Dimensions of Pancake Graphs, and the Fibonacci Sequence Arising in Differences between n and P_n

A Draft

Ryan O'Rourke
orourke.ryan@seznam.cz

2025

"This is how the problem was posed: The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest at the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function of n) that I will ever have to use to rearrange them?" [1]

Pancake sorting is a well known problem in discrete mathematics arising from the question of how to optimally sort a disordered stack of pancakes in order of size. The goal, starting with a disordered stack, is by a certain operation - flipping substacks of pancakes - to arrive at a stack sorted in order from the smallest pancake, at the top, to the largest at the bottom. At every step of the process, any subset of pancakes starting at the current top of the stack may be chosen and then flipped. The pancakes may be formalized as graphs (the aptly named pancake graphs), or as numerical sequences in which flipping is performed as prefix reversals. The pancake problem was first devised in this form by Jacob E. Goodman. It has a range of applications.

A pancake number is the minimum number of flips required for a given number n of pancakes. Finding it is equivalent to finding the diameter of the pancake graph with n nodes. The pancake number P_n has been shown to lie between $\frac{15n}{14}$ and $\frac{18n}{11}$ [3] [4], or about $1.07n$ and $1.64n$, but the exact value remains an open problem. The values of graphs 1 till 19 are known, but above these we have been in rather indefinite territory. A stack of one pancake, or graph with one node, is trivial, since it already has the maximum order. Two pancakes, if not ordered, need at most one flip to achieve the desired formation. A stack of three needs at most three discrete flippings to get in into order. Further on, for instance the pancake number of a pancake graph with 17 nodes is $P_{17} = 19$, then $P_{18} = 20$, $P_{19} = 22$ and – as yet – the number of flips required to reorder 20 pancakes in the worst case scenario has not been proven.

In this paper I propose a function that would, if proven, allow one to find the pancake number of any arbitrarily large pancake graph. This solution is derived from an observation I made which points to there being an underlying structure to the sequence of pancake numbers, tying them closely to the Fibonacci sequence. When reading the book ***The Simpsons and Their Mathematical Secrets***[2]. I had a sudden feeling of familiarity on seeing a table showing the pancake numbers corresponding to the first nineteen pancake graphs. I noticed what seemed to be a the Fibonacci sequence naturally appearing in sequences of *differences* between successive n and P_n numbers

Table 1 on page 6 below relates some values of n to their corresponding pancake number P_n and also gives the difference d between them. It shows known values of pancake graphs and in addition hypothetical values for P_{20},P_{21} and P_{22} which would be valid if the conjecture I am suggesting holds.

Let us define the difference d between P_n an n as $d = (n - P_n)$. The number of times any particular d repeats can be counted up in a list of subsequent n-graphs. For reasons that will become apparent, and which relate to the Fibonacci sequence, any arbitrary group of equivalent values for this d number will appear adjacently to each other. It is in these sets of equivalent differences between a number of adjacent P_n and n numbers that the sequence emerges...

To be a bit more concrete, we can have a look at Table 1. Starting from an n of 1 (i.e. with one pancake), we see that the case adjacent to it, that is $n = 2$ (i.e. a 2-pancake stack) both have a P_n number that corresponds to $n - 1$, i.e. the difference d is -1. Continuing along the direction of positive integers, we can see that $n = 3, n = 4, n = 5$ all share a P_n number corresponding to $n - 0$, so in these three cases n and P_n are equivalent. In the range $n = 6$ to $n = 10$ we have P_n numbers with a value $n + 1$. Looking to the d values, a pattern can be seen to emerge: the 2 consequent n-numbers that the table begins with have d values of -1, the next 3 have values of 0, the next 5 of 1.

We can define any Set_d as containing all n values having equivalent d values. These groupings can be understood as sets of equivalent values of d

appearing for adjacent values of n and P_n. The cardinalities $|C|_d$ of these sets is the key factor here.

$d = (n - P_n)$

$Set_d = \{\forall n \in \mathbb{Z}^+ : n - P_n = d\}$

so that the first three such sets we generate are: {1,2}, {3,4,5}, {6,7,8,9,10} and the cardinatlity $|C|_d$ or sizes of these sets is 2,3 and 5 respectively. Table 2 on page 6 gives the first five of these sets

Paying attention now to cardinalities of the sets of equivalent values of d, we can see a Fibonacci subsequence emerge : $S = 2, 3, 5, 8...$
Two distinct facts should be taken notice of when looking at Table 2:

- that the subsequence S apparently does not start from the first Fibonacci number F_0, but from the third Fibonacci number F_3, that being 2. It continues with the fourth Fibonacci number F_4, which is 3, and so on. Of course, if one admitted graphs with a negative or zero number of nodes, one could make use of the Fibonacci numbers below F_3

- that the value of the difference d is equal to the index of the Fibonacci number corresponding to the cardinality of the set in question minus four, so:
$d = n(F) - 4$

On the basis of these observations, a hypothetically extended table potentially up to infinity can be extrapolated. I limit myself to three as yet unresolved graphs 20, 21 and 22, with cardinalities given up to graph 31 (Table 1).

Derivation of a function $P(n) = P_n$

We define n as being the number of nodes of a pancake graph, F_n as being a number from the Fibonacci sequence, P_n as being the pancake number, which is the minimum amount of prefix reversals needed to sort all permutations of n-numbered sequences, and d as the difference $d = (n-P_n)$.

φ is the golden ratio:

$\varphi = \frac{1}{2}(1 + \sqrt{5})$

The sets $Set_d = \{\forall n \in \mathbb{Z}+ : n - P_n = d\}$, defined above, will be used too

1) The sequence we are working from is shifted and starts from 2, i.e. from the third Fibonacci number F_3, and not from the first one.

2) It is know that F_n can be found by rounding: $F_n = \lfloor \frac{\varphi^n}{\sqrt{5}} \rceil, n \geq 0$

3) The difference d is equal to $(n - P_n)$, thus $P_n = n + d$

4) Since the connection between the Fibonacci sequence and the pancake numbers P_n is not in as direct as would be the case if there was one Fibonacci number for every one n number, but rather the Fibonacci sequence arises in groups of adjacent differences, we need to perform a couple of in-between steps.

These make use of the cardinality of sets Set_d, as defined before. First the index of the Fibonacci number corresponding to the cardinality $|C|_d$ of a set Set_d containing all n-values sharing a common d-value must be found. It is known that the index of a Fibonacci number F can be determined in the following way:

$$n(F) = \lfloor log_{\varphi} \sqrt{5} F \rceil$$

Since it can be observed that the d-values appear to be always 4 lower than the values of indices that the function above generates, we subtract 4:

$$d = \lfloor log_{\varphi} \sqrt{5} F \rceil - 4$$

5) Looking at Table 1, we can intuit that the position in the Fibonacci subsequence as the cardinality of any particular set of n values with the same d-value can be found for any arbitrary n value by adding the Fibonacci numbers that occur from the beginning of the subsequence at F_3 to the point in the sequence in question. Since any number of n-values can share the same d-value, we surround the function by a floor function to avoid fractional values.

In essence, we sum a list of subsequent values of F_n. What we are looking for is the closest value to n of a sum of an unknown number r of terms.

$$f(r) = \sum_{n=1}^{r} \frac{\varphi^n}{\sqrt{5}}$$

simplifying the fraction out:

$f(r) = \sum_{n=1}^{r} \frac{1}{\sqrt{5}} \varphi^n$

we can solve this by using the closed form for the partial sum of the first $r + 1$ terms of a geometric series:

$\sum_{n=0}^{r} aq^n = a(\frac{1-q^{r+1}}{1-q})$

in the following way:

$S_n = \frac{1}{\sqrt{5}}(\frac{1-\varphi^{r+1}}{1-\varphi})$

$\Rightarrow S_n\sqrt{5}(1-\varphi) + 1 = \varphi^{r+1}$

$\Rightarrow log_{\varphi}(-S_n\sqrt{5}(1-\varphi) - 1) - 1 = r$

S_n we replace by $n + 3$, since it is the value corresponding to n that we are looking for and since the sequence we are working from is shifted and starts from 2, i.e. from the third Fibonacci number F_3, and not from the first one. We also add the previously discussed floor function, and we find r to be

$r = \lfloor (log_{\varphi} - (n+3)\sqrt{5}(1-\varphi) - 1) - 1 \rfloor$

6) Then, since $P_n = n + d$, and we are now able to define d in terms of n, putting all these elements together we get:

$P_n = n + \lfloor \log_{\varphi} \sqrt{5} \lfloor \frac{\varphi^{\lfloor (\log_{\varphi} -(n+3)\sqrt{5}(1-\varphi)-1)-1 \rfloor}+1}{\sqrt{5}} \rceil \rceil - 4$

simplifying away some of the square roots of 5, we get:

$P_n = n + \lfloor \log_{\varphi} \lfloor \varphi^{\lfloor (\log_{\varphi} -(n+3)\sqrt{5}(1-\varphi)-1)-1 \rfloor} + 1 \rceil \rceil - 4$

$\triangle$

Number of pancakes n	1	2	3	4	5	6	7	8	9	10	11
Number of flips P_n	0	1	3	4	5	7	8	9	10	11	13
$d = (n - P_n)$	-1	-1	0	0	0	1	1	1	1	1	2

n	12	13	14	15	16	17	18	19	20	21	22
P_n	14	15	16	17	18	19	20	22	23	24	25
d	2	2	2	2	2	2	2	3	3	3	3

Table 1: Number of nodes of pancake graphs (i.e. number of pancakes) with their pancake numbers P_n and the differences between them

Sets	{1,2}	{3,4,5}	{6,7,8,9,10}	{11...18}	{19...31}
Cardinality	2	3	5	8	13

Table 2: Sets of n-values with equivalent d-values and their cardinalities

References

[1] Simon Singh (November 14, 2013) *Flipping pancakes with mathematics* The Guardian. Retrieved March 25, 2014.

[2] Simon Singh (2013) *The Simpsons and Their Mathematical Secrets.* Bloomsbury

[3] Chitturi, B.; Fahle, W.; Meng, Z.; Morales, L.; Shields, C. O.; Sudborough, I. H.; Voit, W. (August 31, 2009). "An (18/11)n upper bound for sorting by prefix reversals". Theoretical Computer Science. Graphs, Games and Computation: Dedicated to Professor Burkhard Monien on the Occasion of his 65th Birthday. 410 (36): 3372–3390. doi:10.1016/j.tcs.2008.04.045.

[4] Heydari M.H.; Sudborough, I.H. On the diameter of the pancake network. J. Algorithms 25 (1997), no. 1, 67–94.