TRIPOTENT-INVERTIBLE DECOMPOSITION OF MATRICES OVER LOCAL RINGS

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Abstract. We establish the first necessary and sufficient conditions for a 2×2 matrix over a local ring to be decomposable into the sum of a tripotent and an invertible matrix. Building on this decomposition, we derive a novel characterization of the generalized Drazin inverse for such matrices. Our approach hinges on a key relationship between this decomposition and the associated tripotent-quasinilpotent decomposition, thereby offering significant new insights into the spectral theory of matrices over local rings.

1. Introduction

Let R be an associative ring with an identity. It is of interest to write a 2×2 matrix as the sum of an idempotent and a unit. An element in a ring R is strongly clean provided that it is the sum of an idempotent and a unit that commutate. A ring R is local if it has only one maximal right ideal. Strongly clean 2×2 matrices over a local ring have been the object of much investigation over the last decade, e.g., [5, 12].

An element p in a ring R is a tripotent if $p^3 = p$. It is readily seen that idempotents and negatives of idempotents are tripotents and among units only the order 2 units (also called square roots of 1) are tripotents. The motivation of this paper is to ascertain the conditions under which a 2×2 matrix over a local ring can be represented as the sum of a tripotent matrix and a unit that commute with each other. The commutant of $a\in R$ is defined by $comm(a) = \{x\in R \mid xa = ax\}$. The double commutant of $a\in R$ is defined by $comm^2(a) = \{x\in R \mid xy = yx \text{ for all } y\in comm(a)\}$. An element a is a ring R is Hirano clean if there exists $p^3 = p\in comm^2(a)$ such that $a+p\in U(R)$. We shall see that every strongly clean 2×2 matrix over a local ring is Hirano clean.

Suppose R is a local ring, and consider a matrix $A\in M_2(R)$. In Section 2, we examine the properties of Hirano clean matrices within the setting of

2020 *Mathematics Subject Classification.* 16E50, 16U90, 15A09.

Key words and phrases. tripotent matrix; invertible matrix; local ring; generalized Drazin inverse; quadratic equation.

local rings. We prove that A is Hirano clean if and only if there exist two idempotent matrices $E, F \in comm^2(A)$ and $U \in GL_2(R)$ such that $A = E - F + U$. Moreover, Hirano clean matrices are investigated by means of the similarity. We prove that A is Hirano clean if and only if $A \in GL_2(R)$; or $A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or A is similar to a diagonal matrix $\left(\begin{array}{cc} \lambda & 0 \\ 0 & \mu \end{array} \right)$ with $\lambda \in U(R), \mu \in J(R)$ and $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective.

A ring R is uniquely bleached provided that it is both bleached and cobleached. For instance, every commutative local ring and every local ring with nil Jacobson radical are uniquely bleached. If R is cobleached, we further prove that A is Hirano clean if and only if $A \in GL_2(R)$; or $A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or A is similar to $\left(\begin{array}{cc} 0 & \lambda \\ 1 & \mu \end{array} \right)$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - \mu x - \lambda = 0$ has a root in $U(R)$ and a root in $J(R)$.

A element $a \in R$ has generalized Drazin inverse if there exists $b \in R$ such that $b \in comm^2(a), bab = b$ and $a - a^2 b \in R^{qnil}$. Here, $R^{qnil} = \{a \in R \mid 1 + ax \in U(R)$ for every $x \in comm(a)\}$. In the field of generalized inverses theory, the generalized Drazin inverse drew much attention, many authors such as [3] and [4] have studied generalized Drazin inverses in a ring. Following Koliha and Patrick, an element a in a ring R is quasipolar if there exists an idempotent $e \in R$ such that $a + e \in U(R)$ and $ae \in R^{qnil}$ (see [7, 8]). As is well known, an element $a \in R$ has generalized Drazin inverse if and only if it is quasipolar (see [6]). Quasipolar elements form a subclass of that of Hirano clean elements in a ring. In Section 3, we prove that A has generalized Drazin inverse if and only if there exists $E^3 = E \in comm^2(A)$ such that $A + E \in GL_2(R)$ and $AE \in M_2(R)^{qnil}$.

Throughout the paper, all rings are associative with an identity. $\mathbb{N}$ stands for the set of all natural numbers. Let p be a prime, and let $\mathbb{Z}_{(p)} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, (m, n) = 1, p \nmid n\}$ be the localization of $\mathbb{Z}$ at the prime ideal (p). We use $J(R), U(R)$ and $N(R)$ to denote the Jacobson radical, the set of all units and the set of all nilpotents of R, respectively. $GL_2(R)$ denotes the sets of all 2×2 invertible matrices over R.

2. Hirano clean matrices

In this section, we present representations of Hirano clean matrices over a local ring, which are expressed in terms of idempotent and unitary matrices. An element a in a ring R is similar to an element $b \in R$ if there exists some

$u \in U(R)$ such that $u^{-1}au = b$. Moreover, we completely characterize Hirano clean matrices over a local ring in terms of similarity.

Theorem 2.1. *Let R be a local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A is Hirano clean.*
(2) *There exist two idempotent matrices $E, F \in comm^2(A)$ and $U \in GL_2(R)$ such that $A = E - F + U$.*

Proof. (1) $\Rightarrow$ (2) Since R is local, $2 \in J(R)$ or $2 \in U(R)$.

Case 1. $2 \in J(R)$. Then there exists some $P^3 = P \in comm^2(A)$ such that $U := A + P \in GL_2(R)$. Hence, $A = -P + U = P^2 - (P + P^2) + U$. Clearly, $(P + P^2)^2 = 2(P + P^2) \in M_2(J(R))$, and so $-(P + P^2) + U \in GL_2(R)$, as required.

Case 2. $2 \in U(R)$. Then there exist some $P^3 = P \in comm^2(A)$ such that $U := A + P \in GL_2(R)$. Let $E = \frac{1}{2}(P^2 - P)$ and $F = \frac{1}{2}(P^2 + P)$. Then

$$E^2 - E = \frac{1}{4}(P^4 - 2P^3 - P^2 + 2P) = \frac{1}{4}(P - 2I_2)(P^3 - P) = 0;$$

$$F^2 - F = \frac{1}{4}(P^4 + 2P^3 - P^2 - 2P) = \frac{1}{4}(P + 2I_2)(P^3 - P) = 0;$$

This implies that $E^2 = E, F^2 = F$ and $P = F - E$. Therefore $A = E - F + U$, where $E, F \in comm^2(A)$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, There exist two idempotent matrices $E, F \in comm^2(A)$ and $U \in GL_2(R)$ such that $A = E - F + U$. Let $P = F - E$, then $P^3 = P, P \in comm^2(A)$. Moreover, $A + P = U \in GL_2(R)$. Therefore A is Hirano clean. $\square$

Corollary 2.2. *Let R be a local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A is Hirano clean.*
(2) *There exist orthogonal idempotent matrices $E, F \in comm^2(A)$ and $U \in GL_2(R)$ such that $A = E - F + U$.*

Proof. (1) $\Rightarrow$ (2) In light of Theorem 2.1, we have two idempotent matrices $E, F \in comm^2(A)$ and $U \in GL_2(R)$ such that $A = E - F + U$. Hence, $A = E(I_2 - F) - (I_2 - E)F + U$, where $E(I_2 - F)$ and $(I_2 - E)F$ are orthogonal idempotent matrices, as desired.

(2) $\Rightarrow$ (1) This is obvious by Theorem 2.1. $\square$

Example 2.3. *Let* $A = \begin{pmatrix} 0 & 3 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_{(3)})$. *Then* A *is Hirano clean, but it is not strongly clean.*

Proof. Clearly, $\mathbb{Z}_{(3)}$ is a commutative local ring with $J(\mathbb{Z}_{(3)}) = 3\mathbb{Z}_{(3)}$. Clearly, $A, I_2 - A \notin GL_2(\mathbb{Z}_{(3)})$. In addition, $tr(A) = 1$ and $det(A) = -3$. Taking $\sigma : \mathbb{Z}_{(3)} \to \mathbb{Q}$ to be the natural map, and $p(x) = x^2 - x - 3 \in \mathbb{Z}_{(3)}[x]$. Clearly, $p^*(x) = x^2 - x - 3 \in \mathbb{Q}[x]$ is irreducible, and so $x^2 - x - 3 = 0$ is not solvable in $\mathbb{Z}_{(3)}$. In light of [12, Theorem 2.6], $A \in M_2(\mathbb{Z}_{(3)})$ is not strongly clean. One easily checks that $A = -I_2 + U$, where $U = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \in GL_2(\mathbb{Z}_{(3)})$. Therefore $A \in M_2(\mathbb{Z}_{(3)})$ is Hirano clean. □

The Hirano cleanness is invariant under the similarity. That is, we have

Lemma 2.4. *Let* R *be a ring, and let* $a \in R$ *and* $u \in U(R)$. *Then* $a \in R$ *is Hirano clean if and only if* $u^{-1}au \in R$ *is Hirano clean.*

Proof. $\Longrightarrow$ By hypothesis, there exists $p^3 = p \in comm^2(a)$ such that

$$v := a + p \in U(R).$$

Then

$$u^{-1}pu = (u^{-1}pu)^3,$$
$$u^{-1}au + u^{-1}pu = u^{-1}vu \text{ with } u^{-1}vu \in U(R).$$

Let $x \in comm(u^{-1}au)$. Then $u^{-1}aux = xu^{-1}au$; hence, $auxu^{-1} = uxu^{-1}a$. Then $uxu^{-1} \in comm(a)$, and so $uxu^{-1}p = puxu^{-1}$. This shows $xu^{-1}pu = u^{-1}pux$, and therefore $u^{-1}pu \in comm^2(u^{-1}au)$, as needed.

$\Longleftarrow$ This is symmetric. □

Lemma 2.5. *(* [9, Lemma 3.3]*) Let* R *be a local ring, and let* $A \in M_2(R)$. *Then* $A \in GL_2(R)$; *or* $A^2 \in M_2(J(R))$; *or* A *is similar to* $\begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix}$, *where* $j \in J(R), u \in U(R)$.

We note that $a \in R$ is Hirano clean if and only if there exist $p^3 = p \in comm^2(a)$ and $u \in U(R)$ such that $a = p + u$. We now derive

Theorem 2.6. *Let* R *be a local ring, and let* $A \in M_2(R)$. *Then* A *is Hirano clean if and only if*

(1) $A \in GL_2(R)$; *or*
(2) $A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; *or*

(3) *A is similar to a diagonal matrix* $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ *with* $\lambda \in U(R), \mu \in J(R)$ *and* $l_\lambda - r_\mu, l_\mu - r_\lambda$ *are injective.*

Proof. $\Longrightarrow$ Since A is Hirano clean, we may write

$$A = E + V, E^3 = E \in comm^2(A), V \in GL_2(R).$$

In view of Lemma 2.5, we have three cases.

Case I. $E \in GL_2(R)$. Then $E^2 = I_2$. Set $U := E$. Then $A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$.

Case II. $E^2 \in M_2(J(R))$. It follows from $E(I_2 - E^2) = 0$ that $E = 0$, and so $A = V \in GL_2(R)$.

Case III. There exists $P \in GL_2(R)$ such that

$$P^{-1}EP = \begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix},$$

where $j \in J(R), u \in U(R)$. Since $E^3 = E$, we get

$$ju = 0, j^2 + ju^2 = j, j + u^2 = 1 \text{ and } uj + ju + u^3 = u.$$

Hence, $j = 0$ and $u^2 = 1$. Thus

$$P^{-1}EP = \begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix},$$

and so

$$\begin{pmatrix} 1 & 0 \\ u^{-1} & 1 \end{pmatrix} P^{-1}EP \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}.$$

Set $Q = P\begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix}$. Then $Q^{-1}AQ = \begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix} + Q^{-1}UQ$. Write $Q^{-1}UQ = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}.$$

Thus, we have $\beta = \gamma = 0$. Set $\lambda := \alpha$ and $\mu := u + \delta$. Then $\lambda \in U(R)$. If $\mu \in U(R)$, then $A \in GL_2(R)$. We now assume that $\mu \in J(R)$. Therefore A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \in U(R), \mu \in J(R)$. Further,

$u=\mu-\delta\in U(R)$. We easily see that $u^2=1$. Since $E\in comm^2(A)$, we have
$$\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}\in comm^2\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$
Given $\lambda x=x\mu$ with $x\in R$, then
$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}.$$
Hence, we have
$$\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & u \end{pmatrix}.$$
It follows that $xu=0$, and so $x=0$. This shows that $l_\lambda-r_\mu$ is injective. Likewise, $l_\mu-r_\lambda$ is injective, as desired.

$\Longleftarrow$ Case I. $A\in GL_2(R)$. Then A is Hirano clean.

Case II. There exists $U\in comm^2(A)$ such that $U^2=I_2$ and $A-U\in GL_2(R)$. Then $U=U^3$, and so A is Hirano clean.

Case III. A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda\in U(R),\mu\in J(R)$ and $l_\lambda-r_\mu,l_\mu-r_\lambda$ are injective. If $\lambda\in 1+U(R)$, then $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}-I_2\in GL_2(R)$. Thus, we assume that $\lambda\in 1+J(R)$. Clearly, $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}+\begin{pmatrix} \lambda & 0 \\ 0 & \mu-1 \end{pmatrix}$, where $\begin{pmatrix} \lambda & 0 \\ 0 & \mu-1 \end{pmatrix}\in GL_2(R)$. Let $\begin{pmatrix} x & s \\ t & y \end{pmatrix}\in comm\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. Then
$$\lambda s=s\mu \text{ and } \mu t=t\lambda.$$
By hypothesis, we see that $s=t=0$. This implies that
$$\begin{pmatrix} x & s \\ t & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & s \\ t & y \end{pmatrix}.$$
Therefore $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\in comm^2\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, hence the result. □

Corollary 2.7. *Let R be a local ring, and let $A\in M_2(R)$. If R is cobleached, then A is Hirano clean if and only if*

(1) *$A\in GL_2(R)$; or*
(2) *$A=U+V,U\in comm^2(A),U^2=I_2,V\in GL_2(R)$; or*
(3) *A is similar to a diagonal matrix.*

Proof. $\Longrightarrow$ This is obvious by Theorem 2.6.

$\Longleftarrow$ Suppose that A is similar to $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$. If $\lambda, \mu \in U(R)$, then $A \in GL_2(R)$. If $\lambda, \mu \in J(R)$, then $A \in M_2(J(R))$. Thus we may assume that $\lambda \in U(R), \mu \in J(R)$. This completes the proof by Theorem 2.6. □

Corollary 2.8. *Let R be a local ring, and let $A \in M_2(R)$. If R is cobleached, then A is Hirano clean if and only if*

(1) *$A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or*
(3) *A is strongly clean.*

Proof. Obviously, A is strongly clean if and only if $A \in GL_2(R)$, or $I_2 - A \in GL_2(R)$, or A is similar to a diagonal matrix. This completes the proof, by Corollary 2.7. □

We come now to characterize Hirano clean 2×2 matrices over a cobleached local ring in terms of solvability of their characteristic equations.

Theorem 2.9. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is Hirano clean if and only if*

(1) *$A \in GL_2(R)$; or*
(2) *$A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or*
(3) *A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$, the equation $x^2 - \mu x - \lambda = 0$ has a root in $U(R)$ and a root in $J(R)$.*

Proof. $\Longrightarrow$ In view of Lemma 2.5, we have three cases. Case 1. $A \in GL_2(R)$. Case 2. $A = U + V$ where $U^2 = I_2, U \in comm^2(A), V \in GL_2(R)$. Case 3, A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$, where $\lambda \in J(R), \mu \in U(R)$. It suffices to consider Case 3. In view of Theorem 2.6, there exists $U \in GL_2(R)$ such that

$$U^{-1} \begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} U = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where $\alpha \in U(R), \beta \in J(R)$. Write $U = \begin{pmatrix} x & y \\ s & t \end{pmatrix}$. Then

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \begin{pmatrix} x & y \\ s & t \end{pmatrix} = \begin{pmatrix} x & y \\ s & t \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix};$$

hence, we have

$$\begin{aligned} \lambda s &= x\alpha; \\ \lambda t &= y\beta; \\ x + \mu s &= s\alpha; \\ y + \mu t &= t\beta. \end{aligned}$$

Clearly, $x \in J(R)$. Since $U \in GL_2(R)$, we see that $y, s \in U(R)$, and so $t \in U(R)$. Let $\delta = s\alpha s^{-1}$ and $\gamma = t\beta t^{-1}$ Then $\delta \in U(R), \gamma \in J(R)$. One easily checks that

$$\begin{aligned} \delta^2 - \mu\delta &= s\alpha^2 s^{-1} - \mu s\alpha s^{-1} \\ &= (s\alpha - \mu s)(\alpha s^{-1}) \\ &= x\alpha s^{-1} \\ &= \lambda; \end{aligned}$$

hence, $\delta^2 - \mu\delta - \lambda = 0$. Likewise, $\gamma^2 - \mu\gamma - \lambda = 0$. Therefore the equation $x^2 - \mu x - \lambda = 0$ has a root $\delta \in U(R)$ and a root $\gamma \in J(R)$, as desired.

$\Longleftarrow$ Suppose that the equation $x^2 - \mu x - \lambda = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in J(R)$. Then

$$\begin{aligned} \alpha^2 - \mu\alpha - \lambda = 0; \\ \beta^2 - \mu\beta - \lambda = 0. \end{aligned}$$

Then

$$\begin{aligned} (\alpha - \mu)\alpha = \lambda; \\ (\beta - \mu)\beta = \lambda. \end{aligned}$$

One easily checks that

$$\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix} \begin{pmatrix} \alpha - \mu & \beta - \mu \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha - \mu & \beta - \mu \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Obviously,

$$\begin{pmatrix} \alpha - \mu & \beta - \mu \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta - \mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha - \beta & 0 \\ 1 & 1 \end{pmatrix} \in GL_2(R).$$

Therefore $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ is similar to $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$, where $\alpha \in U(R)$ and a root $\beta \in J(R)$. In light of Lemma 2.5 and Corollary 2.7, we complete the proof. $\Box$

Corollary 2.10. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is Hirano clean if and only if*

1. *$A \in GL_2(R)$;*
2. *$A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$;*
3. *$x^2 - tr(A)x + det(A) = 0$ has a root $\alpha \in U(R)$ and a root $\beta \in J(R)$.*

Proof. $\Longrightarrow$ In view of Theorem 2.9, we may assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in U(R)$, and the equation $x^2 - \mu x - \lambda = 0$ has a root in $J(R)$ and a root in $U(R)$. This implies that $\mu = tr(A)$ and $-\lambda = det(A)$. Therefore the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $U(R)$.

$\Longleftarrow$ Suppose that the equation $x^2 - tr(A)x + det(A) = 0$ has a root in $J(R)$ and a root in $U(R)$. In light of Lemma 2.5, we may assume that A is similar to $\begin{pmatrix} 0 & \lambda \\ 1 & \mu \end{pmatrix}$ where $\lambda \in J(R), \mu \in U(R)$. This implies that $\mu = tr(A)$ and $-\lambda = det(A)$. As a result, the equation $x^2 - x\mu - \lambda = 0$ has a root in $J(R)$ and a root in $U(R)$. According to Theorem 2.9, we complete the proof. □

Example 2.11. *Let* $A = \begin{pmatrix} 1 & 1 \\ -\frac{2}{9} & 0 \end{pmatrix} \in M_2(\mathbb{Z}_{(2)})$. *Then A is not Hirano clean.*

Proof. Clearly, $\mathbb{Z}_{(2)}$ is a commutative local ring with $J(\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{(2)}$. Obviously, $A \notin GL_2(\mathbb{Z}_{(2)})$. We easily see that $p(x) = x^2 - x + \frac{2}{9} \in \mathbb{Z}_{(2)}[x]$ is irreducible. Since $tr(A) = 1$ and $det(A) = \frac{2}{9}$, $x^2 - tr(A)x + det(A) = 0$ is not solvable in $\mathbb{Z}_{(2)}$. If there exist $U \in comm^2(A), V \in GL_2(R)$ such that $A = U + V$ and $U^2 = I_2$, then $A^2 - U^2 = (2U + V)V \in GL_2(\mathbb{Z}_{(2)})$, as $2 \in J(\mathbb{Z}_{(2)})$. This implies that $A - I_2 = \begin{pmatrix} 0 & 1 \\ -\frac{2}{9} & -1 \end{pmatrix}$ is invertible, an absurd. Therefore $A \in M_2(\mathbb{Z}_{(2)})$ is not Hirano clean, by Corollary 2.10. □

3. Generalized Drazin Inverses

The objective of this section is to identify the conditions under which a matrix over a local ring possesses a generalized Drazin inverse, characterized by tripotent matrices. Additionally, the related properties for pseudo Drazin inverses are established.

Theorem 3.1. *Let R be a local ring, and let $A \in M_2(R)$. Then the following are equivalent:*

(1) *A has generalized Drazin inverse.*
(2) *There exists $E^3 = E \in comm^2(A)$ such that $A + E \in GL_2(R)$ and $AE \in M_2(R)^{qnil}$.*

Proof. (1) $\Rightarrow$ (2) This is obvious (see [6]).

(2) $\Rightarrow$ (1) For a tripotent E, we have $-E$ is tripotent. We may write

$$A = E + U, E^3 = E \in comm^2(A), AE \in M_2(R)^{qnil}, U \in GL_2(R).$$

In view of Lemma 2.5, we have three cases.

Case I. $E \in GL_2(R)$. Then $E^2 = I_2$. If $X \in comm(A)$, then $XE = EX$, and so $X \in comm(AE)$. Thus, $I_2 - (AE)X \in GL_2(R)$. Similarly, $I_2 + (AE)X \in GL_2(R)$. Hence $I_2 - ((AE)X)^2 \in GL_2(R)$. This implies that $I_2 - (AX)^2 \in GL_2(R)$. Thus $I_2 - AX \in GL_2(R)$. This shows that $A \in M_2(R)^{qnil}$.

Case II. $E^2 \in M_2(J(R))$. Then $I_2 - E^2 \in GL_2(R)$. It follows from $E(I_2 - E^2) = 0$ that $E = 0$, and so $A \in GL_2(R)$.

Case III. There exists $P \in GL_2(R)$ such that

$$P^{-1}EP = \begin{pmatrix} 0 & j \\ 1 & u \end{pmatrix},$$

where $j \in J(R), u \in U(R)$. As in the proof of Theorem 2.6, we prove that A is similar to a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \in U(R), \mu \in J(R)$ and $l_\lambda - r_\mu, l_\mu - r_\lambda$ are injective. According to [8, Theorem 3.4], $A \in M_2(R)$ is quasipolar, as asserted. □

Corollary 3.2. *Let R be a cobleached local ring, and let $A \in M_2(R)$. Then A is Hirano clean if and only if*

(1) *$A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or*
(2) *A has generalized Drazin inverse.*

Proof. This is obvious, by Theorem 3.1, Theorem 2.6 and [8, Theorem 3.4].

$\Longleftarrow$ This is obvious. □

Corollary 3.3. *Let R be a commutative local ring, and let $A \in M_2(R)$. Then A is Hirano clean if and only if*

(1) *$A = U + V, U \in comm^2(A), U^2 = I_2, V \in GL_2(R)$; or*
(2) *A has generalized Drazin inverse.*

Example 3.4. *Let $A = \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_{(5)})$. Then A has no generalized Drazin inverse, but it is Hirano clean.*

We are now ready to prove:

Theorem 3.5. *Let R be a local ring, and let $A \in M_2(R)$. If R is uniquely bleached, then A has generalized Drazin inverse if and only if*

(1) $A \in GL_2(R)$;
(2) $A \in M_2(R)^{qnil}$;
(3) A *is similar to* $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, *where* $a \in U(R), b = u + \delta \in J(R), u \in comm^2(b), u^2 = 1$ *and* $c = u^{-1}(\delta - a) + 1$.

Proof. $\Longrightarrow$ Since A is quasipolar, In view of [8, Theorem 3.4], we have three cases.

Case I. $A \in GL_2(R)$.

Case II. $A \in M_2(R)^{qnil}$.

Case III. A is similar to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a \in U(R), b \in J(R)$. Obviously, we have

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix}.$$

Choose $u = 1$ and $c = b - a$. Then A is similar to $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, where $a \in U(R), b = u + \delta \in J(R), u \in comm^2(b), u^2 = 1$ and $c = u^{-1}(\delta - a) + 1$.

$\Longleftarrow$ If $A \in GL_2(R)$, then $A = 0 + A$ is an quasipolar decomposition of A. If $A \in M_2(R)^{qnil}$, then $A = I_2 + (I_2 - A)$ is an quasipolar decomposition of A.

Suppose that A is similar to $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, where $a \in U(R), b = u + \delta \in J(R), u \in comm^2(b), u^2 = 1$ and $c = u^{-1}(\delta - a) + 1$. In view of [8, Lemma 2.1], we will suffice to prove $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$ is quasipolar. Obviously, we have

$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix} + \begin{pmatrix} a & 0 \\ \gamma & \delta \end{pmatrix},$$

where $\gamma = u^{-1}(\delta - a)$. We see that

$$\begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix}, \begin{pmatrix} a & 0 \\ \gamma & \delta \end{pmatrix} \in GL_2(R).$$

Let $\begin{pmatrix} x & y \\ s & t \end{pmatrix} \in comm \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$. Then

$$\begin{gathered} ax = xa + yc, \\ ay = yb, \\ cx + bs = sa + tc, \\ cy + bt = tb. \end{gathered}$$

It follows that
$$y = 0, ax = xa, bt = tb \text{ and } bs - sa = tc - cx.$$
As $u \in comm^2(b)$, we see that $tu = ut$, and so
$$b(us) - (us)a = t(uc) - (uc)x.$$
Since $u(c-1) = \delta - a$, we have $uc = u + \delta - a = b - a$. Thus, we have
$$t(uc) - uc(x) = t(b-a) - (b-a)x = bt - ta - bx + ax = b(t-x) - (t-x)a.$$
By the uniqueness, we get $us = t - x$, and so $t = x + us$. Therefore
$$\begin{pmatrix} x & 0 \\ s & t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix}\begin{pmatrix} x & 0 \\ s & t \end{pmatrix}.$$
Accordingly, $\begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix} \in comm^2\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$. Moreover, we check that
$$\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & u \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & bu \end{pmatrix} \in M_2(J(R)) \subseteq M_2(R)^{qnil},$$
as required. □

Conflict of interest

No potential conflict of interest was reported by the authors.

Data Availability Statement

Not applicable.

References

[1] H. Chen and M. Sheibani, Strongly 2-nil-clean rings, *J. Algebra Appl.*, **16**(2017), Article ID 1750178, 12 p.

[2] H. Chen and M. Sheibani, *Theory of Clean Rings and Matrices*, Singapore: World Scientific, 2023.

[3] H. Chen and M. Sheibani, Block representations for the g-Drazin inverse in Banach algebras, *Miskolc Math. Notes*, **24**(2023), 1259–1271.

[4] H. Chen and M. Sheibani, The generalized Drazin inverse of an operator matrix with commuting entries, *Georgian Math. J.*, **31**(2024), 195–204.

[5] J. Chen; X. Yang and Y. Zhou, When is the 2×2 matrix ring over a commutative local ring strongly clean? *J. Algebra*, **301**(2006), 280–293.

[6] J. Cui and J. Chen, When is a 2×2 matrix ring over a commutative local ring quasipolar? *Comm. Algebra*, **39**(2011), 3212–3221.

[7] J. Cui and J. Chen, Characterizations of quasipolar rings, *Comm. Algebra*, **41**(2013), 3207–3217.

[8] J. Cui and J. Chen, Quasipolar matrix rings over local rings, *Bull. Korean Math. Soc.*, **51**(2014), 813–822.

[9] J. Cui and J. Chen, Pseudopolar matrix rings over local rings, *J. Algebra Appl.*, **13**(2014), Article ID 1350109, 12 p.

[10] P. Danchev; E. García and M.G. Lozano, Decompositions of matrices into diagonalizable and square-zero matrices, *Linear Multilinear Algebra*, **70**(2022), 4056-4070.

[11] P. Danchev; E. García and M.G. Lozano, Decompositions of matrices into a sum of invertible matrices and matrices of fixed nilpotence, *Electron. J. Linear Algebra*, **39**(2023), 460–471.

[12] B. Li, Strongly clean matrix rings over noncommutative local rings, *Bull. Korean Math. Soc.*, **46**(2009), 71–78.

[13] D. Mosić and H. Zou, Extension of generalized strong Drazin inverse, *Oper. Matrices*, **15**(2021), 1563–1573.

[14] H. Zou; J. Chen; H. Zhu and Y. Wei, Characterizations for the n-strong Drazin invertibility in a ring, *J. Algebra Appl.*, **20**(2021), Article ID 2150141, 18 p.

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