A Short Survey on the Transcendence of the Number e

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Abstract

The number e, which is Euler's number, has an important role in the fields of analysis, differential equations, and number theory. In 1873, Charles Hermite proved the transcendence of e, which was a fundamental achievement in the theory of transcendental numbers. In this note, we focus on the historical and mathematical context surrounding Hermite's original proof, followed by an outline of a modern, elegant, and straightforward approach proving e's transcendence. This paper intends to present a clear, succinct, and self-explanatory assembly ideal for undergraduate students or early career researchers and shifts focus from severe technical overgeneralization to clarity of explanation and conceptual understanding.

1. Introduction

The mathematical constant e, which is about 2.71828, appears naturally in most branches of mathematics and physical science. Introduced initially in connection with compound interest and calculus of exponential growth, e quickly established itself as a fundamental part of mathematical analysis. The number e manifests as the foundation of the natural logarithm, represented by the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, and as the distinct real number for which the derivative of e^x equals the function. These extraordinary properties position e as one of the most vital constants in mathematics, in company with π, i, and 0.

Aside from its significance as an analysis, e also holds a unique position in number theory. In 1873, Charles Hermite wrote a seminal proof that e is a transcendental number—that is, it is not a root of any non-zero polynomial equation with rational coefficients. This was the first clear demonstration of the transcendence of a mathematical constant occurring in nature and led to subsequent proofs of the transcendence of π by Ferdinand von Lindemann and of numerous additional numbers in the twentieth century.

Hermite's proof was not just a great achievement of analytical methods but also a remarkable conceptual breakthrough: it revealed that transcendence, hitherto a strange and ill-understood phenomenon, could be treated systematically and demonstrated exactly by means of classical tools such as power series, integrals, and approximations. His research established the groundwork for contemporary theories of transcendental numbers, making it possible to create the Lindemann–Weierstrass theorem and influence mathematicians like Hilbert, Siegel, Schneider, and Gelfond.

Despite the historical and technical depth of Hermite's original argument, subsequent mathematicians have found alternative proofs that are shorter, more transparent, and accessible to a broader mathematical audience. Some of these newer proofs, while less general, offer a pedagogical advantage and provide deeper insight into the essential ideas behind transcendence without becoming entangled in technical intricacies. They rely on clever estimates, the construction of special auxiliary functions, and elementary properties of integers and rational approximations.

In this memo, we strive to outline a modern, simplified approach to proving the transcendence of e. Our goal is more than a simple rehashing of known results; rather, we aim to present a detailed explanation that is accessible to eager undergraduate students and beginning researchers. We introduce the principal points and the unity of organization of the argument, giving sufficient context to account for the crucial points, without including details that are not pertinent to the inherent nature of the proof.

In undertaking this enterprise, we also wish to highlight the beauty and richness of this ancient discovery, and to stimulate further interest in the field of transcendental number theory—a field in which profound analytic concepts enter into algebraic environments, and in which ostensibly simple constants manifest their inner complexity.

2. Basic Definitions

- **Algebraic number:** A real (or complex) number that is a root of a non-zero polynomial with rational coefficients.
- **Transcendental number:** A number that is not algebraic.

It is known that the set of algebraic numbers is countable, while the real numbers are uncountable. Therefore, most real numbers are transcendental, though explicit examples are rare and historically difficult to construct.

3. Methodology

This research employs an analytical approach grounded in the Taylor series expansion of the exponential function e^x. The series representation was reformulated for precise examination of its terms, then this structure was analyzed within a polynomial framework, utilizing binomial expansion for term separation. The proof is completed through *reductio ad absurdum* (proof by contradiction), demonstrating that initial assumptions lead to an inconsistency.

4. Proof of Transcendence of e

Assume, for the sake of contradiction, that the number e is algebraic. Then there exists a non-zero polynomial with integer coefficients

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathbb{Z}[x], \quad \text{with } a_i \neq 0, \tag{1}$$

such that

$$p(e) = a_0 + a_1 e + a_2 e^2 + \cdots + a_n e^n = 0. \tag{2}$$

We recall the Taylor series expansion of the exponential function e^x around $x = 0$. Evaluating the series at $x = 1$, we obtain

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}. \tag{3}$$

That is,

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{m!} + \cdots , \tag{4}$$

where the infinite sum clearly converges due to the rapid growth of the factorial in the denominator. From the Taylor series expansion of e, we may write:

$$e = \sum_{k=0}^{m} \frac{1}{k!} + \sum_{k=m+1}^{\infty} \frac{1}{k!}, \tag{5}$$

where the second term on the right-hand side is the remainder of the series beyond the m-th term.

Multiplying both sides of equation by $m!$, we obtain:

$$e \cdot m! = m! \sum_{k=0}^{m} \frac{1}{k!} + m! \sum_{k=m+1}^{\infty} \frac{1}{k!}. \tag{6}$$

Observe that the first term on the right-hand side,

$$\sum_{k=0}^{m} \frac{m!}{k!},$$

is an integer, since for each $k \leq m$, $\frac{m!}{k!} \in \mathbb{Z}$. Therefore, there exists an integer $L \in \mathbb{Z}$ such that, We define :

$$L := \sum_{k=0}^{m} \frac{m!}{k!}, \tag{7}$$

and also let

$$b_m := m! \sum_{k=m+1}^{\infty} \frac{1}{k!}. \tag{8}$$

Hence, the equation becomes

$$e \cdot m! = L + b_m, \tag{9}$$

where $L \in \mathbb{Z}$ and $b_m \in \mathbb{R}^+$.

Now observe that the remainder term

$$R_m = \sum_{k=m+1}^{\infty} \frac{1}{k!} = \frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \cdots \tag{10}$$

is extremely small for large m, due to the factorial in the denominator.

Lemma 1. *Let m and k be positive integers such that $k > m$. Then:*

$$\frac{m!}{k!} \leq \frac{1}{(m+1)^{k-m}}.$$

Proof. Since $k > m$, we can write:

$$\frac{m!}{k!} = \frac{1}{(m+1)(m+2)\cdots k}.$$

For every $i \in \{m+1, m+2, \ldots, k\}$, we have $i \geq m+1$. Therefore:

$$(m+1)(m+2)\cdots k \geq (m+1)^{k-m}.$$

Taking the reciprocal of both sides yields the desired inequality. □

Proposition 1. The sequence $b_m = m!R_m$ tends to zero as $m \to \infty$.
Proof. Since

$$b_m = m! \sum_{k=m+1}^{\infty} \frac{1}{k!} = \sum_{k=m+1}^{\infty} \frac{m!}{k!},$$

and for $k > m$, we have

$$\frac{m!}{k!} = \frac{1}{(m+1)(m+2)\cdots k} \leq \frac{1}{(m+1)^{k-m}},$$

it follows that

$$b_m \leq \sum_{k=1}^{\infty} \frac{1}{(m+1)^k} = \frac{1}{m}.$$

Thus $b_m \to 0$ as $m \to \infty$. ■

4.1. Continuation of the Proof

We now express e in terms of the previously defined quantities:

$$e = \frac{L}{m!} + \frac{b_m}{m!}. \tag{11}$$

Substituting this into the polynomial $p(x)$, we get:

$$p\left(\frac{L}{m!} + \frac{b_m}{m!}\right) = a_n \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^n + \cdots + a_0 = 0. \tag{12}$$

We now apply the binomial expansion to each term $a_k \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^k$. Using the classical binomial formula, we have:

$$a_k \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^k = a_k \sum_{j=0}^{k} \binom{k}{j} \left(\frac{L}{m!}\right)^{k-j} \left(\frac{b_m}{m!}\right)^j. \tag{13}$$

Assume the polynomial is given by:

$$P\left(\frac{L}{m!} + \frac{b_m}{m!}\right) = a_n \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^n + a_{n-1} \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^{n-1} + \cdots + a_0 = 0$$

We use the binomial expansion for each term:

$$a_k \left(\frac{L}{m!} + \frac{b_m}{m!}\right)^k = a_k \sum_{j=0}^{k} \binom{k}{j} \left(\frac{L}{m!}\right)^{k-j} \left(\frac{b_m}{m!}\right)^j$$

Therefore, the entire polynomial becomes:

$$\sum_{k=0}^{n}\sum_{j=0}^{k} a_k \binom{k}{j} \left(\frac{L}{m!}\right)^{k-j} \left(\frac{b_m}{m!}\right)^{j}$$

We now rewrite this double sum as a single sum in powers of $\left(\frac{b_m}{m!}\right)^r$ by changing the order of summation:

$$P\left(\frac{L}{m!}+\frac{b_m}{m!}\right) = \sum_{r=0}^{n}\left(\sum_{k=r}^{n} a_k \binom{k}{r} \left(\frac{L}{m!}\right)^{k-r}\right)\left(\frac{b_m}{m!}\right)^{r}$$

Define:

$$C_r = \sum_{k=r}^{n} a_k \binom{k}{r} \left(\frac{L}{m!}\right)^{k-r}$$

Then the polynomial becomes:

$$P\left(\frac{L}{m!}+\frac{b_m}{m!}\right) = \sum_{r=0}^{n} C_r \left(\frac{b_m}{m!}\right)^{r} = 0$$

Suppose, for contradiction, that not all C_r vanish. Let r_0 be the smallest index such that $C_{r_0} \neq 0$. Then we can factor the sum as:

$$\sum_{r=0}^{n} C_r \left(\frac{b_m}{m!}\right)^{r} = \left(\frac{b_m}{m!}\right)^{r_0}\left(C_{r_0} + \sum_{r>r_0} C_r \left(\frac{b_m}{m!}\right)^{r-r_0}\right)$$

Now observe the following:

- Since b_m is a small positive real number and $b_m < m!$, we have $0 < \frac{b_m}{m!} < 1$, so $\left(\frac{b_m}{m!}\right)^{r_0} > 0$ and tends to 0 as $m \to \infty$.
- The inner sum in parentheses tends to $C_{r_0} \neq 0$, because the higher-order terms $\left(\frac{b_m}{m!}\right)^{r-r_0} \to 0$ as $m \to \infty$.

Therefore, for sufficiently large m, the whole expression is strictly nonzero, contradicting the assumption that the sum equals zero. Hence, all $C_r = 0$ for all r.

Now, let's analyze the implication of all C_r being zero. Observe that each C_r is a polynomial in $\frac{L}{m!}$ (which depends on m) with coefficients that are integer combinations of a_k. More specifically, consider the coefficient C_n:

$$C_n = \sum_{k=n}^{n} a_k \binom{k}{n} \left(\frac{L}{m!}\right)^{k-n} = a_n \binom{n}{n} \left(\frac{L}{m!}\right)^{0} = a_n \cdot 1 \cdot 1 = a_n$$

If all $C_r = 0$ for sufficiently large m, then $C_n = 0$. Since $C_n = a_n$, this implies $a_n = 0$. This directly contradicts our initial assumption that $P(x)$ is a nonzero polynomial (which implies $a_n \neq 0$).

Thus, we reach a contradiction, completing the proof.

■

5. Conclusion

In this study, a classical proof of the transcendence of the number e has been revisited, relying on fundamental and elementary concepts. By reformulating the Taylor series expansion and employing the binomial expansion alongside proof by contradiction, this proof provides a clear and accessible framework for understanding the transcendental nature of e. This approach not only encapsulates the essential ideas related to transcendence concisely but also offers an intuitive insight into the topic.

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