Vacuum polarization with the proper-time

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Abstract

The Schwinger proper-time method with the avoiding the direct use of the wave function is used in order to calculate the vacuum polarization in the presence of the homogenous magnetic field. The derivation gives the same results as obtained by Adler (1971) and Minguzi (1956).

1 Introduction

Vacuum polarization consists in the formation of virtual electron-positron pairs under the influence of a quantum of the electromagnetic field of a photon, or, under the influence of a peak electric field. At the same time, if an external field acts on the vacuum, then the creation of real particles is possible due to its energy because the vacuum is not a virtual but a real physical object and has a structure. The vacuum polarization leads not to virtual, but real radiation corrections to the laws of quantum electrodynamics (Konstantinov, 2018).

The vacuum polarization in quantum field theory is represented as the one-loop radiative correction to the photon propagator. It can be graphically described by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \to (e^- + e^+) \to \gamma$, where γ is denotation for photon, and e^-, e^+ is the electron-positron pair. It means that photon can exist in the intermediate state with e^+, e^- virtual particles. The photon propagation function based on such process with electron-positron pair e^-, e^+, is determined from the effective emission and absorption sources. The Schwinger source methods of quantum field theory is adequate for determination of the vacuum polarization.

2 Vacuum polarization in a constant magnetic field by the Schwinger proper-time method

Here, the Schwinger proper-time method with the avoiding the direct use of the wave function is used in order to calculate the vacuum polarization in the presence of the homogenous magnetic field.

We will follow of the article of Tsai and Erber (1974) with regard to Urrutia (1978) in order to present the derivation of the results of Adler (1971) and Minguzi (1956).

We know from the source theory (Schwinger, 1973) that the Lagrangian for the electromagnetic process with all external particles to being photons is of the form:

$$\mathcal{L}(\tilde{A}) = \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2}\left(\mathrm{Tr} e^{-is\mathcal{H}} + C.T.\right), \tag{1}$$

where Tr operates both on the spin indices and on the space-time coordinates. To consider the process in an external electric field, we define new quantities

$$\mathcal{H} = \tilde{\Pi}^2 - e\sigma\tilde{F} \tag{2}$$

$$\tilde{\Pi}_\mu = p_\mu - e\tilde{A}_\mu; \quad p_\mu = \frac{1}{i}\partial_\mu \tag{3}$$

$$\sigma\tilde{F} = \frac{1}{2}\sigma_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{4}$$

where

$$\tilde{\Pi}_\mu = \Pi_\mu - ea_\mu; \quad \Pi_\mu = p_\mu - eA_\mu \tag{5}$$

$$\sigma\tilde{F} = \sigma F + \sigma f \tag{6}$$

and $\tilde{F}$ is the contribution of the external fields

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{7}$$

and radiation field

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \tag{8}$$

We write the operator $\mathcal{H}$ in the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \tag{9}$$

with

$$\mathcal{H}_0 = \Pi^2 - e\sigma F \tag{10}$$

$$\mathcal{H}_1 = e(\Pi a + a\Pi + \sigma f) + e^2a^2. \tag{11}$$

The action of the vacuum polarization can be obtained from $\mathrm{L}^{(2)} by expanding Tr e^{-is\mathcal{H}}$ to quadratic power in a, while treating the external field A exactly, using the formula (Schwinger, 1951):

$$\mathrm{Tr} e^{-is\mathcal{H}} = \mathrm{Tr}\left(e^{-is\mathcal{H}_0}\right) - is\mathrm{Tr}\left(e^{-is\mathcal{H}_0}\mathcal{H}_1\right) \quad -$$

$$\frac{s^2}{2}\int_{-1}^{1}\frac{dv}{2}\mathrm{Tr}\left(e^{-is(1-v)\mathcal{H}_0/2}\mathcal{H}_1 e^{-is(1+v)\mathcal{H}_0/2}\mathcal{H}_1\right) \quad + \cdots. \tag{12}$$

We then have

$$\mathcal{L}^{(2)} = -\frac{1}{4}ie^2\int_0^\infty sdse^{-ism^2}(I_a + I_b + C.T.), \tag{13}$$

where

$$I_a = \frac{2i}{s}\mathrm{Tr}\left(e^{-is\mathcal{H}_0}a^2\right) \tag{14}$$

$$I_b = \int_{-1}^{1}\frac{dv}{2}\mathrm{Tr}\left(e^{-is(1-v)\mathcal{H}_0/2}(\Pi a + a\Pi + \sigma f)e^{-is(1+v)\mathcal{H}_0/2}(\Pi a + a\Pi + \sigma f)\right). \tag{15}$$

In the following text we specify the the external constant magnetic field to be in the $+z$-direction such that $F_{12} = -F_{21} = H$. In particular, we have for I_a (Schwinger, 1951):

$$I_a = \frac{2i}{s}\mathrm{Tr}\left(e^{-is\mathcal{H}_0}a^2\right) = \frac{2i}{s}\mathrm{Tr}\int(dx')\langle x'(s)|x'\rangle a^2(x') \quad =$$

$$4Jz\cos z\frac{2i}{s}\int(dk)a^\mu(-k)a_\mu(k), \tag{16}$$

where

$$J = -\frac{i}{(4\pi)^2}\frac{1}{s^2}\frac{z}{\sin z}; \quad z = seH; \quad a_\mu(k) = \int(dx)e^{-ikx}a_\mu(x). \tag{17}$$

The evaluation of I_b proceeds as follows. Using relations

$$\Pi_\mu e^{-i\tau\mathcal{H}_0} = e^{-i\tau\mathcal{H}_0}\left(\Pi e^{-2eF\tau}\right)_\mu \tag{18}$$

$$\gamma_\mu e^{-i\tau\mathcal{H}_0} = e^{-i\tau\mathcal{H}_0}\left(\gamma e^{-2eF\tau}\right)_\mu \tag{19}$$

$$x(\tau) = e^{-i\tau\mathcal{H}_0}xe^{-i\tau\mathcal{H}_0} = x + D(\tau)\Pi, \tag{20}$$

where

$$D(\tau) = \frac{e^{2eF\tau} - 1}{eF}; \quad \tau = s\frac{1+v}{2} \tag{21}$$

and

$$a_\mu(x) = \int\frac{(dk)}{(2\pi)^4}e^{ikx}a_\mu(k), \tag{22}$$

we obtain

$$I_b = \int \frac{(dq)(dk)}{(2\pi)^4} a^\mu(q) a^\nu(k) \int_{-1}^{1} \frac{dv}{2} \mathrm{Tr} \quad \times$$

$$\left\{ \int (dx') \langle x'(s) | \left[\left(2\Pi e^{-2eF\tau} - q \right)_\mu + \left(\gamma e^{-2eF\tau} \right)_\mu \left(\gamma e^{-2eF\tau} q \right) + q_\mu \right] e^{iqx(\tau)} \quad \times \right.$$

$$\left. [(2\Pi - k)_\nu + \gamma_\nu \gamma k + k_\nu] \, x' \rangle \right\}. \tag{23}$$

Using identities for a, b commuting with [a,b],

$$e^{a+b} = e^a e^b e^{-[a,b]/2} \tag{24}$$

$$e^a b \ = (b + [a,b]) e^a, \tag{25}$$

we get

$$e^{iqx(\tau)} = e^{iqdx(s)} e^{iq(1-d)x} e^{\delta}, \tag{26}$$

where

$$d_{\mu\nu} = \left[\frac{D(\tau)}{D(s)} \right]_{\mu\nu} \tag{27}$$

$$\delta = -\frac{1}{2} iq(1-d)^T D(\tau) q. \tag{28}$$

Then, we move $e^{iqdx(s)}$ in formula (23) to the far left and $e^{iq(1-d)x}$ to the far right by by using eq. (25). Then, using

$$\langle x'(s) | \Pi_\mu | x' \rangle = 0 \tag{29}$$

and

$$\langle x'(s) | \Pi_\mu \Pi_\nu | x' \rangle = \langle x'(s) | x' \rangle \left(-\frac{i}{D(s)} \right)_{\mu\nu} \tag{30}$$

and identity

$$e^{2eF\tau} [d + eFD(\tau)]^T = d, \tag{31}$$

we get after taking the trace

$$I_b = 4J \cos z \int (dk) a^\mu(-k) a^\nu(k) \int_{-1}^{1} \frac{dv}{2} e^{\delta'} \quad \times$$

$$\left[-4i \left(\frac{e^{2eF\tau}}{D(s)} \right)_{\mu\nu} + [(1-2d)k]_\mu [k(1-2d)]_\nu + 2 \tan z \frac{\cos zv - \cos z}{\sin z} \tilde{k}_\mu \tilde{k}_\nu + R_{\mu\nu} \right]. \tag{32}$$

where

$$R_{\mu\nu} = \left(ke^{-2eF\tau}k\right)\left(e^{2eF\tau}\right)_{\mu\nu} - \left(ke^{-2eF\tau}\right)_{\mu}\left(e^{-2eF\tau}\right)_{\nu} \quad +$$

$$\tan z\left[\left(\tilde{k}e^{2eF\tau}k\right)\left(e^{2eF\tau}\right)_{\mu\nu} - \left(ke^{-2eF\tau}k\right)\left(e^{-2eF\tau}\frac{eF}{eH}\right)_{\mu\nu} \quad -\right.$$

$$\left.\left(\tilde{k}e^{-2eF\tau}\right)_{\mu}\left(e^{-2eF\tau}k\right)_{\nu} + \left(ke^{-2eF\tau}\right)_{\mu}\left(\tilde{k}e^{2eF\tau}\right)_{\nu}\right] \tag{33}$$

$$\delta' = \delta(q \to -k) = -is\left[\frac{1-v^2}{4}k^2\left(\frac{\cos zv - \cos z}{2z\sin z} - \frac{1-v^2}{4}\right)\mathbf{k}_{\perp}^2\right] \tag{34}$$

$$\tilde{k}^{\mu} = k^{\lambda}\left(\frac{eF}{eH}\right)_{\lambda\mu}; \quad \mathbf{k}^2 = k_x^2 + k_y^2. \tag{35}$$

These expression have been simplified by using the fact that the integrand should be symmetric in v. Substituting I_a and I_b into eq. $\mathcal{L}^{(2)}$, and performing integration by parts,

$$\frac{2i}{s}g_{\mu\nu} + \int_{-1}^{1}\frac{dv}{2}e^{\delta'}\left[-4i\left(\frac{e^{2eF\tau}}{D(s)}\right)_{\mu\nu}\right] = \int_{-1}^{1}\frac{dv}{2}e^{\delta'}2k(1-2d)kd_{\mu\nu}, \tag{36}$$

we obtain the result

$$\mathcal{L}^{(2)} = -\frac{1}{2}a^{\mu}(-k)a^{\nu}(k)M_{\mu\nu}(k), \tag{37}$$

where

$$M_{\mu\nu}(k) = \frac{\alpha}{2\pi}\int_0^{\infty}\frac{ds}{s}e^{-ism^2}\int_{-1}^{1}\frac{dv}{2} \quad \times$$

$$\left\{z\cot z e^{\delta'}\left([(1-2d)k]_{\mu}[k(1-2d)]_{\nu} + 2k(1-2d)kd_{\mu\nu} \quad +\right.\right.$$

$$\left.\left.2\tan z\frac{\cos zv - \cos z}{\sin z}\tilde{k}_{\mu}\tilde{k}_{\nu} + R_{\mu\nu}\right) + C.T.\right\}. \tag{38}$$

For $H \to 0$ and $k^2 \to 0$, $M_{\mu\nu} \to 0$ which is the normalization condition determining the contact term. The result is

$$C.T. = -(k^2 g_{\mu\nu} - k_{\mu}k_{\nu})(1-v^2). \tag{39}$$

Let us notice that $M_{\mu\nu}$ is gauge invariant, since

$$k^{\mu}M_{\mu\nu}(k) \sim \int_{-1}^{1}\frac{dv}{2}e^{\delta'}k(1-2d)kk_{\nu} = 0 \tag{40}$$

by the requirement that the integrand is symmetric in v, which is involved in

$$k(1-2d)k = -\left[vk^2 + \left(\frac{\sin zv}{\sin z} - v\right)\mathbf{k}_{\perp}^2\right] \tag{41}$$

being odd under $v \to -v$.

For $H = 0$, we get

$$M_{\mu\nu} = (g_{\mu\nu}k^2 - k_\mu k_\nu)\frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\int_0^1 dv(1-v^2)e^{-ism^2}\left(e^{-is(1-v^2)k^2/4} - 1\right) \quad =$$

$$(g_{\mu\nu}k^2 - k_\mu k_\nu)\left(-\frac{\alpha}{4\pi}k^2\right)\int_0^1 dv\frac{v^2(1-v^2/3)}{m^2+(1-v^2)k^2/4}, \tag{42}$$

which is the result obtained by Schwinger (1951). The furthermore discussion is in the article of Tsai and Erber (1974).

3 Discussion

The standard explanation of the vacuum polarization which differs from the new Schwinger approach can be found in the famous monograph and textbook on quantum electrodynamics (Akhiezer et al., 1965; Berestetskii et al., 1982). The introduction to the Schwinger source theory is presented in the well-known monographs (Dittrich, 1978; Dittrich at al., 1985; Schwinger, 1969; 1970; 1973; 1989). The purpose of this paper was to present a complete and explicit result by using a different approach that is an extension of the simple and transparent method proposed by Tsai an Erber to calculate the photon mass operator in an external homogeneous magnetic field.

Let us remark that eq. (42) is valid for general values of ω. The change of contour, $s \to -is$, is not permissible unless the range of photon frequencies is restricted to lie below the pair creation threshold, ($\omega < 2m$. The results of Minguzzi (1956) and Adler (1971) were obtained by making this change of contour and therefore are valid only in the region $0 < \omega < 2m$, as they have cautioned. Even though our result, eq. (42), differs from theirs by a simple change of contour $s \to -is$, the range of physical situations encompassed is quite different and requires careful treatment (Tsai et al., 1974).

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