

# ON THE PROJECTION ANALOGUE OF THE SLICING PROBLEM

JOHAN ASPEGREN

ABSTRACT. We give a proof for a sharp projection analogue of the slicing problem. Moreover, we show a geometric proof the slicing problem.

## CONTENTS

1.	Introduction	1
2.	An elementary proof of the slicing problem	2
3.	An elementary bounds for the random projections	3
	References	3

## 1. INTRODUCTION

The isotropic conjecture or Bourgain’s slicing problem asks for the existence of a universal constant  $c$  such that

**Theorem 1.1.** *There exists an affine hyperplane  $H$  and a universal constant  $c$  such that*

$$m_{n-1}(E \cap K) > c,$$

for convex bodies  $K$  of unit volume.

A classic reference for these questions is [2]. The isotropic constant conjecture has already been proven by Klartag and Lehec [1]. We prove that

**Theorem 1.2** (Angular Thickness Condition and Sharpness). *Let  $K \subset \mathbb{R}^n$  be a bounded measurable set of measure  $|K| = 1$  that is star-shaped with respect to the origin. Then there exists a measurable radial function  $R(\theta) \in [0, \infty]$  on  $S^{n-1}$  such that*

$$K = \{ r\theta : \theta \in S^{n-1}, 0 \leq r \leq R(\theta) \}.$$

Let  $R > 0$  be such that  $|B(0, R)| = 1$ . If the set of directions where  $K$  reaches at least radius  $R$  satisfies

$$(1.1) \quad \omega(\{\theta \in S^{n-1} : R(\theta) \geq R\}) \geq \frac{1}{2} \omega(S^{n-1}),$$

then

$$|K \cap B(0, R)| \geq \frac{1}{2}.$$

Moreover, the constant  $\frac{1}{2}$  is sharp: if the left-hand side of (1.1) is strictly less than  $\frac{1}{2} \omega(S^{n-1})$ , the conclusion may fail.

---

*Date:* October, 2025.

*2020 Mathematics Subject Classification.* 52A23.

This, simple theorem 1.2 implies the slicing theorem [2] via the theorem of Milman ellipsoids. We then obtain 1.1. The isotropic position is a position when the covariance matrix of the convex body is diagonal and all the diagonal entries are the same. This kind of position exists [2]. Another position that always exists is John's position. It is the position of a convex body, where the minimal circumscribed ellipsoid is the unit ball.

*Remark 1.3.* It would be interesting to know whether an isotropic position equal a scaled John's position.

Our theorem for the projection analog is the following:

**Theorem 1.4.** *For any  $K$  convex body there exists an universal constant  $c$  such that*

$$\mathbb{E}(m_{n-1}(Proj_\phi(K))) \geq c|K|^{(n-1)/n},$$

*for convex bodies  $K$  of unit volume. Moreover the claim is sharp, with the extremizer being the euclidean ball.*

## 2. AN ELEMENTARY PROOF OF THE SLICING PROBLEM

In this section we proof the theorem 1.2.

*Proof.* Since  $K$  is star-shaped, polar coordinates yield

$$|K \cap B(0, R)| = \int_{S^{n-1}} \int_0^{\min(R, R(\theta))} r^{n-1} dr d\omega(\theta) = \frac{1}{n} \int_{S^{n-1}} \min(R^n, R(\theta)^n) d\omega(\theta).$$

The volume of the ball is

$$|B(0, R)| = \frac{\omega(S^{n-1})}{n} R^n = 1 \quad (\text{by assumption}).$$

Define the density

$$f(R) := \frac{|K \cap B(0, R)|}{|B(0, R)|} = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\left(1, \left(\frac{R(\theta)}{R}\right)^n\right) d\omega(\theta).$$

Let

$$A := \{\theta \in S^{n-1} : R(\theta) \geq R\}.$$

On  $A$  we have  $(R(\theta)/R)^n \geq 1$ , so the integrand equals 1. Therefore

$$f(R) \geq \frac{1}{\omega(S^{n-1})} \int_A 1 d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} \geq \frac{1}{2} \quad \text{by assumption (1.1)}.$$

Thus

$$|K \cap B(0, R)| = f(R) \cdot |B(0, R)| \geq \frac{1}{2},$$

which proves the main statement.

**Sharpness.** Let  $0 < \varepsilon < \frac{1}{2}$ . Choose a measurable set  $A \subset S^{n-1}$  with

$$\omega(A) = \left(\frac{1}{2} - \varepsilon\right)\omega(S^{n-1}).$$

Define a star-shaped set

$$K = \{r\theta : \theta \in A, 0 \leq r \leq L\},$$

where  $L > 0$  is chosen so that  $|K| = 1$ . Then

$$|K| = \int_A \int_0^L r^{n-1} dr d\omega(\theta) = \frac{\omega(A)}{n} L^n = 1 \quad \implies \quad L^n = \frac{n}{\omega(A)}.$$

Using  $|B(0, R)| = 1$  gives  $\frac{\omega(S^{n-1})}{n} R^n = 1$ , so  $R^n = \frac{n}{\omega(S^{n-1})}$ . Therefore

$$\left(\frac{L}{R}\right)^n = \frac{n/\omega(A)}{n/\omega(S^{n-1})} = \frac{\omega(S^{n-1})}{\omega(A)} = \frac{1}{\frac{1}{2} - \varepsilon} > 1,$$

and hence  $L \geq R$ , so  $R(\theta) \geq R$  for all  $\theta \in A$  and  $R(\theta) = 0$  otherwise. Thus

$$f(R) = \frac{1}{\omega(S^{n-1})} \int_{S^{n-1}} \min\left(1, \left(\frac{R(\theta)}{R}\right)^n\right) d\omega(\theta) = \frac{\omega(A)}{\omega(S^{n-1})} = \frac{1}{2} - \varepsilon < \frac{1}{2}.$$

Consequently

$$|K \cap B(0, R)| = f(R) \cdot |B(0, R)| = \frac{1}{2} - \varepsilon,$$

which violates the desired inequality. Since  $\varepsilon$  is arbitrary, the constant  $\frac{1}{2}$  in (1.1) is sharp.  $\square$

### 3. AN ELEMENTARY BOUNDS FOR THE RANDOM PROJECTIONS

We use the Cauchy's surface area formula:

$$|\partial K| = \frac{1}{\omega_{n-1}} \int_{\mathbf{S}^{n-1}} m_{n-1}(Proj_\phi(K)) d\sigma(\phi).$$

The constant  $\omega_{n-1}$  is the volume of the  $n-1$  dimensional unit ball. Combining the Cauchy Surface area formula with the isoperimetric inequality we have that

$$\begin{aligned} |K|^{n/(n-1)} &\leq \frac{\omega_n^{1/n}}{n} |\partial K| \\ &= \frac{\omega_n^{-1/n}}{n} \frac{|\mathbf{S}^{n-1}|}{|\mathbf{S}^{n-1}| \omega_{n-1}} \int_{\mathbf{S}^{n-1}} m_{n-1}(Proj_\phi(K)) d\sigma(\phi) \\ &= \frac{\omega_n^{1/n}}{n \omega_{n-1}} |\mathbf{S}^{n-1}| \mathbb{E}(m_{n-1}(Proj_\phi(K))). \end{aligned}$$

Now, we can calculate and obtain  $\frac{\omega_n^{1/n}}{n \omega_{n-1}} |\mathbf{S}^{n-1}| \leq C$  directly. It's clear that the inequality must be sharp because the isoperimetric inequality is sharp. This ends the proof of the theorem 1.4.

### REFERENCES

1. B.Klartag *Logarithmic bounds for isoperimetry and slices of convex sets* Ars Inveniendi Analytica (2023), Paper No. 4, 17 pp.
2. D. Milman and A. Pajor, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed  $n$ -dimensional space*, Lecture Notes in Mathematics 1376, Springer, Berlin (1989), 64-104.

*Email address:* jaspegren@outlook.com