$ .

On the other hand, prey-taxis model (1.2) can also be regarded as the extension of the classical Keller-Segel model ([17])

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \quad (1.3)$$

where  $\Omega$  is either a bounded domain in  $\mathbb{R}^N$  with smooth boundary  $\partial\Omega$  or the whole space  $\mathbb{R}^N$ , which describes a chemotactic feature, the aggregation of some organisms (cellular slime molds) denoted by  $u$  sensitive to the gradient of a chemical substance denoted by  $v$ . The possibly most striking feature of this model is the occurrence of a critical mass phenomenon in two dimensions, namely there exists a threshold value  $M_c > 0$  with respect to initial datum  $\|u_0\|_{L^1}$  such that all solutions exist globally in time for  $\|u_0\|_{L^1} \leq M_c$ , while finite time blow-up occurs if  $\|u_0\|_{L^1} > M_c$ . Here  $M_c$  is either  $\frac{8\pi}{\chi}$  or  $\frac{4\pi}{\chi}$  according to  $\Omega$  is the whole space or the bounded domain [15, 32, 27]. Notice that the approach to dealing with the dynamical behavior of (1.3) in the whole space  $\mathbb{R}^N$  ( $N \geq 3$ ) is slightly different from that on the bounded domain. For instance, it is well known that the functional

$$\mathcal{F}(u, v) := \int u \log u - \chi \int uv + \frac{\chi}{2} \int v^2 + \frac{\chi}{2} \int |\nabla v|^2$$

satisfies

$$\frac{d}{dt} \mathcal{F}(u, v) = -\mathcal{D}(u, v) := - \int |\Delta v - v + u|^2 - \int \left| \frac{\nabla u}{\sqrt{u}} - \chi \sqrt{u} \nabla v \right|^2 \quad (1.4)$$

along reasonably regular solution curves of (1.3), which then allows us for identifying the circumstances under which solutions exist globally and remain bounded. Going beyond that, (1.4) can also serve as a fundamental ingredient in the detection of blow-up phenomena. Indeed, if there are some  $C > 0$  and  $\vartheta \in (0, 1)$  such that

$$\mathcal{F}(u, v) \geq -C \cdot (\mathcal{D}^\vartheta(u, v) + 1) \quad (1.5)$$

throughout some radially symmetric functions  $(u, v)$ , (1.4) is then capable of identifying large sets of initial data which lead to finite-time blow-up in Neumann problems for (1.3) ([49]). However, as point out in [48], term  $\int_{\mathbb{R}^2} u \log u$  in  $\mathcal{F}(u, v)$  may be no longer bounded from below, hence the energy argument above is not accessible for the Cauchy problem.

The investigation of long-time dynamic properties in biological models is crucial for comprehending the fundamental processes of population growth, death, and migration within biological populations. Additionally, it sheds light on how these processes impact the stability and structure of ecosystems, this research also serves as a valuable resource for ecological conservation and management practices. Regarding the global solution of this kind of model on a bounded domain, the homogeneous Neumann boundary value problem has been thoroughly investigated as we stated above. It is well-known that the asymptotic behavior under zero Neumann boundary condition is nearly the same as the corresponding ODE systems, that is the solution will tend to some constant steady state eventually. In such cases, numerous studies have demonstrated, see for example [16, 44] or the references therein. In addition, some necessary conditions for pattern formation have also been explored [18, 23]. However, the long-time asymptotic behavior of the Cauchy problem for solutions in the whole space differs significantly from that observed on bounded domains. As an illustrative example, considering the 2D Patlak-Keller-Segel system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \Delta v + u = 0, \end{cases}$$

an important observation in this model is its scaling invariance property, meaning that if  $(u(x, t), v(x, t))$  is a solution, so is  $(u_\lambda(x, t), v_\lambda(x, t))$ , where

$$u_\lambda(x, t) = \lambda^{-2} u(\lambda^{-1} x, \lambda^{-2} t), \quad v_\lambda(x, t) = v(\lambda^{-1} x, \lambda^{-2} t).$$

This is a crucial property that significantly influences its dynamic behavior. Biler et al. [2, 3, 4, 29, 30] have delved into the existence and characteristics of self-similar solutions in the form of  $t^{-1} G_\alpha \left( \frac{|x|}{\sqrt{t}} \right)$  when the initial mass  $\alpha$  is below the critical threshold in two dimensional space. It has been confirmed that such self-similar solutions also act as global attractors [1, 3], that is

$$\lim_{t \rightarrow \infty} \left\| u(x, t) - t^{-1} G_\alpha \left( \frac{|x|}{\sqrt{t}} \right) \right\|_{L^p} = 0$$

for  $p \in [1, \infty]$  if  $\|u\|_{L^1} = \alpha$ . Furthermore, the optimal rate of convergence towards this self-similar profile has been presented [5, 10]. Similarly, for the following parabolic-parabolic Keller-Segel system,

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) \\ \frac{\partial v}{\partial t} = \Delta v + u, \end{cases}$$

it is shown that the solution will finally converge to the self-similar solution of this system for small mass in  $\mathbb{R}^2$  [21, 35]. Similar results have also been observed in the two-dimensional Navier-Stokes equations [11], and some reaction-diffusion equations [9, 12, 19]. While for the following Keller-Segel system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \frac{\partial v}{\partial t} = \Delta v - v + u, \end{cases}$$

it is evident that this equation does not exhibit scaling invariance, hence no self-similar solutions exist. Regarding the long-time asymptotic behavior of this model, Nagai [31] in 2001 investigated the asymptotic profile of bounded solutions in  $\mathbb{R}^2$ , and demonstrated that the solution converges to the Gauss kernel in terms of the  $L^\infty$ -norm. Later, in 2003, Nagai, Syukuinn et.al [33] extended this result to  $\mathbb{R}^N$  with  $N \geq 2$ , demonstrating convergence in  $L^p$ -norm for  $1 < p \leq \infty$ . Recently, Carrillo et al. [7] studied a general aggregation-diffusion equation of the form

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla W * u),$$

and proved that if the interaction potential is bounded and its first and second derivatives decay fast enough at infinity, then the linear diffusion eventually dominates its attractive or repulsive effects, and thereby solutions finally behave like the fundamental solution of the heat equation.

Motivated by the above works, this paper considers the long-time dynamical behavior of strong solutions to the Cauchy problem of (1.1) in the higher dimensions, which reads as

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } Q, \\ \frac{\partial v}{\partial t} = \Delta v + \xi \nabla \cdot (v \nabla u) & \text{in } Q, \\ u(x, 0) = u_0(x), v(x, 0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (1.6)$$

where  $Q = \mathbb{R}^N \times \mathbb{R}^+$  with  $N = 1, 2, 3$ , and the initial value  $u_0, v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$  are nonnegative. Our focus is that in the whole space, how far the attractive and repulsive interaction of the species  $u$  and  $v$  affect their long dynamical behavior in the situation that their populations dynamics is neglected.

It is observed that the system (1.6) adheres to the law of conservation of mass, that is

$$\int_{\mathbb{R}^N} u(x, t) dx = \int_{\mathbb{R}^N} u_0(x) dx, \quad \int_{\mathbb{R}^N} v(x, t) dx = \int_{\mathbb{R}^N} v_0(x) dx \quad (1.7)$$

for all  $t > 0$ , whereas our findings reveals that the bounded strong solutions of this system converge to zero in the sense of  $L^p$ -norm with any  $1 < p \leq \infty$ . Apart from that, due to the fact that the  $L^p$ -norm of the heat kernel  $G(\cdot, t)$  decreases to 0 as  $t \rightarrow \infty$  for any  $p > 1$ , that is

$$\|G(\cdot, t)\|_{L^p} = (4\pi t)^{-\frac{N}{2}} \left( \int_{\mathbb{R}^N} \exp\left(-\frac{p|x|^2}{4t}\right) dx \right)^{\frac{1}{p}} = \pi^{-\frac{N}{2}} (4t)^{-\frac{N}{2}(1-\frac{1}{p})} \left( \int_{\mathbb{R}^N} \exp(-py^2) dy \right)^{\frac{1}{p}} \rightarrow 0,$$

the strong solutions hence converges to the heat kernel in  $L^p$ -space for any  $p > 1$ . Therefore a natural question is whether this solution also converges to the heat kernel in the sense of  $L^1$ -norm. To this end, the more rigorous analysis is required. Indeed, we not only prove that for any  $p$  with  $1 \leq p \leq \infty$ , the solutions converge to the heat kernel in  $L^p$ -norm sense, but also provide accurate convergence rate estimates and the decay rate, which appears to be optimal in the sense that the rate achieved herein is consistent with that of the heat equation in  $\mathbb{R}^N$  ( $N = 2, 3$ ) except for  $N = 1$ . In addition, as the complement to the above results, it is shown that the global existence of strong solutions to (1.6) in the space of  $L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N))$  for some small initial data. Here, we define a strong solution as a pair  $(u, v)$  belonging to  $L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N))$ , which not only satisfies the system of equations (1.6) in the sense of distributions, but also meets the equations almost everywhere.

In Section 2, we employ a scaling transformation and  $L^p - L^q$  estimates for  $e^{t\Delta}$  to derive the decay estimate of any strong solution  $(u, v) \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$  of (1.6), and thereby obtain the following convergence results by means of a bootstrap argument.

**Theorem 1.1** Let  $N = 2, 3$  and  $(u, v) \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$  be the global strong solution of (1.6). Then there exists  $C > 0$  such that

$$\begin{aligned}\|u(t)\|_{L^p} &\leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})}\|u_0\|_{L^1}, \\ \|v(t)\|_{L^p} &\leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})}\|v_0\|_{L^1}\end{aligned}$$

where  $p \in (1, \infty]$ .

Moreover, for any  $p \in [1, \infty]$

$$\begin{aligned}\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{p})} \left\| u(\cdot, t) - \int_{\mathbb{R}^N} u_0(y) dy G(\cdot, t) \right\|_{L^p} &= 0, \\ \lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{p})} \left\| v(\cdot, t) - \int_{\mathbb{R}^N} v_0(y) dy G(\cdot, t) \right\|_{L^p} &= 0,\end{aligned}$$

where  $G(x, t)$  is the heat kernel, i.e.  $G(x, t) = (4\pi t)^{-\frac{N}{2}} \exp\left(-\frac{|x|^2}{4t}\right)$ .

In the above theorem, we obtained an estimate of the optimal decay rate for the solutions in two and three dimensions. However, for the one-dimensional spatial case, the bootstrap technique cannot be employed to achieve an optimal decay rate, and only the following conclusion has been reached.

**Theorem 1.2** Let  $N = 1$ . Suppose that  $(u, v) \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}) \cap L^1(\mathbb{R}))$  is a global strong solution of (1.6). Then there exists  $C > 0$ , such that

$$\begin{aligned}\|u(t)\|_{L^p} &\leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})}\|u_0\|_{L^1}, \\ \|v(t)\|_{L^p} &\leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})}\|v_0\|_{L^1},\end{aligned}$$

where  $p \in (1, \infty]$ . Additionally, for any  $0 < m < \frac{9}{32}$

$$\lim_{t \rightarrow \infty} t^{m(1-\frac{1}{p})} \left\| u(\cdot, t) - \int_{\mathbb{R}^N} u_0(y) dy G(\cdot, t) \right\|_{L^p} = 0,$$

where  $p \in (1, \infty]$ .

As a necessary complement to the above result, we make sure that the strong solution of (1.6) in the space of  $L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N))$  actually exists in Section 3.

**Theorem 1.3** Assume that  $N = 1, 2, 3$ , the nonnegative initial value  $(u_0, v_0)$  satisfies  $u_0, v_0 \in L^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$  and  $|x|^r u_0(x) \in L^2(\mathbb{R}^N)$  with some constant  $r \in (0, 1)$ . Denote

$$\begin{aligned}\phi(t) &= \|u(\cdot, t)\|_{H^2}^2 + \|v(\cdot, t)\|_{H^2}^2, \\ h(t) &= \|\nabla u(\cdot, t)\|_{H^2}^2 + \|\nabla v(\cdot, t)\|_{H^2}^2.\end{aligned}$$

Then if  $\phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)}$  for some constant  $C^*$  depending only on  $N$ , the problem (1.6) admits a unique global strong solution  $(u, v)$  with  $u, v \in \mathcal{X}$  such that

$$\frac{1}{2} \int_0^\infty h(t) dt \leq \phi(0), \quad \phi(t) \leq \left( \phi^{-\frac{2}{N}}(0) + \frac{2}{N} A^* t \right)^{-\frac{N}{2}}$$

with some constant  $A^* > 0$ , where

$$\mathcal{X} = \{\varphi \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)); \nabla \varphi \in L^2(\mathbb{R}^+; H^2(\mathbb{R}^N)); \varphi_t \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))\}.$$

## 2 Long-time dynamic behavior and decay rate estimation

In this section, we are devoted to investigating the long-time asymptotic behavior of strong solutions to problem (1.6). To this end, we begin by recalling following results which come from [5] and [14], respectively.

**Lemma 2.1** (*Gagliardo-Nirenberg inequality in  $\mathbb{R}^N$* ) For function  $u : \mathbb{R}^N \rightarrow \mathbb{R}$ , we have

$$\|D^j u\|_{L^p(\mathbb{R}^N)} \leq C \|D^m u\|_{L^r(\mathbb{R}^N)}^\alpha \|u\|_{L^q(\mathbb{R}^N)}^{1-\alpha},$$

where  $1 \leq q, r \leq \infty$ ,  $\frac{j}{m} \leq \alpha \leq 1$ ,  $\frac{1}{p} = \frac{j}{N} + (\frac{1}{r} - \frac{m}{N})\alpha + \frac{1-\alpha}{q}$ .

Next, we introduce the Gagliardo-Nirenberg inequality on ball  $B_R$  centered at 0 with radius  $R \geq 1$ . The crucial aspect is to elucidate the relationship between the embedding constants and the ball radius.

**Lemma 2.2** (*Gagliardo-Nirenberg inequality in a bounded domain*) For function  $u : B_R \rightarrow \mathbb{R}$  with  $B_R \subset \mathbb{R}^N$ , we have

$$\|D^j u\|_{L^p(B_R)} \leq C_1 \|D^m u\|_{L^r(B_R)}^\alpha \|u\|_{L^q(B_R)}^{1-\alpha} + C_2 R^{\frac{N}{p} - \frac{N}{s} - j} \|u\|_{L^s(B_R)},$$

where  $C_1 > 0, C_2 > 0$  are independent of  $R$ ,  $1 \leq q, r \leq \infty$ ,  $\frac{j}{m} \leq \alpha \leq 1$ ,  $\frac{1}{p} = \frac{j}{N} + (\frac{1}{r} - \frac{m}{N})\alpha + \frac{1-\alpha}{q}$ , and  $s > 0$  is arbitrary.

**Proof.** We use scaling techniques to prove the inequality. Firstly, it is well-known that there exist constants  $C_1 > 0$  and  $C_2 > 0$  such that the inequality

$$\|D^j f\|_{L^p(B_1)} \leq C_1 \|D^m f\|_{L^r(B_1)}^\alpha \|f\|_{L^q(B_1)}^{1-\alpha} + C_2 \|f\|_{L^s(B_1)} \quad (2.1)$$

holds for any  $f \in W^{m,r}(B_1)$ . Let  $\tilde{u}(y) = u(Ry) = u(x)$  for  $y \in B_1$ . Then

$$\|D^j \tilde{u}\|_{L^p(B_1)} = R^{j - \frac{N}{p}} \|D_j u\|_{L^p(B_R)}, \quad \|D^m \tilde{u}\|_{L^r(B_1)} = R^{m - \frac{N}{r}} \|D_j u\|_{L^r(B_R)}, \quad \|\tilde{u}\|_{L^q(B_1)} = R^{-\frac{N}{q}} \|u\|_{L^q(B_R)},$$

which in conjunction with (2.1), results in

$$\|D^j u\|_{L^p(B_R)} \leq C_1 R^{(m - \frac{N}{r} + \frac{N}{q})\alpha - \frac{N}{q} + \frac{N}{p} - j} \|D^m u\|_{L^r(B_R)}^\alpha \|u\|_{L^q(B_R)}^{1-\alpha} + C_2 R^{\frac{N}{p} - \frac{N}{s} - j} \|u\|_{L^s(B_R)}.$$

From  $\frac{1}{p} = \frac{j}{N} + (\frac{1}{r} - \frac{m}{N})\alpha + \frac{1-\alpha}{q}$ , we have

$$(m - \frac{N}{r} + \frac{N}{q})\alpha - \frac{N}{q} + \frac{N}{p} - j = 0,$$

and hence obtain that

$$\|D^j u\|_{L^p(B_R)} \leq C_1 \|D^m u\|_{L^r(B_R)}^\alpha \|u\|_{L^q(B_R)}^{1-\alpha} + C_2 R^{\frac{N}{p} - \frac{N}{s} - j} \|u\|_{L^s(B_R)}.$$

□

**Lemma 2.3** Let  $T \leq \infty$ ,  $b > 0$ ,  $\beta > 0$  and  $a(t)$  be a nonnegative function locally integrable on  $(0, T)$ . If  $u(t)$  is nonnegative and locally integrable on  $(0, T)$  with

$$u(t) \leq a(t) + b \int_0^t (t-s)^{\beta-1} u(s) ds$$

for all  $t \in (0, T)$ , then

$$u(t) \leq a(t) + \theta \int_0^t E'_\beta(\theta(t-s)) a(s) ds, \quad \text{for } 0 \leq t < T,$$

where  $\theta = (b\Gamma(\beta))^\frac{1}{\beta}$ ,  $E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^{n\beta}}{\Gamma(n\beta+1)}$ ,  $E'_\beta(z) \simeq \frac{z^{\beta-1}}{\Gamma(\beta)}$  as  $z \rightarrow 0^+$ .

In order to analyze the long-time behavior of solutions, we employ a scaling transformation on the solution by letting

$$u_\lambda(x, t) = \lambda^N u(\lambda x, \lambda^2 t), \quad v_\lambda(x, t) = v(\lambda x, \lambda^2 t).$$

It is observed that if  $(u, v)$  is the solution of (1.6), then  $(u_\lambda, v_\lambda)$  is the solution to the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \chi \nabla \cdot (u \nabla v) & \text{in } Q, \\ \frac{\partial v}{\partial t} = \Delta v + \xi \lambda^{-N} \nabla \cdot (v \nabla u) & \text{in } Q, \\ u(x, 0) = u_{\lambda 0}(x) = \lambda^N u_0(\lambda x), \quad v(x, 0) = v_{\lambda 0}(x) = v_0(\lambda x) & \text{in } \mathbb{R}^N. \end{cases} \quad (2.2)$$

We reformulate problem (2.2) into the following integral equation format:

$$\begin{cases} u_\lambda(x, t) = e^{t\Delta} u_{\lambda 0} - \chi \int_0^t \nabla \cdot (e^{(t-s)\Delta} (u_\lambda \nabla v_\lambda)) ds, \\ v_\lambda(x, t) = e^{t\Delta} v_{\lambda 0} + \xi \lambda^{-N} \int_0^t \nabla \cdot (e^{(t-s)\Delta} (v_\lambda \nabla u_\lambda)) ds, \end{cases} \quad (2.3)$$

where

$$(e^{t\Delta} f)(x) = \int_{\mathbb{R}^N} G(x - y, t) f(y) dy.$$

As the preparation of the following analysis, we recall the following  $L^p - L^q$  estimates for  $e^{t\Delta}$ .

**Lemma 2.4** *Let  $1 \leq q \leq p \leq \infty$  and  $f \in L^q(\mathbb{R}^N)$ . Then*

$$\begin{aligned} \|e^{t\Delta} f\|_{L^p} &\leq (4\pi t)^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p})} \|f\|_{L^q}, \\ \|\nabla e^{t\Delta} f\|_{L^p} &\leq C t^{-\frac{N}{2}(\frac{1}{q} - \frac{1}{p}) - \frac{1}{2}} \|f\|_{L^q}, \end{aligned}$$

where  $C$  is a positive constant depending only on  $p$  and  $q$ .

For convenience, we denote by  $C$  some constants in the proofs below, which may be different from each other, but do not depend on time  $t$ .

By using the scaling technique, we can derive following estimates.

**Lemma 2.5** *Let  $N \leq 3$ ,  $r > N$  and  $p > 1$  with  $1 - \frac{2}{N} < \frac{1}{p} < 1 + \frac{1}{r} - \frac{1}{N}$ . Then there exists constant  $C = C(N, r, p) > 0$  such that for all  $t > 0$ , we have*

$$\|u(t)\|_{L^p} \leq C t^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1} + C t^{\frac{1}{2} - \frac{N}{2}(1-\frac{1}{p} + \frac{1}{r})} \|u_0\|_{L^1} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r}. \quad (2.4)$$

**Proof.** From the first equation of (2.3), and applying Lemma 2.4, we can conclude that

$$\begin{aligned} \|u_\lambda(t)\|_{L^p} &\leq \|e^{t\Delta} u_{\lambda 0}\|_{L^p} + \chi \int_0^t \left\| \nabla \cdot (e^{(t-s)\Delta} (u_\lambda \nabla v_\lambda)) \right\|_{L^p} ds \\ &\leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + C \chi \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \int_0^t (t-s)^{-\frac{1}{2} - \frac{N}{2r}} \|u_\lambda(s)\|_{L^p} ds \end{aligned}$$

for  $0 < t \leq 1$ . Thanks to Lemma 2.3, we get

$$\begin{aligned}
& \|u_\lambda(t)\|_{L^p} \\
& \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + C \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} ds \\
& \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + C \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \|u_{\lambda 0}\|_{L^1} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} \int_0^1 s^{-\frac{N}{2}(1-\frac{1}{p})} (1-s)^{-\frac{1}{2}-\frac{N}{2r}} ds \\
& \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + CB \left(1 - \frac{N}{2}(1 - \frac{1}{p}), \frac{1}{2} - \frac{N}{2r}\right) \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \|u_{\lambda 0}\|_{L^1} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})}
\end{aligned} \tag{2.5}$$

for all  $t \leq 1$ ,  $p > 1$  with  $\frac{N}{2}(1 - \frac{1}{p}) < 1$ , which is warranted by  $\frac{1}{p} > 1 - \frac{2}{N}$ .

Now taking  $t = 1$  in the above inequality, we obtain that

$$\|u_\lambda(1)\|_{L^p} \leq C \|u_{\lambda 0}\|_{L^1} + C \|u_{\lambda 0}\|_{L^1} \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r}. \tag{2.6}$$

Noticing that

$$\begin{aligned}
\|u_{\lambda 0}\|_{L^1} &= \lambda^N \int_{\mathbb{R}^N} u_0(\lambda x) dx = \int_{\mathbb{R}^N} u_0(y) dy = \|u_0\|_{L^1}, \\
\|u_\lambda(1)\|_{L^p} &= \left( \int_{\mathbb{R}^N} \lambda^{Np} |u(\lambda x, \lambda^2)|^p dx \right)^{\frac{1}{p}} = \left( \int_{\mathbb{R}^N} \lambda^{Np-N} |u(x, \lambda^2)|^p dx \right)^{\frac{1}{p}} = \lambda^{N(1-\frac{1}{p})} \|u(\lambda^2)\|_{L^p},
\end{aligned}$$

as well as

$$\|\nabla v_\lambda(s)\|_{L^r} = \left( \int_{\mathbb{R}^N} \lambda^r |\nabla v(\lambda x, \lambda^2 s)|^r dx \right)^{\frac{1}{r}} = \left( \int_{\mathbb{R}^N} \lambda^{r-N} |\nabla v(x, \lambda^2 s)|^r dx \right)^{\frac{1}{r}} = \lambda^{1-\frac{N}{r}} \|\nabla v(\lambda^2 s)\|_{L^r},$$

the choice of  $\lambda = \sqrt{t}$  in (2.6) then yields

$$t^{\frac{N}{2}(1-\frac{1}{p})} \|u(t)\|_{L^p} \leq C \|u_0\|_{L^1} + C t^{\frac{1}{2}-\frac{N}{2r}} \|u_0\|_{L^1} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r},$$

which readily implies that

$$\|u(t)\|_{L^p} \leq C t^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1} + C t^{\frac{1}{2}-\frac{N}{2}(1-\frac{1}{p}+\frac{1}{r})} \|u_0\|_{L^1} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r},$$

and thus completes the proof.  $\square$

Similarly, we also have

**Lemma 2.6** *Let  $1 \leq N \leq 3$ ,  $r > N$  and  $(u, v)$  be the solution of (1.6). Then for all  $p > 1$  with  $1 - \frac{2}{N} < \frac{1}{p} < 1 + \frac{1}{r} - \frac{1}{N}$ , there exists constant  $C = C(N, r, p) > 0$  such that for all  $t > 0$ ,*

$$\|v(t)\|_{L^p} \leq C t^{-\frac{N}{2}(1-\frac{1}{p})} \|v_0\|_{L^1} + C t^{\frac{1}{2}-\frac{N}{2}(1-\frac{1}{p}+\frac{1}{r})} \|v_0\|_{L^1} \sup_{0 < s < t} \|\nabla u(s)\|_{L^r}. \tag{2.7}$$

In order to derive the optimal decay rate estimate of  $u, v$  via bootstrap argument, we need the following lemmas in which  $\sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r}$  and  $\sup_{\frac{t}{2} < s < t} \|\nabla u(s)\|_{L^r}$  are instead of  $\sup_{0 < s < t} \|\nabla v(s)\|_{L^r}$ ,  $\sup_{0 < s < t} \|\nabla u(s)\|_{L^r}$  in (2.6) and (2.7), respectively. However, the approaches adapted here seems to be inaccessible in the case  $N = 1$ .

**Lemma 2.7** Let  $N = 2, 3$  and  $N < r \leq 4$ . Suppose that  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  are bounded, then for all  $p > 1$  with  $\frac{1}{p} > 1 - \frac{2}{N}$ , there exists  $C = C(p, r, N) > 0$  such that

$$\|u(t)\|_{L^p} \leq C\|u_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} (1 + t^{-\frac{r-N}{2r}}) + C\|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})} (t^{\frac{1}{2}-\frac{N}{2r}} + t^{1-\frac{N}{r}}), \quad (2.8)$$

$$\|v(t)\|_{L^p} \leq C\|v_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} (1 + t^{-\frac{r-N}{2r}}) + C\|v_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla u(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})} (t^{\frac{1}{2}-\frac{N}{2r}} + t^{1-\frac{N}{r}}), \quad (2.9)$$

for all  $t > 0$ .

**Proof.** According to (2.5), the following inequality

$$\|u_\lambda(t)\|_{L^q} \leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{q})} \|u_{\lambda 0}\|_{L^1} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^1} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{q})} \quad (2.10)$$

holds for  $0 < t \leq 1$ ,  $r > N$  and  $q > 1$  with  $\frac{1}{q} > 1 - \frac{2}{N}$ .

On the other hand, using Lemma 2.4, we have

$$\begin{aligned} \|u_\lambda(t)\|_{L^p} &\leq \|e^{t\Delta} u_{\lambda 0}\|_{L^p} + \chi \int_0^t \left\| \nabla \cdot \left( e^{(t-s)\Delta} (u_\lambda \nabla v_\lambda) \right) \right\|_{L^p} ds \\ &\leq \|e^{t\Delta} u_{\lambda 0}\|_{L^p} + C\chi \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{N}{2}(1-\frac{1}{p})} \|u_\lambda(s) \nabla v_\lambda(s)\|_{L^1} ds + C\chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^p} \|\nabla v_\lambda(s)\|_{L^r} ds \\ &\leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + C\chi \sup_{0 < s < \frac{t}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} t^{-\frac{1}{2}-\frac{N}{2}(1-\frac{1}{p})} \int_0^{\frac{t}{2}} \|u_\lambda(s)\|_{L^{\frac{Nr}{Nr+N-2r}}} ds \\ &\quad + C\chi \sup_{\frac{t}{2} < s < t} \|\nabla v_\lambda(s)\|_{L^r} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^p} ds \end{aligned} \quad (2.11)$$

for  $0 < t \leq 1$ .

Now as the application of (2.10) to  $q = \frac{Nr}{Nr+N-2r}$  and  $q = p$  respectively, one can have

$$\begin{aligned} \int_0^{\frac{t}{2}} \|u_\lambda(s)\|_{L^{\frac{Nr}{Nr+N-2r}}} ds &\leq \int_0^{\frac{t}{2}} \left( (4\pi s)^{\frac{N}{2r}-1} \|u_{\lambda 0}\|_{L^1} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^1} s^{-\frac{1}{2}} \right) ds \\ &\leq C\|u_{\lambda 0}\|_{L^1} t^{\frac{N}{2r}} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^1} t^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} &\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^p} ds \\ &\leq \|u_{\lambda 0}\|_{L^1} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \left( (4\pi s)^{-\frac{N}{2}(1-\frac{1}{p})} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} s^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} \right) ds \\ &= \|u_{\lambda 0}\|_{L^1} (4\pi)^{-\frac{N}{2}(1-\frac{1}{p})} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} \int_{\frac{1}{2}}^1 (1-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{-\frac{N}{2}(1-\frac{1}{p})} ds \\ &\quad + C\|u_{\lambda 0}\|_{L^1} \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} t^{1-\frac{N}{r}-\frac{N}{2}(1-\frac{1}{p})} \int_{\frac{1}{2}}^1 (1-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} ds \\ &\leq C\|u_{\lambda 0}\|_{L^1} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} + C\|u_{\lambda 0}\|_{L^1} \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} t^{1-\frac{N}{r}-\frac{N}{2}(1-\frac{1}{p})} \end{aligned}$$

for  $0 < t \leq 1$ , due to  $r > N$ .

Substituting the above two inequalities into (2.11) yields

$$\begin{aligned} \|u_\lambda(t)\|_{L^p} &\leq (4\pi t)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_{\lambda 0}\|_{L^1} + C \|u_{\lambda 0}\|_{L^1} \sup_{0 < s < \frac{t}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} t^{-\frac{1}{2}-\frac{N}{2}(1-\frac{1}{p})+\frac{N}{2r}} \\ &\quad + C \|u_{\lambda 0}\|_{L^1} \sup_{0 < s < \frac{t}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})} \\ &\quad + C \|u_{\lambda 0}\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v_\lambda(s)\|_{L^r} \left( t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} + \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} t^{1-\frac{N}{r}-\frac{N}{2}(1-\frac{1}{p})} \right). \end{aligned}$$

So taking  $t = 1$  in the above inequality, we have

$$\begin{aligned} \|u_\lambda(1)\|_{L^p} &\leq C \|u_{\lambda 0}\|_{L^1} + C \|u_{\lambda 0}\|_{L^1} \sup_{0 < s < \frac{1}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} \left( 1 + \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \right) \\ &\quad + C \|u_{\lambda 0}\|_{L^1} \sup_{\frac{1}{2} < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \left( 1 + \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \right). \end{aligned} \quad (2.12)$$

Letting  $\lambda = \sqrt{t}$  in (2.12) and noticing that

$$\|u_{\lambda 0}\|_{L^1} = \|u_0\|_{L^1}, \quad \|u_\lambda(1)\|_{L^p} = \lambda^{N(1-\frac{1}{p})} \|u(\lambda^2)\|_{L^p}, \quad \|\nabla v_\lambda(t)\|_{L^q} = \lambda^{1-\frac{N}{q}} \|\nabla v(\lambda^2 t)\|_{L^q},$$

we arrive at

$$\begin{aligned} t^{\frac{N}{2}(1-\frac{1}{p})} \|u(t)\|_{L^p} &\leq C \|u_0\|_{L^1} + C \|u_0\|_{L^1} \sup_{0 < s < \frac{t}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}} \left( 1 + t^{\frac{1}{2}-\frac{N}{2r}} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} \right) t^{\frac{N-r}{2r}} \\ &\quad + C \|u_0\|_{L^1} t^{\frac{1}{2}-\frac{N}{2r}} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} \left( 1 + t^{\frac{1}{2}-\frac{N}{2r}} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} \right), \end{aligned}$$

that is

$$\begin{aligned} \|u(t)\|_{L^p} &\leq C \|u_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} + C \|u_0\|_{L^1} \sup_{0 < s < \frac{t}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}} t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{r-N}{2r}} \\ &\quad + C \|u_0\|_{L^1} \sup_{0 < s < \frac{t}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})} + C \|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}(1-\frac{1}{p})} \\ &\quad + C \|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} t^{1-\frac{N}{r}-\frac{N}{2}(1-\frac{1}{p})}. \end{aligned} \quad (2.13)$$

In addition, by Lemma 2.1 (the Gagliardo-Nirenberg inequality), there exists  $C > 0$  such that for any  $q \in [\frac{4}{3}, \frac{2N}{(N-2)_+}]$ ,

$$\|\nabla v\|_{L^q} \leq C \|v\|_{L^1}^{1-\alpha} \|\nabla^2 v\|_{L^2}^\alpha, \quad (2.14)$$

where  $\alpha = \frac{1+\frac{1}{N}-\frac{1}{q}}{1+\frac{2}{N}-\frac{1}{2}} \geq \frac{1}{2}$ . Noticing that  $\frac{4}{3} \leq \frac{Nr}{2r-N} < r < \frac{2N}{(N-2)_+}$  due to  $N < r \leq 4$ , hence,  $\sup_{0 < s < \frac{t}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}}$  and  $\sup_{0 < s < t} \|\nabla v(s)\|_{L^r}$  are bounded since  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  is bounded. Therefore combining (2.13) with (2.14), we readily have

$$\|u(t)\|_{L^p} \leq C \|u_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} (1 + t^{-\frac{r-N}{2r}}) + C \|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})} (t^{\frac{1}{2}-\frac{N}{2r}} + t^{1-\frac{N}{r}}),$$

which is the desired inequality (2.8). It is observed that the inequality (2.9) can be proved similarly.  $\square$

At this position, we utilize Lemma 2.7 to derive the optimal decay rate estimate of  $u, v$  which can be stated as follows.

**Lemma 2.8** *Let  $N = 2, 3$ , and  $(u, v)$  be the solution of (1.6) such that  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  are bounded. Then there exists  $C > 0$  such that for all  $t \geq 1$ ,*

$$\begin{aligned} \|u(t)\|_{L^p} &\leq Ct^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1} \\ \|v(t)\|_{L^p} &\leq Ct^{-\frac{N}{2}(1-\frac{1}{p})} \|v_0\|_{L^1} \end{aligned} \quad (2.15)$$

is valid for  $p \in (1, 2]$ .

**Proof.** Recalling (2.8) and (2.9), we see that for any  $N < r \leq 4$ ,  $p > 1$  with  $1 - \frac{2}{N} < \frac{1}{p} < 1 - \frac{2}{N} + \frac{2}{r}$ ,

$$\|u(t)\|_{L^p} \leq C\|u_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} + C\|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})+1-\frac{N}{r}} \quad (2.16)$$

as well as

$$\|v(t)\|_{L^p} \leq C\|v_0\|_{L^1} t^{-\frac{N}{2}(1-\frac{1}{p})} + C\|v_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|\nabla u(s)\|_{L^r} t^{-\frac{N}{2}(1-\frac{1}{p})+1-\frac{N}{r}} \quad (2.17)$$

for  $t \geq \frac{1}{2}$ . From Lemma 2.1, we infer that

$$\|\nabla v\|_{L^r} \leq C\|v\|_{L^p}^\alpha \|\nabla^2 v\|_{L^2}^{1-\alpha} \quad (2.18)$$

with  $\alpha = \frac{\frac{1}{N} + \frac{1}{r} - \frac{1}{2}}{\frac{1}{p} + \frac{2}{N} - \frac{1}{2}}$ . It is observed that  $\alpha < \frac{1}{2}$  can be warranted by  $p < \frac{2r}{(4-r)_+}$ .

Noticing that  $\|\nabla^2 v\|_{L^2}$  is bounded, substituting (2.18) into (2.16) then yields

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1} + Ct^{-\frac{N}{2}(1-\frac{1}{p})+1-\frac{N}{r}} \|u_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|v(s)\|_{L^p}^\alpha \quad (2.19)$$

for all  $t \geq \frac{1}{2}$ . Similarly, we have

$$\|v(t)\|_{L^p} \leq Ct^{-\frac{N}{2}(1-\frac{1}{p})} \|v_0\|_{L^1} + Ct^{-\frac{N}{2}(1-\frac{1}{p})+1-\frac{N}{r}} \|v_0\|_{L^1} \sup_{\frac{t}{2} < s < t} \|u(s)\|_{L^p}^\alpha. \quad (2.20)$$

Furthermore, taking  $r = N + \varepsilon$ , one can find  $\varepsilon_1 > 0$  such that for  $\varepsilon < \varepsilon_1$ ,  $1 - \frac{N}{r} = 1 - \frac{N}{N+\varepsilon} < \frac{N}{4}(1 - \frac{1}{p})$ . In addition, by Lemma 2.1,

$$\|u\|_{L^q} \leq \|u\|_{L^1}^{1-\beta} \|D^2 u\|_{L^2}^\beta, \quad \text{for any } 1 < q \leq \infty, \quad (2.21)$$

with  $\beta = \frac{p-1}{p(\frac{2}{N} + \frac{1}{2})}$  for all  $N \leq 3$ . Hence from (2.20), it follows that

$$\|v(t)\|_{L^p} \leq Ct^{-\frac{N}{4}(1-\frac{1}{p})} \|v_0\|_{L^1} \quad (2.22)$$

for  $t \geq \frac{1}{2}$ .

Now at this position, we insert (2.22) into (2.19) to get

$$\|u(t)\|_{L^p} \leq Ct^{-\frac{N}{2}(1-\frac{1}{p})}\|u_0\|_{L^1} + Ct^{-\frac{N}{2}(1-\frac{1}{p})+1-\frac{N}{N+\varepsilon}-\frac{N}{4}(1-\frac{1}{p})\cdot\frac{\frac{1}{N}+\frac{1}{N+\varepsilon}-\frac{1}{2}}{\frac{1}{p}+\frac{2}{N}-\frac{1}{2}}}\|u_0\|_{L^1} \quad (2.23)$$

and thus readily arrive at the decay estimate of  $u$  in (2.15) for  $t \geq 1$  due to the fact that

$$\lim_{\varepsilon \rightarrow 0} 1 - \frac{N}{N+\varepsilon} - \frac{N}{4}\left(1 - \frac{1}{p}\right) \cdot \frac{\frac{1}{N} + \frac{1}{N+\varepsilon} - \frac{1}{2}}{\frac{1}{p} + \frac{2}{N} - \frac{1}{2}} = -\frac{N}{4}\left(1 - \frac{1}{p}\right) \cdot \frac{\frac{2}{N} - \frac{1}{2}}{\frac{1}{p} + \frac{2}{N} - \frac{1}{2}}.$$

The decay estimate of  $v$  in (2.15) can be achieved similarly, and the proof of this lemma is thus completed.  $\square$

Our subsequent goal is to verify the result of Lemma 2.8 for all  $p$ -value, inter alia  $p = \infty$ . To this end, we achieve following decay result in the style of reasoning from Lemmas 2.5 and 2.7.

**Lemma 2.9** *Assume that  $N = 2, 3$  and  $(u, v)$  is the solution of (1.6). If  $\sup_{t>0} \|\nabla u(t)\|_{L^r}$  and  $\sup_{t>0} \|\nabla v(t)\|_{L^r}$  are bounded for some  $r \in (N, 4]$ , then for any  $q \in (1, r)$  with  $\frac{1}{q} < \frac{2}{N} + \frac{1}{r}$ , there exists a constant  $C = C(q, r, N) > 0$  such that*

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2q}}(1 + t^{\frac{N}{2r}-\frac{1}{2}})\|u_0\|_{L^q} + Ct^{-\frac{N}{2q}+\frac{1}{2}-\frac{N}{2r}}\|u_0\|_{L^q} \sup_{\frac{1}{2}<s<t} \|\nabla v(s)\|_{L^r} \left(1 + t^{\frac{1}{2}-\frac{N}{2r}}\right), \quad (2.24)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2q}}(1 + t^{\frac{N}{2r}-\frac{1}{2}})\|v_0\|_{L^q} + Ct^{-\frac{N}{2q}+\frac{1}{2}-\frac{N}{2r}}\|v_0\|_{L^q} \sup_{\frac{1}{2}<s<t} \|\nabla u(s)\|_{L^r} \left(1 + t^{\frac{1}{2}-\frac{N}{2r}}\right) \quad (2.25)$$

for any  $t > 0$ .

**Proof.** From the first equation of (2.3), we infer that

$$\begin{aligned} \|u_\lambda(t)\|_{L^p} &\leq \|e^{t\Delta}u_{\lambda 0}\|_{L^p} + \chi \int_0^t \left\| \nabla \cdot \left( e^{(t-s)\Delta}(u_\lambda \nabla v_\lambda) \right) \right\|_{L^p} ds \\ &\leq \|e^{t\Delta}u_{\lambda 0}\|_{L^p} + C\chi \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^p} \|\nabla v_\lambda(s)\|_{L^r} ds \\ &\leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{s \leq t} \|\nabla v_\lambda(s)\|_{L^r} \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^p} ds \end{aligned}$$

for  $1 < q < p$ ,  $r > N$ . Furthermore, with the help of Lemma 2.3, we can see that if  $\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right) < 1$ , then for  $0 < t \leq 1$ ,

$$\begin{aligned} \|u_\lambda(t)\|_{L^p} &\leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{0<t<1} \|\nabla v_\lambda(t)\|_{L^r} \int_0^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_{\lambda 0}\|_{L^q} ds \\ &\leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{0<t<1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^q} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \int_0^1 s^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} (1-s)^{-\frac{1}{2}-\frac{N}{2r}} ds \\ &\leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{0<t<1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^q} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}. \end{aligned} \quad (2.26)$$

Similarly, the following inequality is also valid

$$\|v_\lambda(t)\|_{L^p} \leq (4\pi t)^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} \|v_{\lambda 0}\|_{L^q} + C\chi \lambda^{-N} \sup_{0<t<1} \|\nabla u_\lambda(t)\|_{L^r} \|v_{\lambda 0}\|_{L^q} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}. \quad (2.27)$$

On the other hand, we can also obtain

$$\begin{aligned}
\|u_\lambda(t)\|_{L^\infty} &\leq \|e^{t\Delta}u_{\lambda 0}\|_{L^\infty} + \chi \int_0^t \left\| \nabla \cdot \left( e^{(t-s)\Delta} (u_\lambda \nabla v_\lambda) \right) \right\|_{L^\infty} ds \\
&\leq \|e^{t\Delta}u_{\lambda 0}\|_{L^\infty} + C\chi \int_0^{\frac{t}{2}} (t-s)^{-\frac{3}{2}} \|u_\lambda(s) \nabla v_\lambda(s)\|_{L^{\frac{N}{2}}} ds + C\chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^\infty} \|\nabla v_\lambda(s)\|_{L^r} ds \\
&\leq (4\pi t)^{-\frac{N}{2q}} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{0 < s < \frac{t}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} t^{-\frac{3}{2}} \int_0^{\frac{t}{2}} \|u_\lambda(s)\|_{L^r} ds \\
&\quad + C\chi \sup_{\frac{t}{2} < s < t} \|\nabla v_\lambda(s)\|_{L^r} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^\infty} ds, \tag{2.28}
\end{aligned}$$

where the last two integral terms can be estimated as follows. Indeed, by (2.26) and  $\frac{N}{2} \left( \frac{1}{q} - \frac{1}{r} \right) \in (0, 1)$  due to  $q \in (1, r)$  with  $\frac{1}{q} < \frac{2}{N} + \frac{1}{r}$ , we have

$$\begin{aligned}
\int_0^{\frac{t}{2}} \|u_\lambda(s)\|_{L^r} ds &\leq \int_0^{\frac{t}{2}} \left( (4\pi s)^{-\frac{N}{2} \left( \frac{1}{q} - \frac{1}{r} \right)} \|u_{\lambda 0}\|_{L^q} + C\chi \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^q} s^{\frac{1}{2}-\frac{N}{2q}} \right) ds \\
&\leq C \|u_{\lambda 0}\|_{L^q} t^{1-\frac{N}{2} \left( \frac{1}{q} - \frac{1}{r} \right)} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \|u_{\lambda 0}\|_{L^q} t^{\frac{3}{2}-\frac{N}{2q}}
\end{aligned}$$

and

$$\begin{aligned}
\int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u_\lambda(s)\|_{L^\infty} ds &\leq \|u_{\lambda 0}\|_{L^q} \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \left( (4\pi s)^{-\frac{N}{2q}} + C \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} s^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2q}} \right) ds \\
&\leq C \|u_{\lambda 0}\|_{L^q} t^{\frac{1}{2}-\frac{N}{2r}-\frac{N}{2q}} + C \|u_{\lambda 0}\|_{L^q} \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} t^{1-\frac{N}{r}-\frac{N}{2q}}
\end{aligned}$$

for  $0 < t \leq 1$ .

Substituting the above two inequalities into (2.28), and setting  $t = 1$ , we then get

$$\begin{aligned}
\|u_\lambda(1)\|_{L^\infty} &\leq C \|u_{\lambda 0}\|_{L^q} + C \|u_{\lambda 0}\|_{L^q} \sup_{0 < s < \frac{1}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} + C \|u_{\lambda 0}\|_{L^q} \sup_{0 < s < \frac{1}{2}} \|\nabla v_\lambda(s)\|_{L^{\frac{Nr}{2r-N}}} \sup_{0 < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \\
&\quad + C \|u_{\lambda 0}\|_{L^q} \sup_{\frac{1}{2} < s < 1} \|\nabla v_\lambda(s)\|_{L^r} \left( 1 + \sup_{0 < t < 1} \|\nabla v_\lambda(t)\|_{L^r} \right). \tag{2.29}
\end{aligned}$$

Further, thanks to the fact that

$$\|u_{\lambda 0}\|_{L^q} = \lambda^{N(1-\frac{1}{q})} \|u_0\|_{L^q}, \quad \|u_\lambda(t)\|_{L^p} = \lambda^{N(1-\frac{1}{p})} \|u(\lambda^2 t)\|_{L^p}, \quad \|\nabla v_\lambda(t)\|_{L^r} = \lambda^{1-\frac{N}{r}} \|\nabla v(\lambda^2 t)\|_{L^r},$$

we then choose  $\lambda = \sqrt{t}$  in (2.29) to arrive at

$$\begin{aligned}
t^{\frac{N}{2}} \|u(t)\|_{L^\infty} &\leq C t^{\frac{N}{2}-\frac{N}{2q}} \|u_0\|_{L^q} + C t^{\frac{N}{2}-\frac{N}{2q}+\frac{N}{2r}-\frac{1}{2}} \|u_0\|_{L^q} \sup_{0 < s < \frac{1}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}} \\
&\quad + C t^{\frac{N}{2}-\frac{N}{2q}} \|u_0\|_{L^q} \sup_{0 < s < \frac{1}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} \\
&\quad + C t^{\frac{N}{2}-\frac{N}{2q}+\frac{1}{2}-\frac{N}{2r}} \|u_0\|_{L^q} \sup_{\frac{1}{2} < s < t} \|\nabla v(s)\|_{L^r} \left( 1 + t^{\frac{1}{2}-\frac{N}{2r}} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r} \right). \tag{2.30}
\end{aligned}$$

Recalling (2.14),  $\sup_{0 < s < \frac{t}{2}} \|\nabla v(s)\|_{L^{\frac{Nr}{2r-N}}}$  and  $\sup_{0 < s < t} \|\nabla v(s)\|_{L^r}$  are bounded since  $\frac{4}{3} < \frac{Nr}{2r-N} < \frac{2N}{(N-2)_+}$  if  $N < r \leq 4$ . Therefore, from (2.30), it follows that

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2q}}(1 + t^{\frac{N}{2r}-\frac{1}{2}})\|u_0\|_{L^q} + Ct^{-\frac{N}{2q}+\frac{1}{2}-\frac{N}{2r}}\|u_0\|_{L^q} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} \left(1 + t^{\frac{1}{2}-\frac{N}{2r}}\right).$$

The proof of (2.25) can be proved similarly, and the proof of this lemma is thus completed.  $\square$

Similarly as in the proof Lemma 2.8, one can sharp the decay results of  $(u, v)$  in Lemma 2.9, which is stated as

**Lemma 2.10** *Assume that  $N = 2, 3$ . Let  $(u, v)$  be the solution of (1.6) with  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  being bounded. Then for any  $t \geq 1$ ,*

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^2}, \quad (2.31)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|v_0\|_{L^2}, \quad (2.32)$$

where  $C$  is independent of  $t$ .

**Proof.** Recalling (2.24) and (2.25) with  $q = 2$ , we see that for any  $N < r < 4$ ,

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}}\|u_0\|_{L^2} \sup_{\frac{t}{2} < s < t} \|\nabla v(s)\|_{L^r} \quad (2.33)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|v_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}}\|v_0\|_{L^2} \sup_{\frac{t}{2} < s < t} \|\nabla u(s)\|_{L^r} \quad (2.34)$$

when  $t \geq \frac{1}{2}$ . From Lemma 2.1, there exists a constant  $C > 0$  such that

$$\|\nabla v\|_{L^r} \leq C\|v\|_{L^{\frac{2r}{4-r}}}^{\frac{1}{2}} \|\nabla^2 v\|_{L^2}^{\frac{1}{2}},$$

which along with

$$\|v\|_{L^{\frac{2r}{4-r}}} \leq \|v\|_{L^1}^{\frac{4-r}{2r}} \|v\|_{L^\infty}^{\frac{3r-4}{2r}},$$

leads to

$$\|\nabla v\|_{L^r} \leq C\|v\|_{L^1}^{\frac{4-r}{4r}} \|v\|_{L^\infty}^{\frac{3r-4}{4r}} \|\nabla^2 v\|_{L^2}^{\frac{1}{2}}. \quad (2.35)$$

Noticing that  $\|\nabla^2 v\|_{L^2}$  is bounded, substituting (2.35) into (2.33) then yields

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}}\|u_0\|_{L^2} \sup_{\frac{t}{2} < s < t} \|v\|_{L^\infty}^{\frac{3r-4}{4r}} \quad \text{for } t \geq \frac{1}{2}. \quad (2.36)$$

Similarly, we also have

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}}\|v_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}}\|v_0\|_{L^2} \sup_{\frac{t}{2} < s < t} \|u\|_{L^\infty}^{\frac{3r-4}{4r}}, \quad \text{for } t \geq \frac{1}{2}. \quad (2.37)$$

While recalling (2.21) and (2.22), and taking  $r = N + \varepsilon$  in (2.36) and (2.37), one can see that if  $\varepsilon$  is sufficiently small such that  $1 - \frac{N}{N+\varepsilon} < \frac{N}{8}$ , then (2.36) and (2.37) implies that

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{8}}\|u_0\|_{L^2}, \quad (2.38)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{8}}\|v_0\|_{L^2} \quad (2.39)$$

for any  $t \geq \frac{1}{2}$ , respectively. Furthermore, substituting (2.38) and (2.39) into (2.37) and (2.36) yields

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}} \|u_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}-\frac{N}{8}\frac{3r-4}{4r}} \|u_0\|_{L^2} \quad \text{for } t \geq 1, \quad (2.40)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}} \|v_0\|_{L^2} + Ct^{-\frac{N}{4}+1-\frac{N}{r}-\frac{N}{8}\frac{3r-4}{4r}} \|v_0\|_{L^2} \quad \text{for } t \geq 1 \quad (2.41)$$

with  $r = N + \varepsilon$ , respectively. Noticing that

$$\lim_{\varepsilon \rightarrow 0^+} 1 - \frac{N}{N + \varepsilon} - \frac{N}{8} \frac{3(N + \varepsilon) - 4}{4(N + \varepsilon)} = -\frac{3N - 4}{32} < 0,$$

one can see that for the suitably small  $\varepsilon > 0$ ,  $-\frac{N}{4} + 1 - \frac{N}{r} - \frac{N}{8}\frac{3r-4}{4r} < -\frac{N}{4}$  with  $r = N + \varepsilon$  and thereby achieve the desired inequalities (2.31) and (2.32).  $\square$

Now with the help of Lemma 2.8 and Lemma 2.10, one can obtain the optimal decay rate of  $u$  and  $v$ .

**Lemma 2.11** *Let the assumption of Lemma 2.10 hold. Then there exists  $C > 0$  such that*

$$\|u(t)\|_{L^p} \leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1}, \quad (2.42)$$

$$\|v(t)\|_{L^p} \leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})} \|v_0\|_{L^1} \quad (2.43)$$

for all  $p \in (1, \infty]$  and  $t > 0$ .

**Proof.** From Lemma 2.10, it is observed that for any  $t > 2$ ,

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}} \left\| u\left(\frac{t}{2}\right) \right\|_{L^2},$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{4}} \left\| v\left(\frac{t}{2}\right) \right\|_{L^2}.$$

On the other hand, by leveraging Lemma 2.8, one can conclude that for any  $t > 2$ ,

$$\left\| u\left(\frac{t}{2}\right) \right\|_{L^2} \leq Ct^{-\frac{N}{4}} \|u_0\|_{L^1},$$

$$\left\| v\left(\frac{t}{2}\right) \right\|_{L^2} \leq Ct^{-\frac{N}{4}} \|v_0\|_{L^1}.$$

Hence as the combination of the above inequalities, one deduces that for any  $t > 2$ ,

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2}} \|u_0\|_{L^1}, \quad (2.44)$$

$$\|v(t)\|_{L^\infty} \leq Ct^{-\frac{N}{2}} \|v_0\|_{L^1},$$

which implies that

$$\|u(t)\|_{L^\infty} \leq C(t+1)^{-\frac{N}{2}} \|u_0\|_{L^1}, \quad (2.45)$$

$$\|v(t)\|_{L^\infty} \leq C(t+1)^{-\frac{N}{2}} \|v_0\|_{L^1}$$

for any  $t > 0$ , due to the fact that both  $\|u(t)\|_{L^\infty}$  and  $\|u(t)\|_{L^1}$  are bounded by Gagliardo-Nirenberg inequality.

As a directed application of  $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{\frac{p-1}{p}} \|f\|_{L^1}^{\frac{1}{p}}$ , (2.45) gives

$$\|u(t)\|_{L^p} \leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})} \|u_0\|_{L^1}, \quad \|v(t)\|_{L^p} \leq C(t+1)^{-\frac{N}{2}(1-\frac{1}{p})} \|v_0\|_{L^1}$$

for any  $t > 0$  and thus completes the proof.  $\square$

In contrast to the case of  $N = 2, 3$ , the decay estimate in Lemma 2.7 seems to be invalid for the one-dimensional case, and thus the optimal decay rate of solution  $(u, v)$  of (1.6) is likely inaccessible.

**Lemma 2.12** *Let  $N = 1$  and  $(u, v)$  be the solution of (1.6) with  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  being bounded. Then there exists  $C > 0$  such that*

$$\begin{aligned} \|u(t)\|_{L^p} &\leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})} \|u_0\|_{L^1}, \\ \|v(t)\|_{L^p} &\leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})} \|v_0\|_{L^1} \end{aligned} \quad (2.46)$$

for all  $p \in (1, \infty]$  and  $t > 0$ .

**Proof.** As the application of (2.4) to  $N = 1$ , we can see that for  $r > 1$  there exists  $C > 0$  such that

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1} + Ct^{-\frac{1}{2r}} \|u_0\|_{L^1} \sup_{0 < s < t} \|\nabla v(s)\|_{L^r}. \quad (2.47)$$

In addition, according to Lemma 2.1, the inequality  $\|\nabla v\|_{L^r} \leq \|v\|_{L^1}^{1-\alpha} \|\nabla^2 v\|_{L^2}^\alpha$  with  $\alpha = \frac{4}{5} - \frac{2}{5r}$  is valid for all  $r \geq \frac{4}{3}$ .

Taking  $r = \frac{4}{3}$  in (2.47), we arrive at

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{1}{2}} \|u_0\|_{L^1} + Ct^{-\frac{3}{8}} \|u_0\|_{L^1}$$

and thus for any  $t > 0$ ,

$$\|u(t)\|_{L^\infty} \leq C(t+1)^{-\frac{3}{8}} \|u_0\|_{L^1}$$

since  $\|u(t)\|_{L^\infty}$  is bounded, which readily implies that

$$\|u\|_{L^p} \leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})} \|u_0\|_{L^1}$$

since  $\|u\|_{L^p} \leq \|u\|_{L^\infty}^{\frac{p-1}{p}} \|u\|_{L^1}^{\frac{1}{p}}$ . Similar to the above proof, and using Lemma 2.6, we also have

$$\|v\|_{L^p} \leq C(t+1)^{-\frac{3}{8}(1-\frac{1}{p})} \|v_0\|_{L^1}.$$

$\square$

In what follows we show that the solution converges to heat kernel as time tends to infinity. As a prerequisite, it is observed that the difference between  $t^{\frac{N}{2}(1-\frac{1}{p})} e^{t\Delta} f$  and  $t^{\frac{N}{2}(1-\frac{1}{p})} \int_{\mathbb{R}^N} f(y) dy G(\cdot, t)$  decays to zero in  $L^p(\mathbb{R}^N)$ . The proof thereof may be for convenient of some readers.

**Lemma 2.13** *Assume that  $f \geq 0$ , and  $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ . Then for any  $1 \leq p \leq \infty$ ,*

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{p})} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^p} = 0, \quad (2.48)$$

where  $G(x, t)$  is the heat kernel.

**Proof.** Given that  $f(x) \in L^1(\mathbb{R}^N)$ , then for any  $\varepsilon > 0$ , there exists  $R > 0$  such that

$$\int_{\mathbb{R}^N \setminus B_R} f(y) dy < \varepsilon.$$

We also note that

$$|G(x-y, t) - G(x, t)| = |y \cdot \nabla G(x - \theta y, t)| = (4\pi t)^{-\frac{N}{2}} \frac{|y|}{\sqrt{t}} \left| \frac{x - \theta y}{2\sqrt{t}} e^{-\frac{|x-\theta y|^2}{4t}} \right| \leq \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} (4\pi t)^{-\frac{N}{2}} \frac{|y|}{\sqrt{t}}.$$

Then by a direct calculation, we see that

$$\begin{aligned} (4\pi t)^{\frac{N}{2}} \left| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(x, t) \right| &= (4\pi t)^{\frac{N}{2}} \left| \int_{\mathbb{R}^N} (G(x-y, t) - G(x, t)) f(y) dy \right| \\ &\leq (4\pi t)^{\frac{N}{2}} \int_{\mathbb{R}^N \setminus B_R} |G(x-y, t) - G(x, t)| f(y) dy + (4\pi t)^{\frac{N}{2}} \int_{B_R} |G(x-y, t) - G(x, t)| f(y) dy \\ &\leq 2 \int_{\mathbb{R}^N \setminus B_R} f(y) dy + e^{-\frac{1}{2}} \frac{R}{\sqrt{2t}} \int_{B_R} f(y) dy \\ &\leq 2\varepsilon + e^{-\frac{1}{2}} \frac{R}{\sqrt{2t}} \int_{\mathbb{R}^N} f(y) dy, \end{aligned}$$

which leads to

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^\infty} = 0. \quad (2.49)$$

Next, we show that  $\left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^1} \rightarrow 0$ . For this purpose, we first prove the following inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |G(x-y, t) - G(x, t)| dx &= \int_{\mathbb{R}^N} \left| \int_0^1 \frac{d}{ds} G(x - sy, t) ds \right| dx \\ &\leq \int_{\mathbb{R}^N} \int_0^1 |y \nabla G(x - sy, t)| ds dx \\ &\leq \int_0^1 \int_{\mathbb{R}^N} (4\pi t)^{-\frac{N}{2}} \frac{|y|}{\sqrt{t}} \left| \frac{x - sy}{2\sqrt{t}} e^{-\frac{|x-sy|^2}{4t}} \right| dx ds \\ &= (\pi)^{-\frac{N}{2}} \frac{|y|}{\sqrt{t}} \int_{\mathbb{R}^N} |x e^{-x^2}| dx \\ &= (\pi)^{-\frac{N}{2}} \frac{|y|}{\sqrt{t}} |\partial B_1| \int_0^\infty r^N e^{-r^2} dr. \end{aligned}$$

It follows from the latter that

$$\begin{aligned} &\int_{\mathbb{R}^N} \left| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(x, t) \right| dx \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |G(x-y, t) - G(x, t)| f(y) dy dx \\ &\leq \int_{\mathbb{R}^N} f(y) dy \int_{\mathbb{R}^N} |G(x-y, t) - G(x, t)| dx \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{B_R} \frac{|y|}{\sqrt{t}} f(y) dy + \int_{\mathbb{R}^N \setminus B_R} f(y) dy \int_{\mathbb{R}^N} (|G(x-y, t)| + |G(x, t)|) dx \\
&\leq C \frac{R}{\sqrt{t}} \int_{\mathbb{R}^N} f(y) dy + 2 \int_{\mathbb{R}^N \setminus B_R} f(y) dy \\
&\leq C \frac{R}{\sqrt{t}} \int_{\mathbb{R}^N} f(y) dy + 2\varepsilon.
\end{aligned}$$

As a result of this, we have

$$\lim_{t \rightarrow \infty} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^1} = 0. \quad (2.50)$$

Therefore according to (2.49) and (2.50), (2.48) can be identified by the fact that

$$\left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^p} \leq \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^1}^{\frac{1}{p}} \left\| e^{t\Delta} f - \int_{\mathbb{R}^N} f(y) dy G(\cdot, t) \right\|_{L^\infty}^{1-\frac{1}{p}}.$$

□

**Lemma 2.14** *Assume that  $N = 2, 3$ , and  $(u, v)$  is the solution of (1.6) with  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  being bounded. Then*

$$\begin{aligned}
\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{p})} \left\| u(\cdot, t) - \int_{\mathbb{R}^N} u_0(y) dy G(\cdot, t) \right\|_{L^p} &= 0, \\
\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1-\frac{1}{p})} \left\| v(\cdot, t) - \int_{\mathbb{R}^N} v_0(y) dy G(\cdot, t) \right\|_{L^p} &= 0,
\end{aligned} \quad (2.51)$$

where  $p \in [1, \infty]$ .

**Proof.** According to

$$u(x, t) = e^{t\Delta} u_0 - \chi \int_0^t \nabla \cdot (e^{(t-s)\Delta} (u \nabla v)) ds,$$

for any  $t > 2$ , using (2.14) and (2.42)

$$\begin{aligned}
&\|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} \\
&\leq \chi \int_0^t \left\| \nabla \cdot (e^{(t-s)\Delta} (u \nabla v)) \right\|_{L^\infty} ds \\
&\leq C\chi \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{N}{2}} \|u(s)\|_{L^4} \|\nabla v(s)\|_{L^{\frac{4}{3}}} ds + C\chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} \|u(s)\|_{L^\infty} \|\nabla v(s)\|_{L^r} ds \\
&\leq C\chi \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}-\frac{N}{2}} (1+s)^{-\frac{3N}{8}} \|u_0\|_{L^1} ds + C\chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{-\frac{N}{2}} \|\nabla v(s)\|_{L^r} \|u_0\|_{L^1} ds \\
&\leq Ct^{-\frac{1}{2}-\frac{N}{2}} \left( (2+t)^{1-\frac{3N}{8}} + 1 \right) + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{N}{2r}} s^{-\frac{N}{2}} \|\nabla v(s)\|_{L^r} ds.
\end{aligned}$$

By Lemma 2.1, for  $r \in [4, \frac{2N}{N-2}]$ ,

$$\|\nabla v\|_{L^r} \leq C \|v\|_{L^\infty}^\alpha \|D^2 v\|_{L^2}^{1-\alpha}$$

with  $\alpha = \frac{\frac{1}{N} + \frac{1}{r} - \frac{1}{2}}{\frac{2}{N} - \frac{1}{2}} \leq \frac{1}{2}$ , which along with Lemma 2.11, implies that

$$\|\nabla v(s)\|_{L^r} \leq C \|v\|_{L^\infty}^\alpha \leq C s^{-\frac{N}{2}\alpha}$$

for  $s > 1$ . Hence one can conclude that for any  $t > 2$ ,

$$\begin{aligned} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} &\leq C t^{-\frac{1}{2} - \frac{N}{2}} \left( (2+t)^{1 - \frac{3N}{8}} + 1 \right) + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2} - \frac{N}{2r}} s^{-\frac{N}{2} - \frac{N}{2}\alpha} ds \\ &\leq C t^{-\frac{1}{2} - \frac{N}{2}} \left( (2+t)^{1 - \frac{3N}{8}} + 1 \right) + C t^{\frac{1}{2} - \frac{N}{2r} - \frac{N}{2} - \frac{N}{2}\alpha} \int_{\frac{1}{2}}^1 (1-s)^{-\frac{1}{2} - \frac{N}{2r}} s^{-\frac{N}{2} - \frac{N}{2}\alpha} ds, \end{aligned}$$

which means that

$$t^{\frac{N}{2}} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} \leq C t^{-\frac{1}{2}} \left( (2+t)^{1 - \frac{3N}{8}} + 1 \right) + C t^{\frac{1}{2} - \frac{N}{2r} - \frac{N}{2}\alpha}. \quad (2.52)$$

Furthermore, when fixing  $r = 5$ , we have  $\alpha = \frac{2}{5}$  for  $N = 2$  and  $\alpha = \frac{1}{5}$  for  $N = 3$ . The latter ensures that  $\frac{1}{2} - \frac{N}{2r} - \frac{N}{2}\alpha = -\frac{1}{10}$  for  $N = 2, 3$ . Therefore from (2.52), it thereby follows that

$$t^{\frac{N}{2}} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} \leq C t^{-\frac{1}{10}}, \quad \text{for } t > 2. \quad (2.53)$$

On the other hand, similar to the above proof, we use (2.14) and (2.42) again to see that

$$\begin{aligned} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^1} &\leq \chi \int_0^t \left\| \nabla \cdot \left( e^{(t-s)\Delta} (u \nabla v) \right) \right\|_{L^1} ds \\ &\leq C \chi \int_0^{\frac{t}{2}} (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^4} \|\nabla v(s)\|_{L^{\frac{4}{3}}} ds + C \chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} \|u(s)\|_{L^4} \|\nabla v(s)\|_{L^{\frac{4}{3}}} ds \\ &\leq C t^{-\frac{1}{2}} \left( (2+t)^{1 - \frac{3N}{8}} + 1 \right) + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}} s^{-\frac{3N}{8}} ds \\ &\leq C t^{-\frac{1}{2}} \left( (2+t)^{1 - \frac{3N}{8}} + 1 \right) + C t^{-\frac{3N-4}{8}} \end{aligned} \quad (2.54)$$

for any  $t > 2$ , which together with (2.53) yields

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1 - \frac{1}{p})} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^p} = 0 \quad (2.55)$$

for any  $1 \leq p \leq \infty$  since  $\|f\|_{L^p} \leq \|f\|_{L^\infty}^{1 - \frac{1}{p}} \|f\|_{L^1}^{\frac{1}{p}}$ . Combining (2.55) and (2.48), we can conclude that

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1 - \frac{1}{p})} \left\| u(\cdot, t) - \int_{\mathbb{R}^N} u_0(y) dy G(\cdot, t) \right\|_{L^p} = 0. \quad (2.56)$$

Similarly,

$$\lim_{t \rightarrow \infty} t^{\frac{N}{2}(1 - \frac{1}{p})} \left\| v(\cdot, t) - \int_{\mathbb{R}^N} v_0(y) dy G(\cdot, t) \right\|_{L^p} = 0 \quad (2.57)$$

for  $1 \leq p \leq \infty$ . This lemma is proved.  $\square$

Upon the decay results for the one-dimensional space asserted in Lemma 2.12, we have

**Lemma 2.15** *Let  $N = 1$ , and  $(u, v)$  be the strong solution of (1.6) such that  $\sup_{t>0} \|\nabla^2 u(t)\|_{L^2}$  and  $\sup_{t>0} \|\nabla^2 v(t)\|_{L^2}$  are bounded. Then for any  $0 < m < \frac{9}{32}$ , we have*

$$\lim_{t \rightarrow \infty} t^{m(1-\frac{1}{p})} \left\| u(\cdot, t) - \int_{\mathbb{R}^N} u_0(y) dy G(\cdot, t) \right\|_{L^p} = 0, \quad (2.58)$$

where  $p \in (1, \infty]$ .

*Proof.* Similar to the proof of Lemma 2.14, using (2.14) and (2.46), we arrive at

$$\begin{aligned} \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} &\leq \chi \int_0^t \left\| \nabla \cdot (e^{(t-s)\Delta} (u \nabla v)) \right\|_{L^\infty} ds \\ &\leq C \chi \int_0^{\frac{t}{2}} (t-s)^{-1} \|u(s)\|_{L^4} \|\nabla v(s)\|_{L^{\frac{4}{3}}} ds + C \chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{p}+\frac{p+2}{4p})} \|u(s)\|_{L^p} \|\nabla v(s)\|_{L^{\frac{4p}{p+2}}} ds \\ &\leq C \sup_{s>0} \|\nabla v(s)\|_{L^{\frac{4}{3}}} \int_0^{\frac{t}{2}} (t-s)^{-1} (1+s)^{-\frac{9}{32}} ds + C \chi \int_{\frac{t}{2}}^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{p}+\frac{p+2}{4p})} \|u(s)\|_{L^p} \|v(s)\|_{L^{\frac{1}{2}}} \|D^2 v(s)\|_{L^2}^{\frac{1}{2}} ds \\ &\leq C(1+t)^{-\frac{9}{32}} + C \int_{\frac{t}{2}}^t (t-s)^{-\frac{5p+6}{8p}} s^{-\frac{9}{16}(1-\frac{1}{p})} ds \\ &\leq C t^{-\frac{9}{32}} + C t^{1-\frac{5p+6}{8p}-\frac{9}{16}(1-\frac{1}{p})} \\ &\leq C t^{-\frac{9}{32}} + C t^{-\frac{3p+3}{16p}}, \end{aligned}$$

for  $t > 2$  and any  $p > 2$ . It is easy to see that  $\frac{3p+3}{16p}$  is decreasing on  $p$  and  $\frac{3p+3}{16p} \rightarrow \frac{9}{32}$  as  $p \searrow 2$ . Hence for any  $0 < m < \frac{9}{32}$ , we have

$$\lim_{t \rightarrow \infty} t^m \|u(\cdot, t) - e^{t\Delta} u_0\|_{L^\infty} = 0. \quad (2.59)$$

Combining (2.48) and (2.59), we arrive at

$$\lim_{t \rightarrow \infty} t^m \left\| u(\cdot, t) - \int_{\mathbb{R}} u_0(y) dy G(\cdot, t) \right\|_{L^\infty} = 0, \quad (2.60)$$

and hence (2.58) results from the inequality

$$\begin{aligned} &\left\| u(\cdot, t) - \int_{\mathbb{R}} u_0(y) dy G(\cdot, t) \right\|_{L^p} \\ &\leq \left\| u(\cdot, t) - \int_{\mathbb{R}} u_0(y) dy G(\cdot, t) \right\|_{L^1}^{\frac{1}{p}} \left\| u(\cdot, t) - \int_{\mathbb{R}} u_0(y) dy G(\cdot, t) \right\|_{L^\infty}^{1-\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}} \|u_0\|_{L^1}^{\frac{1}{p}} \left\| u(\cdot, t) - \int_{\mathbb{R}} u_0(y) dy G(\cdot, t) \right\|_{L^\infty}^{1-\frac{1}{p}}. \end{aligned}$$

□

In conclusion, Theorem 1.1 and Theorem 1.2 are the consequence of combining Lemma 2.11 with Lemma 2.14, and Lemma 2.14 with Lemma 2.15, respectively.

### 3 Existence of global strong solutions

In Section 2, we have investigated the long-time dynamic behavior of the global strong solution  $(u, v) \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))$  of (1.6). Hence this section concentrates on the existence of

such strong solutions to the Cauchy problem of (1.6). For this purpose, we consider the following approximation problem of (1.6)

$$\begin{cases} u_t + \varepsilon \Delta^2 u - \Delta u = -\chi \nabla \cdot (u \nabla v) & \text{in } Q, \\ v_t - \Delta v = \xi \nabla \cdot (v \nabla u) & \text{in } Q, \\ u(x, 0) = u_{\varepsilon 0}(x) \geq 0, v(x, 0) = v_{\varepsilon 0}(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \quad (3.1)$$

where  $Q = \mathbb{R}^N \times \mathbb{R}^+$ ,  $\varepsilon \in (0, 1)$ ,  $u_{\varepsilon 0}, v_{\varepsilon 0} \in C_0^\infty(\mathbb{R}^N)$  are the smooth approximations of  $u_0$  and  $v_0$  in  $H^2(\mathbb{R}^N)$ , respectively. Let

$$\mathcal{D} = L^\infty((0, T); H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N)),$$

$$\tilde{\mathcal{D}} = \{u \in L^\infty((0, T); H^3(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \cap L^2((0, T); H^5(\mathbb{R}^N)), u_t \in L^2(Q_T), xu \in L^\infty((0, T); L^2(\mathbb{R}^N))\}.$$

Define

$$\mathcal{F} : \mathcal{D} \times [0, 1] \rightarrow \mathcal{D}, \quad \mathcal{F}(\tilde{u}, \sigma) = u,$$

where  $u$  is the solution of

$$\begin{cases} u_t + \varepsilon \Delta^2 u - \Delta u = -\chi \nabla \cdot (u \nabla v) & \text{in } Q, \\ v_t - \Delta v = \xi \nabla \cdot (v \nabla \tilde{u}) & \text{in } Q, \\ u(x, 0) = \sigma u_{\varepsilon 0}(x) \geq 0, v(x, 0) = \sigma v_{\varepsilon 0}(x) \geq 0, & x \in \mathbb{R}^N. \end{cases} \quad (3.2)$$

According to Proposition A.1, Proposition A.2 and Lemma A.1 in Appendix, we see that for  $\tilde{u} \in \mathcal{D}$ , the problem (3.2) admits a unique solution  $u \in \tilde{\mathcal{D}}$ . Therefore,  $\mathcal{F}$  is well-defined. By Aubin-Lions Lemma and the compact results in [51] one also see that  $\mathcal{F}$  is a compact mapping. Next, it is easy to see that

$$\mathcal{F}(\tilde{u}, 0) = 0.$$

According to the Leray-Schauder fixed point theorem, to show the existence of strong solutions to the problem (3.1), it remains only to show that there exists constant  $C > 0$  independent of  $\sigma \in [0, 1]$  such that  $\|u\|_{\mathcal{D}} \leq C$  for any  $\mathcal{F}(u, \sigma) = u$ .

**Lemma 3.1** *Assume that  $\mathcal{F}(u, \sigma) = u$ . Then we have*

$$\begin{aligned} & \frac{d}{dt} \left( \|u(\cdot, t)\|_{H^2}^2 + \|v(\cdot, t)\|_{H^2}^2 \right) + 2\varepsilon \|\Delta u(\cdot, t)\|_{H^2}^2 + \|\nabla u(\cdot, t)\|_{H^2}^2 + \|\nabla v(\cdot, t)\|_{H^2}^2 \\ & \leq C^* (\chi^2 + \xi^2) \left( \|u(\cdot, t)\|_{H^2}^2 + \|v(\cdot, t)\|_{H^2}^2 \right) \left( \|\nabla u(\cdot, t)\|_{H^2}^2 + \|\nabla v(\cdot, t)\|_{H^2}^2 \right), \end{aligned} \quad (3.3)$$

where constant  $C^* > 1$  depends only on the spatial dimension  $N$ .

**Proof.** Multiplying the first equation of (3.2) by  $u$ , integrating the results over  $\mathbb{R}^N$  yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} u^2 dx + \varepsilon \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla u|^2 dx \\ & = \chi \int_{\mathbb{R}^N} u \nabla v \nabla u dx \\ & \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\chi^2}{2} \int_{\mathbb{R}^N} u^2 |\nabla v|^2 dx. \end{aligned}$$

Similarly testing the second equation of (3.2) by  $v$  and integrating the corresponding result over  $\mathbb{R}^N$  yield

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |v|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx \\ &= -\xi \int_{\mathbb{R}^N} v \nabla v \nabla u dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{\xi^2}{2} \int_{\mathbb{R}^N} v^2 |\nabla u|^2 dx. \end{aligned}$$

Combining the above two equalities, we have

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} (|u|^2 + |v|^2) dx + 2\varepsilon \int_{\mathbb{R}^N} |\Delta u|^2 dx + \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx \\ &\leq \chi^2 \|u\|_{L^\infty}^2 \|\nabla v\|_{L^2}^2 + \xi^2 \|v\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 \\ &\leq C_1 \chi^2 \|u\|_{H^2}^2 \|\nabla v\|_{L^2}^2 + C_1 \xi^2 \|v\|_{H^2}^2 \|\nabla u\|_{L^2}^2. \end{aligned} \quad (3.4)$$

Furthermore, multiplying the first equation of (3.2) by  $-\Delta u$ , and integrating the resulting equation over  $\mathbb{R}^N$  yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \varepsilon \int_{\mathbb{R}^N} |\nabla \Delta u|^2 dx + \int_{\mathbb{R}^N} |\Delta u|^2 dx \\ &= \chi \int_{\mathbb{R}^N} \nabla \cdot (u \nabla v) \Delta u dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\Delta u|^2 dx + \chi^2 \int_{\mathbb{R}^N} (|\nabla u|^2 |\nabla v|^2 + u^2 |\Delta v|^2) dx. \end{aligned} \quad (3.5)$$

Similarly, we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |\Delta v|^2 dx \\ &= -\xi \int_{\mathbb{R}^N} \nabla \cdot (v \nabla u) \Delta v dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |\Delta v|^2 dx + \xi^2 \int_{\mathbb{R}^N} (|\nabla u|^2 |\nabla v|^2 + v^2 |\Delta u|^2) dx. \end{aligned} \quad (3.6)$$

Combining (3.5) and (3.6), we arrive at

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx + 2\varepsilon \int_{\mathbb{R}^N} |\nabla \Delta u|^2 dx + \int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2) dx \\ &\leq 2\chi^2 \int_{\mathbb{R}^N} (|\nabla u|^2 |\nabla v|^2 + u^2 |\Delta v|^2) dx + 2\xi^2 \int_{\mathbb{R}^N} (|\nabla u|^2 |\nabla v|^2 + v^2 |\Delta u|^2) dx \\ &\leq 2\chi^2 (\|\nabla u\|_{L^4}^2 \|\nabla v\|_{L^4}^2 + \|u\|_{L^\infty}^2 \|\Delta v\|_{L^2}^2) + 2\xi^2 (\|\nabla u\|_{L^4}^2 \|\nabla v\|_{L^4}^2 + \|v\|_{L^\infty}^2 \|\Delta u\|_{L^2}^2) \\ &\leq C(\chi^2 + \xi^2) (\|\nabla u\|_{H^1}^2 + \|\nabla v\|_{H^1}^2) (\|u\|_{H^2}^2 + \|v\|_{H^2}^2). \end{aligned} \quad (3.7)$$

Moreover, applying  $\nabla$  to the equations in (3.2), and multiplying the resulting equations by  $-\nabla \Delta u$  and  $-\nabla \Delta v$ , respectively, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2) dx + \varepsilon \int_{\mathbb{R}^N} |\Delta^2 u|^2 dx + \int_{\mathbb{R}^N} |\nabla \Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla \Delta v|^2 dx$$

$$\begin{aligned}
&= \chi \int_{\mathbb{R}^N} \nabla(\nabla \cdot (u\nabla v)) \nabla \Delta u dx - \xi \int_{\mathbb{R}^N} \nabla(\nabla \cdot (v\nabla u)) \nabla \Delta v dx \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \Delta u|^2 + |\nabla \Delta v|^2) dx + C\chi^2 \int_{\mathbb{R}^N} (|\nabla^2 v|^2 |\nabla u|^2 + |\nabla v|^2 |\nabla^2 u|^2 + u^2 |\nabla \Delta v|^2) dx \\
&\quad + C\xi^2 \int_{\mathbb{R}^N} (|\nabla^2 v|^2 |\nabla u|^2 + |\nabla v|^2 |\nabla^2 u|^2 + v^2 |\nabla \Delta u|^2) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \Delta u|^2 + |\nabla \Delta v|^2) dx \\
&\quad + C(\xi^2 + \chi^2) \left( \|\nabla u\|_{L^6}^2 \|\nabla^2 v\|_{L^3}^2 + \|\nabla v\|_{L^6}^2 \|\nabla^2 u\|_{L^3}^2 + \|u\|_{L^\infty}^2 \|\nabla \Delta v\|_{L^2}^2 + \|v\|_{L^\infty}^2 \|\nabla \Delta u\|_{L^2}^2 \right) \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla \Delta u|^2 + |\nabla \Delta v|^2) dx + C(\xi^2 + \chi^2) \left( \|u\|_{H^2}^2 + \|v\|_{H^2}^2 \right) \left( \|\nabla^2 u\|_{H^1}^2 + \|\nabla^2 v\|_{H^1}^2 \right).
\end{aligned}$$

The latter inequality implies that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}^N} (|\Delta u|^2 + |\Delta v|^2) dx + 2\varepsilon \int_{\mathbb{R}^N} |\Delta^2 u|^2 dx + \int_{\mathbb{R}^N} |\nabla \Delta u|^2 dx + \int_{\mathbb{R}^N} |\nabla \Delta v|^2 dx \\
&\leq C(\xi^2 + \chi^2) \left( \|u\|_{H^2}^2 + \|v\|_{H^2}^2 \right) \left( \|\nabla^2 u\|_{H^1}^2 + \|\nabla^2 v\|_{H^1}^2 \right). \tag{3.8}
\end{aligned}$$

Now combining the inequalities (3.4), (3.7) and (3.8), we then arrive at the desired inequality (3.3) readily.  $\square$

**Lemma 3.2** *Assume that  $\mathcal{F}(u, \sigma) = u$ . Let*

$$\phi(t) = \|u(\cdot, t)\|_{H^2}^2 + \|v(\cdot, t)\|_{H^2}^2,$$

$$h(t) = \|\nabla u(\cdot, t)\|_{H^2}^2 + \|\nabla v(\cdot, t)\|_{H^2}^2.$$

Then if  $\phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)}$  with  $C^*$  given in Lemma 3.1, we have

$$\phi(t) \leq \phi(0),$$

$$2\varepsilon \int_0^\infty \|\Delta u(\cdot, t)\|_{H^2}^2 dt + \frac{1}{2} \int_0^\infty h(t) dt \leq \phi(0).$$

**Proof.** From Lemma 3.1, it follows that

$$\phi'(t) + 2\varepsilon \|\Delta u(\cdot, t)\|_{H^2}^2 + \left(1 - C^*(\chi^2 + \xi^2)\phi(t)\right)h(t) \leq 0.$$

It is observed that whenever

$$\phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)},$$

we have

$$\phi(t) \leq \phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)} \quad \text{for all } t > 0.$$

and then derive

$$2\varepsilon \int_0^\infty \|\Delta u(\cdot, t)\|_{H^2}^2 dt + \frac{1}{2} \int_0^\infty h(t) dt \leq \phi(0),$$

and thereby complete the proof of this lemma.  $\square$

**Lemma 3.3** Suppose that  $\mathcal{F}(u, \sigma) = u$  and  $\phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)}$ . Then we have

$$\int_0^\infty \int_\Omega (|u_t|^2 + |v_t|^2) dx dt \leq C,$$

where  $C$  is independent of  $\varepsilon$ , which depends only on  $\Omega$ ,  $u_0$ ,  $v_0$ ,  $\chi$ , and  $\xi$ .

**Proof.** Multiplying the first equation of (3.2) by  $u_t$ , integrating it over  $\Omega$  then yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (\varepsilon |\Delta u|^2 + |\nabla u|^2) dx + \int_{\mathbb{R}^N} |u_t|^2 dx \\ &= -\chi \int_{\mathbb{R}^N} \nabla \cdot (u \nabla v) u_t dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |u_t|^2 dx + C\chi^2 \|u\|_{H^2}^2 \|\nabla v\|_{H^1}^2. \end{aligned}$$

Using Lemma 3.2 and by a direct integration, we obtain that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}^N} |u_t|^2 dx dt &\leq 2C\chi^2 \sup_{t>0} \|u(\cdot, t)\|_{H^2}^2 \int_0^\infty \|\nabla v\|_{H^1}^2 dt + \int_{\mathbb{R}^N} (\varepsilon |\Delta u_0|^2 + |\nabla u_0|^2) dx \\ &\leq \tilde{C}. \end{aligned}$$

Multiplying the second equation of (3.2) by  $v_t$ , integrating the result over  $\Omega$  then yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \int_{\mathbb{R}^N} |v_t|^2 dx \\ &= \xi \int_{\mathbb{R}^N} \nabla \cdot (v \nabla u) v_t dx \\ &\leq \frac{1}{2} \int_{\mathbb{R}^N} |v_t|^2 dx + C\xi^2 \|v\|_{H^2}^2 \|\nabla u\|_{H^1}^2. \end{aligned}$$

From Lemma 3.2 we infer that

$$\int_0^\infty \int_{\mathbb{R}^N} |v_t|^2 dx dt \leq 2C\xi^2 \sup_{t>0} \|v(\cdot, t)\|_{H^2}^2 \int_0^\infty \|\nabla u\|_{H^1}^2 dt + \int_{\mathbb{R}^N} |\nabla v_0|^2 dx \leq \hat{C},$$

and thereby completes the proof.  $\square$

Furthermore by Lemma A.1, we can also conclude that

**Lemma 3.4** Assume that  $\mathcal{F}(u, \sigma) = u$ ,  $\phi(0) \leq \frac{1}{2C^*(\chi^2 + \xi^2)}$  with  $C^*$  given as in Lemma 3.1. If  $|x|^r u_{\varepsilon 0}(x) \in L^2(\mathbb{R}^N)$  for some a positive constant  $r \leq 1$ , then  $|x|^r u \in L_{loc}^\infty((0, \infty); L^2(\mathbb{R}^N))$ .

This lemma allows us to employ the compactness result in [51] to demonstrate the existence of strong solutions to system (1.6).

**Proof of Theorem 1.3.** By the Leray-Schauder fixed point theorem, and Lemmas 3.2–3.4, the problem (3.1) possesses the strong solution  $(u_\varepsilon, v_\varepsilon)$  for all  $\varepsilon \in [0, 1]$ , such that  $u_\varepsilon \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N)) \cap L^2(\mathbb{R}^+; H^4(\mathbb{R}^N))$ ,  $\partial_t u_\varepsilon \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$ ,  $v_\varepsilon \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N)) \cap L^2(\mathbb{R}^+; H^3(\mathbb{R}^N))$ ,  $\partial_t v_\varepsilon \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$ . Rechecking the proof of Lemma 3.2, and noticing that

$$\|\varepsilon \Delta^2 u_\varepsilon\|_{L^2(Q)}^2 \leq C\varepsilon,$$

$$\sup_{0 < t < T} \int_{\mathbb{R}^N} (1 + |x|^2)^r u^2 dx \leq C_T,$$

letting  $\varepsilon \rightarrow 0$  (taking its subsequence if necessary), and using the compact results in [51], for any  $T > 0$ , we obtain

$$\begin{aligned} \varepsilon \Delta^2 u_\varepsilon &\rightarrow 0 \text{ in } L^2((0, T); H^2(\mathbb{R}^N)); \\ (u_\varepsilon, v_\varepsilon) &\rightarrow (u, v) \text{ in } L^2((0, T); H^3(\mathbb{R}^N)); \\ (\partial_t u_\varepsilon, \partial_t v_\varepsilon) &\rightarrow (\partial_t u, \partial_t v) \text{ in } L^2((0, T); L^2(\mathbb{R}^N)); \\ (u_\varepsilon, v_\varepsilon) &\overset{*}{\rightharpoonup} (u, v) \text{ in } L^\infty((0, T); H^2(\mathbb{R}^N)); \\ (u_\varepsilon, v_\varepsilon) &\rightarrow (u, v) \text{ in } L^2((0, T); H^2(\mathbb{R}^N)); \\ (u_\varepsilon, v_\varepsilon) &\rightarrow (u, v) \text{ uniformly in } Q_T. \end{aligned}$$

Utilizing the respective estimations derived in Lemma 3.2–Lemma 3.3, the limit functions  $(u, v)$  with  $u, v \in L^\infty(\mathbb{R}^+; H^2(\mathbb{R}^N)) \cap L^2(\mathbb{R}^+; H^3(\mathbb{R}^N))$ ,  $\partial_t u, \partial_t v \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$  can be identified to be the strong solution of (1.6). In addition, noticing that  $u_0, v_0 \geq 0$ , then by strong maximum principle,  $u(x, t) > 0$ ,  $v(x, t) > 0$  for  $t > 0, x \in \mathbb{R}^N$ .

Next, we show the uniqueness of the strong solution of (1.6). Indeed, supposed that  $(u_1, v_1), (u_2, v_2)$  are the global strong solutions thereof. Denoting  $\hat{u} = u_1 - u_2, \hat{v} = v_1 - v_2$ , then by a direct calculation, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\hat{u}|^2 dx + \int_{\mathbb{R}^N} |\nabla \hat{u}|^2 dx &= \chi \int_0^t \int_{\mathbb{R}^N} (\hat{u} \nabla v_1 \nabla \hat{u} + u_2 \nabla \hat{v} \nabla \hat{u}) dx, \\ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} |\hat{v}|^2 dx + \int_{\mathbb{R}^N} |\nabla \hat{v}|^2 dx &= -\xi \int_0^t \int_{\mathbb{R}^N} (\hat{v} \nabla u_1 \nabla \hat{v} + v_2 \nabla \hat{u} \nabla \hat{v}) dx. \end{aligned}$$

Combining the above two equalities, we further have

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (|\hat{u}|^2 + |\hat{v}|^2) dx + \int_{\mathbb{R}^N} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) dx \\ &= \chi \int_{\mathbb{R}^N} (\hat{u} \nabla v_1 \nabla \hat{u} + u_2 \nabla \hat{v} \nabla \hat{u}) dx - \xi \int_0^t \int_{\mathbb{R}^N} (\hat{v} \nabla u_1 \nabla \hat{v} + v_2 \nabla \hat{u} \nabla \hat{v}) dx \\ &\leq \chi \|\hat{u}\|_{L^3} \|\nabla v_1\|_{L^6} \|\nabla \hat{u}\|_{L^2} + \xi \|\hat{v}\|_{L^3} \|\nabla u_1\|_{L^6} \|\nabla \hat{v}\|_{L^2} + \chi \|u_2\|_{H^2} \|\nabla \hat{v}\|_{L^2} \|\nabla \hat{u}\|_{L^2} + \xi \|v_2\|_{H^2} \|\nabla \hat{u}\|_{L^2} \|\nabla \hat{v}\|_{L^2} \\ &\leq C \chi \|\hat{u}\|_{L^2}^{\frac{6-N}{6}} \|\nabla \hat{u}\|_{L^2}^{\frac{N}{6}} \|\nabla v_1\|_{H^1} \|\nabla \hat{u}\|_{L^2} + \xi \|\hat{v}\|_{L^2}^{\frac{6-N}{6}} \|\nabla \hat{v}\|_{L^2}^{\frac{N}{6}} \|\nabla u_1\|_{H^1} \|\nabla \hat{v}\|_{L^2} \\ &\quad + \chi \|u_2\|_{H^2} \|\nabla \hat{v}\|_{L^2} \|\nabla \hat{u}\|_{L^2} + \xi \|v_2\|_{H^2} \|\nabla \hat{u}\|_{L^2} \|\nabla \hat{v}\|_{L^2} \\ &\leq \frac{1}{2} \left( \|\nabla \hat{u}\|_{L^2}^2 + \|\nabla \hat{v}\|_{L^2}^2 \right) + C \left( \|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2 \right) + \chi^2 \|u_2\|_{H^2}^2 \|\nabla \hat{v}\|_{L^2}^2 + \xi^2 \|v_2\|_{H^2}^2 \|\nabla \hat{u}\|_{L^2}^2, \end{aligned}$$

which implies that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left( \|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2 \right) - C \left( \|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2 \right) \\ &\quad + \left( \frac{1}{2} - (\chi^2 + \xi^2) (\|u_2\|_{H^2}^2 + \|v_2\|_{H^2}^2) \right) \int_{\mathbb{R}^N} (|\nabla \hat{u}|^2 + |\nabla \hat{v}|^2) dx \leq 0. \end{aligned}$$

It is observed that  $\Phi(t) \leq \frac{1}{2C^*(\chi^2 + \xi^2)}$  with  $C^* > 1$ , and hence

$$\frac{1}{2} \frac{d}{dt} \left( \|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2 \right) - C \left( \|\hat{u}\|_{L^2}^2 + \|\hat{v}\|_{L^2}^2 \right) \leq 0.$$

Therefore,

$$\|\hat{u}(\cdot, t)\|_{L^2}^2 + \|\hat{v}(\cdot, t)\|_{L^2}^2 \equiv 0,$$

since  $\|\hat{u}(\cdot, 0)\|_{L^2}^2 + \|\hat{v}(\cdot, 0)\|_{L^2}^2 = 0$ , which thereby completes the proof of the uniqueness of strong solutions.

At last, we pay our attention to the estimate of decay rate claimed in Theorem 1.3. From (3.3), we see that

$$\frac{d}{dt}\phi(t) + \frac{1}{2}h(t) \leq 0.$$

On the other hand, noting that  $u, v > 0$  and

$$\int_{\mathbb{R}^N} u(x, t) dx \equiv \int_{\mathbb{R}^N} u_0(x) dx = \bar{u}_0, \quad \int_{\mathbb{R}^N} v(x, t) dx \equiv \int_{\mathbb{R}^N} v_0(x) dx = \bar{v}_0,$$

and by the Gagliardo-Nirenberg inequality, we infer that

$$\|u\|_{L^2}^2 \leq C|\bar{u}_0|^{\frac{4}{N+2}} \|\nabla u\|_{L^2}^{\frac{2N}{N+2}}, \quad \|v\|_{L^2}^2 \leq C|\bar{v}_0|^{\frac{4}{N+2}} \|\nabla v\|_{L^2}^{\frac{2N}{N+2}}.$$

Therefore, we have

$$\phi'(t) + A^* \phi(t)^{\frac{N+2}{N}} \leq 0,$$

for some constant  $A^*$ , and then

$$\phi(t) \leq \left( \Phi^{-\frac{2}{N}}(0) + \frac{2}{N} A^* t \right)^{-\frac{N}{2}},$$

which thus completes the proof of Theorem 1.3 readily.  $\square$

## A Appendix

Consider

$$\begin{cases} v_t - \Delta v = \xi \nabla \cdot (v \nabla \tilde{u}) & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ v(x, 0) = \sigma v_0(x) \geq 0, & x \in \mathbb{R}^N, \end{cases} \quad (\text{A.1})$$

where  $\sigma \in [0, 1]$ ,  $v_0(x) \in H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$  with  $N \leq 3$ .

**Proposition A.1** *Assume that  $\tilde{u} \in L^\infty((0, T); H^2(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N))$ . Then (A.1) admits a strong solution  $v$  with  $v \geq 0$ ,  $v \in L^\infty((0, T); H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N))$ , and  $v_t \in L^2((0, T); L^2(\mathbb{R}^N))$ .*

**Proof.** To prove this lemma, we use bounded domains  $B_{R_n}$  to approximate  $\mathbb{R}^N$ , where  $B_{R_n}$  denotes the ball centered at 0 with radius  $R_n > 1$ , and  $R_n \rightarrow \infty$  as  $n \rightarrow \infty$ . In what follows, we denote

$$Q_n := B_{R_n} \times (0, \infty); \quad Q_{n,T} := B_{R_n} \times (0, T);$$

$\mathring{C}^\infty(Q_{n,T})$  denotes the set of functions that are infinitely differentiable and vanish near the lateral boundary of  $Q_{n,T}$ .

Let  $\{v_{n0}\}_{n=1}^\infty$  with  $0 \leq v_{n0} \in C_0^\infty(B_{R_n})$  be the smooth approximation of  $v_0$ ,  $\{\tilde{u}_n\}_{n=1}^\infty$  with  $\tilde{u}_n \in \mathring{C}^\infty(Q_{n,T})$  be the smooth approximation of  $\tilde{u}$ , such that

$$\|v_{n,0}\|_{H^2(B_{R_n})} + \|v_{n,0}\|_{L^1(B_{R_n})} \leq C, \quad (\text{A.2})$$

$$\sup_{0 < t < T} \|\tilde{u}_n(\cdot, t)\|_{H^2(B_{R_n})}^2 + \int_0^T \|\tilde{u}_n(\cdot, t)\|_{H^3(B_{R_n})}^2 dt \leq C, \quad (\text{A.3})$$

with  $C$  is independent of  $n$ . By classical theory of linear parabolic equations, the following problem

$$\begin{cases} v_t - \Delta v = \xi \nabla \cdot (v \nabla \tilde{u}_n), & x \in B_{R_n}, t > 0 \\ \frac{\partial v}{\partial \nu} \Big|_{\partial B_{R_n}} = 0 \\ v(x, 0) = \sigma v_{n0}(x), & x \in B_{R_n} \end{cases} \quad (\text{A.4})$$

admits a unique nonnegative classical solution  $v_n \in C^\infty(\overline{Q}_{n,T})$ . Therefore, to show the existence of solutions claimed in Proposition A.1, we concentrate on derivation of the following a priori estimates on  $v_n$

$$\sup_{0 < t < T} \left( \|v_n(t)\|_{L^1(B_{R_n})} + \|v_n(t)\|_{H^2(B_{R_n})}^2 \right) + \int_0^T \|v_n(t)\|_{H^3(B_{R_n})}^2 dt \leq C$$

with  $C$  independent of  $n$ .

Firstly, it is easy to see that  $v_n$  follows the law of conservation of mass, that is

$$\|v_n(\cdot, t)\|_{L^1(B_{R_n})} \equiv \|v_{n,0}\|_{L^1(B_{R_n})}.$$

Noticing that

$$\partial_t v_n - \Delta v_n + v_n = \xi \nabla \cdot (v_n \nabla \tilde{u}_n) + v_n,$$

by Neumann heat semigroup theory, we obtain that

$$\begin{aligned} & \|v_n(\cdot, t)\|_{L^\infty(B_{R_n})} \leq e^{-t} \|v_{n,0}\|_{L^\infty(B_{R_n})} \\ & + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{8}-\frac{1}{2}} \|v_n \nabla \tilde{u}_n\|_{L^4(B_{R_n})} ds + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{4}} \|v_n\|_{L^2(B_{R_n})} ds \\ & \leq e^{-t} \|v_{n,0}\|_{L^\infty(B_{R_n})} + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{8}-\frac{1}{2}} \|v_n\|_{L^{12}(B_{R_n})} \|\nabla \tilde{u}_n\|_{L^6(B_{R_n})} ds + \int_0^t e^{-(t-s)} (t-s)^{-\frac{N}{4}} \|v_n\|_{L^2(B_{R_n})} ds \\ & \leq e^{-t} \|v_{n,0}\|_{L^\infty(B_{R_n})} + \sup_{t>0} \{ \|v_n(\cdot, t)\|_{L^{12}(B_{R_n})} \|\tilde{u}_n(\cdot, t)\|_{H^2(B_{R_n})} \} \int_0^t e^{-s} s^{-\frac{N}{8}-\frac{1}{2}} ds \\ & + \sup_{t>0} \|v_n(\cdot, t)\|_{L^2(B_{R_n})} \int_0^t e^{-s} s^{-\frac{N}{4}} ds \\ & \leq e^{-t} \|v_{n,0}\|_{L^\infty(B_{R_n})} + C \sup_{t>0} \|v_n(\cdot, t)\|_{L^\infty(B_{R_n})}^{\frac{11}{12}} \|v_n(\cdot, t)\|_{L^1(B_{R_n})}^{\frac{1}{12}} + \sup_{t>0} \|v_n(\cdot, t)\|_{L^\infty(B_{R_n})}^{\frac{1}{2}} \|v_n(\cdot, t)\|_{L^1(B_{R_n})}^{\frac{1}{2}}. \end{aligned}$$

Due to  $H^2 \hookrightarrow L^\infty$  for  $N < 4$ ,  $\|v_{n,0}\|_{L^\infty}$  is bounded by (A.2). Hence from the above inequality, we infer that

$$\sup_{t>0} \|v_n(\cdot, t)\|_{L^\infty(B_{R_n})} \leq C, \quad (\text{A.5})$$

where  $C$  is independent of  $n$ , and thereby

$$\sup_{t>0} \|v_n(\cdot, t)\|_{L^r(B_{R_n})} \leq C, \quad \text{for any } 1 \leq r \leq \infty \quad (\text{A.6})$$

since  $v_n(\cdot, t) \in L^1(B_{R_n})$ , and  $C$  is independent of  $n$ .

Recalling Lemma 2.2, we see that for any  $u \in W^{m,r}(B_R)$ ,

$$\|D^j u\|_{L^p(B_R)} \leq C_1 \|D^m u\|_{L^r(B_R)}^\alpha \|u\|_{L^q(B_R)}^{1-\alpha} + C_2 R^{\frac{N}{p} - \frac{N}{s} - j} \|u\|_{L^s(B_R)}.$$

From the above inequality, we observe that as long as  $\frac{N}{p} < \frac{N}{s} + j$ , we can replace  $R$  with 1 in this inequality when  $R > 1$ . **This ensures that the embedding constants for the Gagliardo-Nirenberg inequality on all bounded domains do not depend on  $R_n$  since  $R_n > 1$ .**

Note that

$$\|\nabla \tilde{u}_n\|_{L^6(B_{R_n})}^2 \leq C \|\nabla \tilde{u}_n\|_{L^2(B_{R_n})}^{\frac{6-2N}{3}} \|\Delta \tilde{u}_n\|_{L^2(B_{R_n})}^{\frac{2N}{3}} + CR_n^{-\frac{2N}{3}} \|\nabla \tilde{u}_n\|_{L^2(B_{R_n})}^2 \leq C \|\nabla \tilde{u}_n\|_{H^1(B_{R_n})}^2. \quad (\text{A.7})$$

Multiplying the equation in (A.1) by  $-\Delta v_n$ , and using (A.7) and Lemma 2.2, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |\nabla v_n|^2 dx + \int_{B_{R_n}} |\Delta v_n|^2 dx = -\xi \int_{B_{R_n}} \Delta v_n \nabla \cdot (v_n \nabla \tilde{u}_n) dx \\ & \leq \frac{1}{4} \int_{B_{R_n}} |\Delta v_n|^2 dx + 2\xi^2 \int_{B_{R_n}} (v_n^2 |\Delta \tilde{u}_n|^2 + |\nabla v_n \nabla \tilde{u}_n|^2) dx \\ & \leq \frac{1}{4} \int_{B_{R_n}} |\Delta v_n|^2 dx + 2\xi^2 \|v_n\|_{L^\infty(B_{R_n})}^2 \|\Delta \tilde{u}_n\|_{L^2(B_{R_n})}^2 + 2\xi^2 \|\nabla v_n\|_{L^3(B_{R_n})}^2 \|\nabla \tilde{u}_n\|_{L^6(B_{R_n})}^2 \\ & \leq \frac{1}{4} \int_{B_{R_n}} |\Delta v_n|^2 dx + 2\xi^2 \|v_n\|_{L^\infty(B_{R_n})}^2 \|\Delta \tilde{u}_n\|_{L^2(B_{R_n})}^2 \\ & + C\xi^2 \left( \|\Delta v_n\|_{L^2(B_{R_n})}^{\frac{N}{3}} \|v_n\|_{L^2(B_{R_n})}^{2-\frac{N}{3}} + \|v_n\|_{L^2(B_{R_n})}^2 \right) \|\nabla \tilde{u}_n\|_{H^1(B_{R_n})}^2 \\ & \leq \frac{1}{2} \int_{B_{R_n}} |\Delta v_n|^2 dx + \tilde{C}, \end{aligned}$$

which implies that

$$\sup_{0 < t < T} \int_{B_{R_n}} |\nabla v_n|^2 dx + \int_0^T \int_{B_{R_n}} |\Delta v_n|^2 dx dt \leq C \quad (\text{A.8})$$

with some constant  $C$  independent of  $n$ .

In addition, testing the equation of (A.1) by  $\partial_t v_n$ , similar to the proof of (A.8), we also have

$$\int_0^T \int_{B_{R_n}} |\partial_t v_n|^2 dx dt \leq C. \quad (\text{A.9})$$

By Lemma 2.2, we obtain the following inequalities,

$$\|\nabla^2 v_n\|_{L^3}^2 \leq C \|v_n\|_{L^\infty}^{\frac{2}{3}} \|\nabla \Delta v_n\|_{L^2}^{\frac{4}{3}} + CR_n^{\frac{N}{3}-2} \|v_n\|_{L^\infty}^2 \leq \tilde{C} \left( \|\nabla \Delta v_n\|_{L^2}^{\frac{4}{3}} + 1 \right). \quad (\text{A.10})$$

Applying  $\nabla$  to the equation of (A.1), and multiplying the resultant equation by  $-\nabla \Delta v$ , we use (A.5), (A.7) and (A.10) to get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |\Delta v_n|^2 dx + \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx = -\xi \int_{B_{R_n}} \nabla (\nabla \cdot (v_n \nabla \tilde{u}_n)) \nabla \Delta v_n dx \\ & \leq \frac{1}{4} \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx + C_1 \int_{B_{R_n}} (|\nabla^2 v_n|^2 |\nabla \tilde{u}_n|^2 + |\nabla v_n|^2 |\nabla^2 \tilde{u}_n|^2 + v_n^2 |\nabla \Delta \tilde{u}_n|^2) dx \\ & \leq \frac{1}{4} \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx + C_1 \left( \|\nabla \tilde{u}_n\|_{L^6(B_{R_n})}^2 \|\nabla^2 v_n\|_{L^3(B_{R_n})}^2 + \|\nabla^2 \tilde{u}_n\|_{L^6(B_{R_n})}^2 \|\nabla v_n\|_{L^3(B_{R_n})}^2 \right) \end{aligned}$$

$$\begin{aligned}
& + \|v_n\|_{L^\infty(B_{R_n})}^2 \|\nabla \Delta \tilde{u}_n\|_{L^2(B_{R_n})}^2) \\
& \leq \frac{1}{4} \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx + C_2 \left( \|\nabla \tilde{u}_n\|_{H^1(B_{R_n})}^2 (\|\nabla \Delta v_n\|_{L^2(B_{R_n})}^{\frac{4}{3}} + 1) \right. \\
& \quad \left. + \|\nabla^2 \tilde{u}_n\|_{H^1(B_{R_n})}^2 (\|\nabla v_n\|_{L^2(B_{R_n})}^2 + \|\Delta v_n\|_{L^2(B_{R_n})}^2) + \|\nabla \Delta \tilde{u}_n\|_{L^2(B_{R_n})}^2 \right) \\
& \leq \frac{1}{2} \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx + C_3 \left( \|\nabla^2 \tilde{u}_n\|_{H^1(B_{R_n})}^2 + \|\nabla^2 \tilde{u}_n\|_{H^1(B_{R_n})}^2 \|\Delta v_n\|_{L^2(B_{R_n})}^2 + 1 \right),
\end{aligned}$$

which implies that

$$\sup_{0 < t < T} \int_{B_{R_n}} |\Delta v_n|^2 dx + \int_0^T \int_{B_{R_n}} |\nabla \Delta v_n|^2 dx dt \leq C, \quad (\text{A.11})$$

where  $C > 0$  is independent of  $n$ . According to (A.5), (A.8), (A.9) and (A.11) and by the Cantor diagonal method, one can conclude that there exists a subsequence of  $\{v_n\}_{n=1}^\infty$ , denoted by itself for simplicity, and a nonnegative function  $v \in L^\infty((0, T); H^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N))$ , and  $v_t \in L^2((0, T); L^2(\mathbb{R}^N))$  such that

$$\begin{aligned}
v_n & \xrightarrow{*} v \quad \text{in } L^\infty((0, T); H_{loc}^2(\mathbb{R}^N) \cap L_{loc}^1(\mathbb{R}^N)); \\
v_n & \rightarrow v \quad \text{in } L^2((0, T); H_{loc}^3(\mathbb{R}^N)); \\
\partial_t v_n & \rightarrow v_t \quad \text{in } L^2((0, T); L_{loc}^2(\mathbb{R}^N)); \\
v_n & \rightarrow v \quad \text{in } L^2((0, T); H_{loc}^2(\mathbb{R}^N)),
\end{aligned}$$

and  $v$  is exactly a strong solution of (A.1). □

Now for the function  $v(x, t)$  obtained in Proposition A.1, we consider

$$\begin{cases} u_t + \varepsilon \Delta^2 u - \Delta u = -\chi \nabla \cdot (u \nabla v) & \text{in } \mathbb{R}^N \times \mathbb{R}^+, \\ u(x, 0) = \sigma u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (\text{A.12})$$

with  $xu_0(x) \in L^2(\mathbb{R}^N)$ ,  $u_0(x) \in H^3(\mathbb{R}^N)$ .

**Proposition A.2** *Assume that  $v \in L^\infty((0, T); H^2(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N))$ . Then (A.12) admits a strong solution  $u \in L^\infty((0, T); H^3(\mathbb{R}^N)) \cap L^2((0, T); H^5(\mathbb{R}^N))$  and  $u_t \in L^2(Q_T)$ .*

From Lemma 2.2 and the proof of Proposition A.2, we can observe that the embedding constants in either the Gagliardo-Nirenberg inequality or the Sobolev inequality (see for example (A.7), (A.10) etc.) can be selected in a manner that is independent of  $R_n$ . Consequently, the subsequent verification of this proposition adheres to this principle. For simplicity, in the subsequent proofs, we will not provide detailed explanations for each inequality as we did in the previous proposition.

**Proof.** As the proof in Proposition A.1, in order to obtain the a priori estimates for solutions of approximate system of (A.12), we consider the initial boundary problem of the form

$$\begin{cases} u_t + \varepsilon \Delta^2 u - \Delta u = -\chi \nabla \cdot (u \nabla \tilde{v}_n), & x \in B_{R_n}, t > 0, \\ \frac{\partial u}{\partial \nu} \Big|_{\partial B_{R_n}} = 0, \quad \frac{\partial \Delta u}{\partial \nu} \Big|_{\partial B_{R_n}} = 0, \\ u(x, 0) = \sigma u_{n0}(x), & x \in B_{R_n}, \end{cases} \quad (\text{A.13})$$

where  $\{u_{n0}\}_{n=1}^\infty$  with  $u_{n0} \in C_0^\infty(B_{R_n})$  are the smooth approximation of  $u_0$ ,  $\{\tilde{v}_n\}_{n=1}^\infty$  with  $\tilde{v}_n \in \dot{C}^\infty(Q_{n,T})$  are the smooth approximation of  $v$ , such that

$$\|u_{n,0}\|_{H^3(B_{R_n})} \leq C,$$

$$\sup_{0 < t < T} \|\tilde{v}_n(\cdot, t)\|_{H^2(B_{R_n})}^2 + \int_0^T \|\tilde{v}_n(\cdot, t)\|_{H^3(B_{R_n})}^2 dt \leq C,$$

with  $C$  independent of  $n$ . By classical theory of linear parabolic equations, the problem (A.13) admits a unique classical solution  $u_n \in C^\infty(\overline{Q_{n,T}})$ . Therefore, to show Proposition A.2, we only need to derive the following a priori estimate

$$\sup_{0 < t < T} (\|u_n(t)\|_{H^3(B_{R_n})}^2) + \int_0^T (\|u_n(t)\|_{H^5(B_{R_n})}^2 + \|\partial_t u_n(t)\|_{L^2(B_{R_n})}^2) dt \leq C,$$

with  $C$  independent of  $n$ .

Multiplying the equation of (A.13) by  $u_n$ , then integrating the resultant equation over  $B_{R_n}$ , and using Sobolev embedding inequality, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |u_n|^2 dx + \varepsilon \int_{B_{R_n}} |\Delta u_n|^2 dx + \int_{B_{R_n}} |\nabla u_n|^2 dx \\ &= -\frac{\chi}{2} \int_{B_{R_n}} u_n^2 \Delta \tilde{v}_n \\ &\leq \frac{\chi}{2} \|\Delta \tilde{v}_n\|_{L^2(B_{R_n})} \|u_n\|_{L^4(B_{R_n})}^2 \\ &\leq \frac{1}{2} \int_{B_{R_n}} |\nabla u_n|^2 dx + C \int_{B_{R_n}} |u_n|^2 dx, \end{aligned}$$

which implies that

$$\sup_{0 < t < T} \|u_n(\cdot, t)\|_{L^2(B_{R_n})}^2 + \int_0^T \int_{B_{R_n}} (\varepsilon |\Delta u_n|^2 + |\nabla u_n|^2) dx dt \leq C, \quad (\text{A.14})$$

where  $C$  is independent of  $n$ .

Testing (A.13) by  $-\Delta u_n$ , and integrating the resultant equation over  $B_{R_n}$ , then the Sobolev embedding inequality gives us

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |\nabla u_n|^2 dx + \varepsilon \int_{B_{R_n}} |\nabla \Delta u_n|^2 dx + \int_{B_{R_n}} |\Delta u_n|^2 dx \\ &= -\chi \int_{B_{R_n}} u_n \nabla \tilde{v}_n \nabla \Delta u_n dx \\ &\leq \frac{\varepsilon}{2} \int_{B_{R_n}} |\nabla \Delta u_n|^2 dx + \frac{\chi^2}{2\varepsilon} \|u_n\|_{L^3}^2 \|\nabla \tilde{v}_n\|_{L^6}^2 \\ &\leq \frac{\varepsilon}{2} \int_{B_{R_n}} |\nabla \Delta u_n|^2 dx + \frac{\chi^2}{2\varepsilon} (C \|u_n\|_{L^2(B_{R_n})}^2 + C \|\nabla u_n\|_{L^2(B_{R_n})}^2) \|\nabla \tilde{v}_n\|_{H^1(B_{R_n})}^2 \\ &\leq \frac{\varepsilon}{2} \int_{B_{R_n}} |\nabla \Delta u_n|^2 dx + C_\varepsilon (\|\nabla u_n\|_{L^2(B_{R_n})}^2 + 1), \end{aligned}$$

which implies that

$$\sup_{0 < t < T} \|\nabla u_n(\cdot, t)\|_{L^2(B_{R_n})}^2 + \int_0^T \int_{B_{R_n}} (\varepsilon |\nabla \Delta u_n|^2 + |\Delta u_n|^2) dx dt \leq C, \quad (\text{A.15})$$

where  $C$  is independent of  $n$ . Noticing that

$$\begin{aligned} \|\nabla \cdot (u_n \nabla \tilde{v}_n)\|_{L^2(B_{R_n})}^2 &\leq \|u_n\|_{L^3}^2 \|\Delta \tilde{v}_n\|_{L^6}^2 + \|\nabla u_n\|_{L^3}^2 \|\nabla \tilde{v}_n\|_{L^6}^2 \\ &\leq C_1 (\|u_n\|_{H^1(B_{R_n})}^2 \|\Delta \tilde{v}_n\|_{H^1(B_{R_n})}^2 + \|\nabla u_n\|_{L^2(B_{R_n})}^2 \|\nabla \tilde{v}_n\|_{H^1(B_{R_n})}^2 + \|\Delta u_n\|_{L^2(B_{R_n})}^2 \|\nabla \tilde{v}_n\|_{H^1(B_{R_n})}^2) \\ &\leq C_2 (\|\Delta \tilde{v}_n\|_{H^1(B_{R_n})}^2 + \|\Delta u_n\|_{L^2(B_{R_n})}^2 + 1), \end{aligned}$$

recalling that  $\tilde{v}_n \in L^2(B_{R_n})((0, T); H^3(B_{R_n}))$ , and (A.15), we then can see that

$$\int_0^T \int_{B_{R_n}} |\nabla \cdot (u_n \nabla \tilde{v}_n)|^2 dx dt \leq C. \quad (\text{A.16})$$

Multiplying the equation in (A.13) by  $\Delta^2 u_n$ , and using (A.14), (A.15), we obtain that

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |\Delta u_n|^2 dx + \varepsilon \int_{B_{R_n}} |\Delta^2 u_n|^2 dx + \int_{B_{R_n}} |\nabla \Delta u_n|^2 dx \\ &= -\chi \int_{B_{R_n}} \nabla \cdot (u_n \nabla \tilde{v}_n) \Delta^2 u_n dx \\ &\leq \frac{\varepsilon}{2} \int_{B_{R_n}} |\Delta^2 u_n|^2 dx + \frac{\chi^2}{\varepsilon} \int_{B_{R_n}} |\nabla \cdot (u_n \nabla \tilde{v}_n)|^2 dx dt. \end{aligned}$$

From (A.16), we infer that

$$\sup_{0 < t < T} \|\Delta u_n(\cdot, t)\|_{L^2(B_{R_n})}^2 + \int_0^T \int_{B_{R_n}} (\varepsilon |\Delta^2 u_n|^2 + |\nabla \Delta u_n|^2) dx dt \leq C, \quad (\text{A.17})$$

where  $C$  is independent of  $n$ .

Similarly, by multiplying (A.13) by  $\partial_t u_n$  and using (A.16), we also have

$$\int_0^T \int_{B_{R_n}} |\partial_t u_n|^2 dx dt \leq C, \quad (\text{A.18})$$

where  $C$  is independent of  $n$ .

Recalling that  $\tilde{v}_n \in L^\infty((0, T); H^2(B_{R_n})) \cap L^2((0, T); H^3(B_{R_n}))$ , using (A.14), (A.15) and (A.17), we obtain that

$$\begin{aligned} \|\nabla \nabla \cdot (u_n \nabla \tilde{v}_n)\|_{L^2(B_{R_n})}^2 &\leq C_1 (\|\nabla^2 u_n\|_{L^3(B_{R_n})}^2 \|\nabla \tilde{v}_n\|_{L^6}^2 + \|\nabla u_n\|_{L^3(B_{R_n})}^2 \|\nabla^2 \tilde{v}_n\|_{L^6(B_{R_n})}^2 + \|u_n\|_{L^\infty(B_{R_n})}^2 \|\nabla \Delta \tilde{v}_n\|_{L^2(B_{R_n})}^2) \\ &\leq C_2 (\|u_n\|_{H^3(B_{R_n})}^2 \|\tilde{v}_n\|_{H^2(B_{R_n})}^2 + \|u_n\|_{H^2(B_{R_n})}^2 \|\tilde{v}_n\|_{H^3(B_{R_n})}^2) \\ &\leq C_3 (\|u_n\|_{H^3(B_{R_n})}^2 + \|\tilde{v}_n\|_{H^3(B_{R_n})}^2 + 1), \end{aligned}$$

which implies that

$$\int_0^T \|\nabla \nabla \cdot (u_n \nabla \tilde{v}_n)\|_{L^2(B_{R_n})}^2 dt \leq C. \quad (\text{A.19})$$

Applying  $\nabla$  to the equation of (A.13), and multiplying the resultant equation by  $\nabla\Delta^2u$ , we arrive at

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{B_{R_n}} |\nabla\Delta u_n|^2 dx + \varepsilon \int_{B_{R_n}} |\nabla\Delta^2 u_n|^2 dx + \int_{B_{R_n}} |\Delta^2 u_n|^2 dx \\ &= -\chi \int_{B_{R_n}} \nabla\nabla \cdot (u_n \nabla \tilde{v}_n) \nabla\Delta^2 u_n dx \\ &\leq \frac{\varepsilon}{2} \int_{B_{R_n}} |\nabla\Delta^2 u_n|^2 dx + \frac{\chi^2}{2\varepsilon} \int_{B_{R_n}} |\nabla\nabla \cdot (u_n \nabla \tilde{v}_n)|^2 dx, \end{aligned}$$

which implies that

$$\sup_{0 < t < T} \|\nabla\Delta u_n(\cdot, t)\|_{L^2(B_{R_n})}^2 + \int_0^T \int_{B_{R_n}} (\varepsilon |\nabla\Delta^2 u_n|^2 + |\Delta^2 u_n|^2) dx dt \leq C, \quad (\text{A.20})$$

where  $C$  is independent of  $n$ . Similar to the proof of Proposition A.1, we thus complete the proof upon letting  $n \rightarrow \infty$  in (A.20).  $\square$

**Lemma A.1** Assume that  $v \in L^\infty((0, T); H^2(\mathbb{R}^N)) \cap L^2((0, T); H^3(\mathbb{R}^N))$ . Let  $u \in L^\infty((0, T); H^3(\mathbb{R}^N)) \cap L^2((0, T); H^5(\mathbb{R}^N))$  be the corresponding strong solution of (A.12) such that

$$\|u(\cdot, t)\|_{H^2} \leq C,$$

where  $C$  is independent of  $\varepsilon$ . Furthermore, if  $(1 + |x|^2)^{r/2} u_0 \in L^2(\mathbb{R}^N)$  for some positive constant  $r \leq 1$ , then

$$\sup_{0 < t < T} \int_{\mathbb{R}^N} (1 + |x|^2)^r u^2 dx + \varepsilon \int_0^T \int_{\mathbb{R}^N} (1 + |x|^2)^r |\Delta u|^2 dx dt + \int_0^T \int_{\mathbb{R}^N} (1 + |x|^2)^r |\nabla u|^2 dx dt \leq C_T,$$

where  $C_T$  is constant depending on  $T$  but independent of  $\varepsilon$ .

**Proof.** For any  $R > 1$ , let  $\eta_R(x) \in C_0^\infty(B_{2R})$  be a cut off function on  $B_{2R}$  such that  $0 \leq \eta_R(x) \leq 1$ ,  $\eta_R(x) = 1$  for  $x \in B_R$ ,  $|\nabla^k \eta_R(x)| \leq \frac{C}{R^k}$ , where  $C$  is a constant independent of  $R$ . Let

$$\varphi(x) = |\eta_R(x)|^q (1 + |x|^2)^r,$$

where  $q$  is a positive constant to be determined below,  $r \in (0, 1)$ . By a direct calculation, we obtain

$$\frac{|\nabla\varphi|^2}{\varphi} \leq 8r^2(1 + |x|^2)^{r-2} |x|^2 \eta_R^q + 2q^2(1 + |x|^2)^r \eta_R^{q-2} |\nabla\eta_R|^2 < 8 + C\eta_R^{q-2} \frac{(1 + 4R^2)^r}{R^2}, \quad (\text{A.21})$$

$$\begin{aligned} \frac{|\Delta\varphi|}{\varphi} &\leq C \left( 1 + \eta_R^{q-2} |\nabla\eta_R|^2 + \eta_R^{q-4} |\nabla\eta_R|^4 (1 + |x|^2)^r + \eta_R^{q-2} |\Delta\eta_R|^2 (1 + |x|^2)^r \right) \\ &\leq C \left( 1 + \frac{1}{R^2} \eta_R^{q-2} + \eta_R^{q-4} \frac{(1 + 4R^2)^r}{R^4} \right). \end{aligned} \quad (\text{A.22})$$

$$|\nabla\varphi| \leq 2r\eta_R^q (1 + |x|^2)^{r-1} |x| + q\eta_R^{q-1} (1 + |x|^2)^r |\nabla\eta_R| \leq 2\varphi + q\eta_R^{q-1} (1 + |x|^2)^r |\nabla\eta|. \quad (\text{A.23})$$

In what follows, we take  $q = 4$ . Then

$$\frac{|\nabla\varphi|^2}{\varphi} \leq C, \quad \frac{|\Delta\varphi|}{\varphi} \leq C. \quad (\text{A.24})$$

Multiplying the first equation of (A.12) by  $\varphi(x)u$ , and integrating it over  $\mathbb{R}^N$ , and using (A.23), (A.24), we arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_{B_{2R}} \varphi(x)u^2 dx + \varepsilon \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx + \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx \\
&= -2\varepsilon \int_{B_{2R}} \nabla \varphi \nabla u \Delta u dx - \varepsilon \int_{B_{2R}} u \Delta \varphi \Delta u dx - \int_{B_{2R}} u \nabla \varphi \nabla u dx + \chi \int_{B_{2R}} u \nabla v (\varphi \nabla u + u \nabla \varphi) dx \\
&\leq \frac{\varepsilon}{2} \int_{B_{2R}} \varphi |\Delta u|^2 dx + 4\varepsilon \int_{B_{2R}} \frac{|\nabla \varphi|^2}{\varphi} |\nabla u|^2 dx + \varepsilon \int_{B_{2R}} \frac{|\Delta \varphi|^2}{\varphi} |u|^2 dx + \frac{1}{2} \int_{B_{2R}} \varphi |\nabla u|^2 dx + \int_{B_{2R}} \frac{|\nabla \varphi|^2}{\varphi} |u|^2 dx \\
&+ \chi^2 \|\nabla v\|_{L^\infty}^2 \int_{B_{2R}} \varphi u^2 dx + 2\chi \|\nabla v\|_{L^\infty} \int_{B_{2R}} \varphi u^2 dx + 4\chi \|\nabla v\|_{L^\infty} \int_{B_{2R}} \eta_R^3 (1 + |x|^2)^r |\nabla \eta| u^2 dx \\
&\leq \frac{\varepsilon}{2} \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx + C\varepsilon \int_{B_{2R}} |\nabla u|^2 dx + C \int_{B_{2R}} |u|^2 dx + \frac{1}{2} \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx \\
&+ (\chi^2 \|\nabla v\|_{H^2}^2 + 2\chi \|\nabla v\|_{L^\infty}) \int_{B_{2R}} \varphi u^2 dx + \chi^2 \|\nabla v\|_{H^2}^2 \int_{B_{2R}} \varphi u^2 dx + 4 \int_{B_{2R}} \eta_R^2 (1 + |x|^2)^r |\nabla \eta|^2 u^2 dx \\
&\leq \frac{\varepsilon}{2} \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx + \frac{1}{2} \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx + (3\chi^2 \|\nabla v\|_{H^2}^2 + 1) \int_{B_{2R}} \varphi u^2 dx \\
&+ C\varepsilon \int_{B_{2R}} |\nabla u|^2 dx + C \int_{B_{2R}} |u|^2 dx + 4 \frac{(1 + |4R|^2)^r}{R^2} \int_{B_{2R}} u^2 dx \\
&\leq \frac{\varepsilon}{2} \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx + \frac{1}{2} \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx + (3\chi^2 \|\nabla v\|_{H^2}^2 + 1) \int_{B_{2R}} \varphi u^2 dx + C.
\end{aligned}$$

That is

$$\begin{aligned}
& \frac{d}{dt} \int_{B_{2R}} \varphi(x)u^2 dx + \varepsilon \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx + \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx \\
&\leq (6\chi^2 \|\nabla v\|_{H^2}^2 + 2) \int_{B_{2R}} \varphi u^2 dx + C,
\end{aligned} \tag{A.25}$$

where  $C$  is independent of  $R$  and  $\varepsilon$ . Ignoring the last two terms on the left side of the above inequality, a direct calculation yields

$$\frac{d}{dt} \left\{ \int_{B_{2R}} \varphi u^2 dx \exp \left\{ - \int_0^t (6\chi^2 \|\nabla v(s)\|_{H^2}^2 + 2) ds \right\} \right\} \leq C \exp \left\{ - \int_0^t (6\chi^2 \|\nabla v(s)\|_{H^2}^2 + 2) ds \right\}.$$

The integration of the above inequality from 0 to  $t$  gives

$$\sup_{0 < t < T} \int_{B_{2R}} \varphi u^2 dx \leq C, \tag{A.26}$$

where  $C$  is independent of  $R$  and  $\varepsilon$ . Recalling (A.25), integrating it from 0 to  $T$ , and using (A.26) gives

$$\begin{aligned}
& \int_{B_{2R}} \varphi(x)u^2 dx + \varepsilon \int_0^T \int_{B_{2R}} \varphi(x)|\Delta u|^2 dx dt + \int_0^T \int_{B_{2R}} \varphi(x)|\nabla u|^2 dx dt \\
&\leq \sup_{0 < t < T} \int_{B_{2R}} \varphi u^2 dx \int_0^T (6\chi^2 \|\nabla v\|_{H^2}^2 + 2) dt + \int_{B_{2R}} \varphi(x)u_0^2 dx + CT \\
&\leq \tilde{C},
\end{aligned} \tag{A.27}$$

where  $\tilde{C}$  is independent of  $R$  and  $\varepsilon$ . Letting  $R \rightarrow \infty$ , we achieve the desired inequality.  $\square$

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