

Foundations of Quantum-Elastic Geometry

A geometric framework for a unified physical description of nature

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Abstract. *This work presents a framework for the derivation of key physical constants and their interrelations, building the basis for the geometric structure underlying Quantum-Elastic Geometry (QEG) model, a unified field theory in which spacetime is described as an elastic substrate. From QEG Lagrangian and a generally covariant action, we assume the foundational principles of homogeneity, isotropy, covariance, and Lorentz invariance for a physical substrate, showing that imposing self-consistency -formalized through a minimal set of geometric normalization conditions compatible with QEG basis— uniquely determines the substrate's emergent structure and properties. The outcome is a deductive framework in which fundamental constants are geometrically enforced, emerging as predictable consequences of a stable and symmetrically constrained geometry.*

"Entia non sunt multiplicanda praeter necessitatem"
— Ockham's Razor

"Padre, Señor del cielo y de la tierra, te doy gracias porque has ocultado todo esto a los sabios y entendidos y se lo has revelado a los que son como niños."
— Matthew 11:25

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I. INTRODUCTION: THE QUEST FOR A UNIFIED PHYSICAL REALITY

Modern theoretical physics rests on two mathematically sophisticated yet conceptually divergent frameworks. The Standard Model encodes the electromagnetic, weak, and strong interactions in the language of quantum gauge fields [1], while General Relativity geometrizes gravitation as the curvature of a Lorentzian manifold governed by the Einstein field equations [2, 3]. These theories exemplify the power of mathematics to capture nature, yet their incompatibility reveals a structural gap [4].

This disunity is not incidental but fundamental. The Einstein–Hilbert action,

$$S_{\text{EH}} = \frac{1}{16\pi G} \int R \sqrt{-g} d^4x,$$

cannot be renormalized within the operator algebra of quantum field theory [5], while the Standard Model requires an external geometric background rather than generating one. The cosmological constant problem further reflects an inconsistency between spectral sums in field theory and geometric volume integrals in differential geometry [6]. Additional tensions, such as the

discrepancy between early- and late-time determinations of the Hubble constant H_0 [7, 8], indicate that our present formalism is incomplete.

Several lines of research suggest that unification must be sought in general mathematical principles rather than in phenomenological adjustments. Jacobson derived the Einstein equations from the Clausius relation $\delta Q = TdS$, interpreting curvature as an equation of state [9]. Verlinde extended this idea within the framework of entropic gravity [10]. Within the AdS/CFT correspondence, Ryu and Takayanagi demonstrated that minimal surfaces compute entanglement entropy, establishing a direct link between geometric variational principles and quantum information [11, 12]. These developments collectively indicate that geometry, thermodynamics, and information are not merely analogous but can be shown to be mathematically equivalent under suitable axioms.

Concrete realizations of emergent geometry arise in analogue gravity models, where the effective metric for excitations in fluids or Bose–Einstein condensates takes the form of a curved Lorentzian manifold [13]. This universality suggests that curved geometry is the generic effective description of collective modes in a substrate, not an exclusive property of gravitation.

Finally, gauge-theoretic formulations have provided evidence that the dimensional character of constants may be representational rather than intrinsic. Recent work has shown that embedding gravity in a $U(1)^4$ gauge structure yields a dimensionless gravitational coupling and a renormalizable effective theory, suggesting that dimensional collapse is not only possible but necessary within a unified framework [14].

II. SCOPE AND RELATION TO THE COMPANION QEG MODEL

This article is the mathematical companion to the unified field model developed in the companion paper developing *Quantum–Elastic Geometry (QEG)* [15]. There, spacetime is modeled as a single, unified elastic–dissipative substrate. The state of this substrate is described by a single symmetric rank-2 tensor field, $\mathcal{G}_{\mu\nu}$, which dynamically defines the spacetime geometry.

The theory is fundamentally background-independent. The dynamics are governed by a generally covariant action where the volume element and the covariant derivative ∇ are determined by $\mathcal{G}_{\mu\nu}$ itself. The full action takes the form [15, cite: 650]:

$$S_{\text{QEG}} = \int d^4x \sqrt{-\det(\mathcal{G})} \left[\frac{\kappa}{2} \mathcal{G}^{\alpha\beta} \mathcal{G}^{\mu\nu} \mathcal{G}^{\rho\sigma} \nabla_\alpha \mathcal{G}_{\mu\nu} \nabla_\beta \mathcal{G}_{\rho\sigma} - V(\mathcal{G}) \right] \quad (\text{II.1})$$

where κ is the universal stiffness of the substrate. For the purpose of analyzing low-energy excitations and deriving the fundamental constants, we study the theory in its weak-field limit. This is achieved by linearizing the field $\mathcal{G}_{\mu\nu}$ as a small perturbation $h_{\mu\nu}$ around a flat Minkowski background, $\mathcal{G}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. In this approximation, the complex kinetic term of Eq. (II.1) reduces to the standard Lorentz-invariant form, gov-

erned by the Lagrangian density:

$$\mathcal{L}_{\text{QEG}}^{(\text{weak-field})} = \frac{\kappa}{2} (\partial^\alpha h^{\mu\nu})(\partial_\alpha h_{\mu\nu}) - V(h), \quad (\text{II.2})$$

where indices are now raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. It is this effective weak-field description that we will use as the starting point for the derivations in this paper.

The goal of this paper is to prove that, *given* the QEG Lagrangian (II.2), its explicit action (II.1), and the intrinsic symmetries (homogeneity, isotropy, Lorentz covariance, and scale freedom), the substrate's response structure is *rigid*. Specifically: (i) the admissible response channels are uniquely fixed; (ii) the admissible normalization conditions reduce to three *compatibility normalizations*; and (iii) the long-wavelength dynamics enforce scaling laws that interrelate the network of fundamental constants (Sec. V). In contrast with earlier formulations I, the constants are not treated as independent empirical inputs: once the QEG action and its symmetries are postulated, the entire structure follows as a consequence of compatibility. This paper thus provides the rigorous mathematical underpinning of the physical QEG model, serving as its formal companion.

A. Dissipative extension and identification of the universal damping ratio

Firstly, we must complete the conservative QEG action by adding a covariant dissipative sector, as a dissipative parameter cannot be obtained from a purely conservative action. We work at the effective level where dissipation emerges from coarse-graining microscopic degrees of freedom of the elastic-geometric substrate.

To introduce *linear* damping in a Lorentz-compatible way, we augment the conservative dynamics with a dissipative force derived from a **Rayleigh functional**, built from the material time derivative along a unit timelike 4-velocity field u^μ (the local rest frame of the medium):

$$\mathcal{R}[G; \gamma] = \frac{\gamma}{2} \int d^4x \sqrt{-g} \left(u^\mu \nabla_\mu G_{\alpha\beta} \right) \left(u^\nu \nabla_\nu G^{\alpha\beta} \right) \quad (\text{II.3})$$

Here $\gamma > 0$ is the (local, low-frequency) **dissipative coefficient**¹. The associated *dissipative four-force density* is the variational derivative with respect to the field “velocity”,

$$F_{\mu\nu}^{(\text{diss})} = -\frac{1}{\sqrt{-g}} \frac{\delta \mathcal{R}}{\delta (u^\rho \nabla_\rho G^{\mu\nu})} = -\gamma \left(u^\rho \nabla_\rho \mathcal{G}_{\mu\nu} \right) \quad (\text{II.4})$$

Adding this force to the conservative Euler–Lagrange

equations yields the *damped* equations of motion,

$$\kappa \square \mathcal{G}_{\mu\nu} - \frac{\partial V}{\partial G^{\mu\nu}} - \gamma \left(u^\rho \nabla_\rho \mathcal{G}_{\mu\nu} \right) = J_{\mu\nu}, \quad (\text{II.5})$$

which implement a linear damping term proportional to the generalized “velocity” $u \cdot \nabla G$, ensuring causal relaxation in the hydrodynamic (IR) regime. This is the *minimal* covariant, local, parity-even choice that yields linear dissipation; higher-order or nonlocal kernels can be included if needed, but are unnecessary at leading order.

Remark II.1 (On minimality and uniqueness). *This Rayleigh form is the simplest and most parsimonious covariant choice: it is the lowest-order local scalar quadratic in the field “velocity” $u^\mu \nabla_\mu G_{\alpha\beta}$ and therefore produces a linear damping force, in direct analogy with Rayleigh dissipation in mechanics and with causal hydrodynamics. Among local, covariant, parity-even quadratic functionals built from G and one material derivative, the form above is unique up to an overall coefficient and boundary terms; alternatives either reduce to it by field redefinitions or introduce unnecessary higher derivatives/scales.*

II.X.4 Definition of the universal damping ratio

We **define** the (dimensionless) universal damping ratio of the substrate as the ratio of dissipative to elastic response at low frequency:

$$\alpha \equiv \frac{\gamma}{\kappa}. \quad (\text{II.6})$$

With this definition, α is not an ad-hoc construct from static invariants but a fundamental **material property** of the quantum-elastic substrate. In later sections we show that, at leading order in the anomaly-free closure regime, this universal ratio matches the fine-structure constant, $\alpha = \alpha_{\text{em}}$, consistently with the impedance/quantum-conductance identity $Z_0 \sigma = 2\alpha$.

II.X.5 Consistency with symmetries and thermodynamics

- **Covariance & causality.** The dissipative term is a scalar under diffeomorphisms; causality is ensured because it depends on the material time derivative $u \cdot \nabla$ (no acausal higher-time derivatives appear at leading order).
- **Positivity & entropy production.** For fluctuations around a homogeneous background, $\mathcal{R} \geq 0$ and the work done by dissipation is $\propto \gamma \int (u \cdot \nabla G)^2 \geq 0$, ensuring non-negative entropy production.
- **Minimality.** No new intrinsic scales are introduced at leading order; α is dimensionless and compatible with the scale-free closure discussed in Sec. IV–V.

¹ We use γ for the dissipative coefficient; it must not be confused with the Lorentz factor.

III. EXPLICIT CONSTRUCTION OF THE GEOMETRIC INVARIANTS

a. Conceptual motivation. We start from a medium in which interactions, propagations, and deformations can occur. In QEG, this medium must obey a minimal set of restrictions dictated by the action and by the intrinsic symmetries of spacetime: homogeneity, isotropy, covariance, Lorentz invariance, and scale freedom. From this standpoint, there must exist *dimensionless invariants* that survive under these symmetries and thus uniquely characterize the medium. Our goal is to derive such invariants directly from the Lagrangian dynamics at minimal (quadratic, scale-free) order, and only thereafter elevate them to physically interpretable composites.

Action and linearized equations

The full QEG action is background-independent, with the dynamical field $G_{\mu\nu}$ defining the spacetime geometry itself, as detailed in the companion paper [15]. For the analysis of low-energy (long-wavelength) excitations, we follow the standard procedure of linearizing the theory around a flat background. We expand the field as a fluctuation $h_{\mu\nu}$ over the Minkowski metric $\eta_{\mu\nu}$:

$$G_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x). \quad (\text{III.1})$$

The action, when expanded to second order in $h_{\mu\nu}$, yields the quadratic Lagrangian for the fluctuations. The kinetic part of this action is given by:

$$S_{\text{kin}}^{(2)} = \int d^4x \left[\frac{\kappa}{2} (\partial_\alpha h_{\mu\nu}) (\partial^\alpha h^{\mu\nu}) + \dots \right], \quad (\text{III.2})$$

where indices are now raised and lowered with the background Minkowski metric $\eta_{\mu\nu}$, and covariant derivatives reduce to partial derivatives ∂_α . The resulting linearized Euler-Lagrange equations in this infrared (IR) regime are:

$$\begin{aligned} \kappa \square h_{\mu\nu} - \left(\partial V / \partial G^{\mu\nu} \right) \Big|_{\bar{G}=\eta} &= J_{\mu\nu}, \\ K h &= J, \quad K := \kappa \square \xrightarrow{\text{static}} -\kappa \nabla^2, \end{aligned} \quad (\text{III.3})$$

where $\square \equiv \partial_\alpha \partial^\alpha$ is the d'Alembertian operator on the flat background. This approximation is sufficient to derive the geometric invariants that characterize the substrate's response.

A. Gauge channel: an explicit projector and unit residue $\Rightarrow N_g = 1/4$

b. Explicit projector. Define the spatial trace-free divergence of h_{ij} (Latin i, j denote spatial indices) by

$$\mathcal{S}_i[h] := \partial^j \left(h_{ij} - \frac{1}{3} \delta_{ij} h^k{}_k \right)$$

Set the 1-form

$$A_i := c_\Pi \mathcal{S}_i[h], \quad A_0 := 0, \quad (\text{III.4})$$

with c_Π a dimensionless constant to be fixed by residue normalization below. (Other linear, local, rotationally-covariant maps $\Pi[h] \rightarrow A$ differ by field redefinitions and surface terms at this order; see Appendix B.) Inserting (III.4) back into the quadratic action generated by (III.2) and integrating by parts gives an effective gauge-sector Lagrangian

$$\begin{aligned} \mathcal{L}_{\text{gauge}}^{(2)} &= \frac{c_g}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \quad (\text{III.5})$$

with $c_g = c_g(c_\Pi; \kappa)$ a *dimensionless* constant determined by the contractions inherited from (III.2).

c. Noether current and residue. Let J^μ denote the source descending from the same quadratic action (e.g., coupling to a probe). The transverse propagator is

$$\tilde{D}_{\mu\nu}(k) = -\frac{i}{c_g} \frac{P_{\mu\nu}^T(k)}{k^2 + i0^+} + (\text{gauge part})$$

Lemma III.0.1. *Demand simultaneously (i) positivity of the Hamiltonian, i.e. $\frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2)$ with + sign, and (ii) unit residue for the transverse massless pole. We use "unit residue" in the effective sense: it is the canonical normalization of the retarded propagator ensuring that the same source J^μ (derived from the variation of the same action) couples with unit strength to the emergent physical mode. This is equivalent to fixing the wave function renormalization factor to one in an effective quantization. Under these conditions, $c_g = 1$.*

Proof. A field rescaling $A \mapsto A/\sqrt{c_g}$ would repair a wrong c_g in the kinetic term but would also rescale the source coupling to $(1/\sqrt{c_g})A \cdot J$, changing the LSZ residue of the J - J correlator. Because J is defined by variation of the *same* quadratic action, the current normalization is not a free knob. Hence $c_g = 1$. \square

With the standard scalar F^2 convention, this implies

$$N_g = \frac{1}{4}.$$

d. Robustness to projector choice.

Theorem III.1. *Any two local, rotationally-covariant linear projectors Π mapping h to a 1-form A that (i) differ by field redefinitions and (ii) yield a positive Hamiltonian, lead—after imposing unit residue on the same J^μ —to the same $c_g = 1$ and thus to $N_g = 1/4$.*

Proof. Composition with an invertible local linear operator induces $A \mapsto \Lambda A$, which rescales *both* kinetic and source terms in the same way; imposing unit residue fixes $\Lambda = 1$. \square

**B. Propagation channel: Noether charge \Rightarrow
Gauss flux $\Rightarrow N_\Delta = 1/(4\pi)$**

From (III.3) the static fundamental solution $G(\mathbf{x})$ of $-\kappa\nabla^2$ obeys

$$-\kappa\nabla^2 G(\mathbf{x}) = \delta^{(3)}(\mathbf{x})$$

Let $Q := \int d^3x J^0$ be the conserved Noether charge associated with the linearized symmetry that generates the 1-form channel (the same J^μ that appears in (III.5)). The static equation for the scalar potential ϕ sourced by J^0 reads

$$-\kappa\nabla^2 \phi(\mathbf{x}) = J^0(\mathbf{x})$$

Integrating over a ball B_R and applying the divergence theorem gives

$$\kappa \int_{S_R^2} \nabla \phi \cdot d\mathbf{S} = \int_{B_R} J^0 d^3x = Q \quad (\text{for } R \text{ large enough})$$

If we define the *unit* Noether charge as the unit coupling to J^0 in (III.5), then the flux condition for the *fundamental solution* is $\kappa \int_{S_R^2} \nabla G \cdot d\mathbf{S} = 1$. The unique rotationally-invariant solution satisfying this normalization is

$$G(\mathbf{x}) = \frac{1}{4\pi r},$$

so the propagation normalization is

$$N_\Delta = \frac{1}{4\pi}$$

Remark. This fixes the Green kernel *by the Noether charge normalization that already appears in the same quadratic action*, without importing Gauss' law from classical electrostatics (though it would agree).

C. Storage channel: two independent derivations of $N_k = 1/N_{\text{self}}$ with $N_{\text{self}} = \frac{3}{5} 4\pi$

Let the quadratic self-energy for a static compact mode with density ρ be

$$U[\rho] = \frac{1}{2} \int d^3x d^3y \rho(\mathbf{x}) K(\mathbf{x} - \mathbf{y}) \rho(\mathbf{y}),$$

$$K(\mathbf{x}) = \lambda G(\mathbf{x}) = \frac{\lambda}{4\pi|\mathbf{x}|} \quad (\text{III.6})$$

e. (i) *Variational/Riesz route (geometric).*

Lemma III.1.1. *For fixed $Q = \int \rho$ and positive radial decreasing kernel K , the symmetric decreasing rearrangement minimizes $U[\rho]$; thus the uniform ball of*

radius R is the minimizer. Direct evaluation yields

$$U_{\text{self}}(Q, R) = \frac{\lambda}{2} \frac{3}{5} \frac{Q^2}{4\pi R} \quad \Longrightarrow$$

$$N_{\text{self}} = \frac{3}{5} 4\pi, \quad N_k = \frac{1}{N_{\text{self}}} \quad (\text{III.7})$$

Proof. Apply the Riesz rearrangement inequality to (III.6). Compute the integral for a uniform ball either by shell assembly or by convolving characteristic functions; the numerical factor $3/5$ appears universally with the 4π coming from G . \square

f. (ii) *Spectral/energy route (operator-theoretic).* Consider the quadratic form $\langle \rho, K * \rho \rangle$ in Fourier space:

$$U[\rho] = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \tilde{K}(k) |\tilde{\rho}(k)|^2,$$

$$\tilde{K}(k) = \lambda \tilde{G}(k) = \lambda \frac{1}{\kappa k^2} \quad (\text{III.8})$$

Among all compact ρ with fixed $Q = \tilde{\rho}(0)$, minimizing U pushes spectral weight to the lowest k allowed by compactness; the extremal configuration that saturates the $k \rightarrow 0$ dominance is the real-space ball (the unique radial extremizer). Evaluating U for the ball reproduces exactly the same coefficient as above. Hence the quadratic storage coefficient is universal and equal to $N_{\text{self}} = \frac{3}{5} 4\pi$.

g. *Conclusion for N_k .* Both independent routes (geometric rearrangement and spectral minimization) fix the same coefficient. No ‘‘spherical ansatz’’ is assumed; the sphere *emerges* as the unique extremizer under the positivity/long-range kernel chosen earlier by the same action.

Remark III.2 (Physical summary of the three invariants). *The three scalar invariants are understood as normalization constraints required for compatibility with the QEG quadratic dynamics:*

1. **Canonical gauge normalization** ($N_g = \frac{1}{4}$): *ensures positive Hamiltonian and canonical quantization of the emergent 1-form (unit LSZ residue).*
2. **Unit-flux Green normalization** ($N_\Delta = \frac{1}{4\pi}$): *fixes the static Green kernel of the unique isotropic second-order operator by Noether charge normalization.*
3. **Spherical self-storage** ($N_{\text{self}} = \frac{3}{5} 4\pi$; $N_k = 1/N_{\text{self}}$): *isoperimetric optimality for homogeneous stable modes at quadratic order.*

They are not additional axioms of nature, nor tunable constants: they are the only normalizations compatible with the action at minimal order.

Remark III.3 (Exclusivity). *At minimal order and under locality, isotropy and scale freedom, no fourth invariant exists: any additional candidate either introduces a scale (breaking Weyl weight zero), reduces to a reparametrization of N_g, N_Δ, N_k , or belongs to higher-order corrections.*

D. The simplest composite invariants: α_0 and Y_0

We have successfully identified the three fundamental dimensionless invariants (\mathcal{N}_g , \mathcal{N}_Δ , \mathcal{N}_k), derived from the static, conservative sector of the action. We now use these to construct two composite parameters that characterize the substrate's response: the *conservative admittance* Y_0 , describing its capacity to store energy, and a geometric composite α_0 , which acts as a structural analogue to the universal damping ratio α that was physically defined in Sec. II A. We will show that the construction rules for these composites are dictated by their physical roles (conservative vs. dissipative) and that the resulting α_0 is indeed consistent with the physically defined α .

The composite invariants Y_0 and α must be built from factors operating at the same physical level: (i) conservative (quadratic) quantities multiply quadratic-level factors; (ii) dissipative (linear) quantities multiply linear-level factors; and (iii) the passage from quadratic stiffness to linear amplitude-level enters via a square root. Otherwise, one introduces spurious freedom: (a) weighted sums inject extra dimensionless coefficients that break channel-democracy and parametric minimality, and (b) mixing quadratic and linear factors violates homogeneity of the functional (and, at this order, Weyl weight 0). Consequently, the *unique* parameter-free composites are:

$$\begin{aligned} Y_{0,0} &\equiv \mathcal{N}_g \mathcal{N}_\Delta \mathcal{N}_k && \text{(conservative, quadratic level),} \\ \alpha_0 &\equiv \mathcal{N}_g \mathcal{N}_\Delta \sqrt{\mathcal{N}_k} && \text{(dissipative, linear level).} \end{aligned} \quad (\text{III.9})$$

As argued in Appendix C 2, these are the only multilinear invariants built from the three one-dimensional channels without introducing extra couplings; a more expanded derivation of the above follows below, together with the dressed, physical quantities adjusted by the perturbative corrections discussed below.

1. The Geometric Admittance of the Vacuum (*A Conservative, Quadratic-Level Quantity*)

The admittance, Y_{vac} , quantifies the substrate's intrinsic ability to support conservative, energy-storing configurations. In any field theory, stored energy is fundamentally a **quadratic** functional of the field amplitudes or potentials (e.g., $U = \frac{1}{2}CV^2$ or $U = \int \frac{1}{2}\epsilon E^2 dV$). Therefore, the total admittance must be constructed from invariants that are defined at this same quadratic level.

Our three invariants, when viewed as coefficients of the conservative response, naturally exist at this level:

- R1:** $\mathcal{N}_g = \frac{1}{4}$ is the normalization of the Lagrangian density term $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, which is quadratic in the field strengths.
- R2:** $\mathcal{N}_\Delta = \frac{1}{4\pi}$ is the normalization of the Green's function, the kernel of the quadratic action.
- R3:** The geometric capacitance, $\mathcal{N}_C = \frac{3}{5} \cdot 4\pi$, is by definition the coefficient in the quadratic energy

formula $U \propto \mathcal{N}_C A^2$. Thus, its reciprocal $\mathcal{N}_k = \frac{1}{\mathcal{N}_{self}}$ inherits its quadratic nature.

The principle of parsimony dictates that the combination must be the simplest one possible, introducing no new arbitrary parameters or unexplained structures. As each channel is one-dimensional at minimal order, the only nontrivial parameter-free multilinear invariant of the three channels is the tensor product (the *product* of their scalars), up to an overall normalization. Any weighted sum introduces extra dimensionless coefficients, violating channel democracy and the parameter-free requirement. We therefore *define* the **Geometric Admittance of the Vacuum** (Y_0) as this product, representing the total conservative compliance of the substrate:

$$Y_0 \equiv \mathcal{N}_g \cdot \mathcal{N}_\Delta \cdot \mathcal{N}_k = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4\pi}\right) \cdot \left(\frac{1}{\frac{3}{5} \cdot 4\pi}\right) \quad (\text{III.10})$$

This multiplicative form is the unique combination that respects the independence and equal status of the three geometric constraints without introducing extraneous arbitrary parameters. Any weighted sum would introduce three non-geometric weights; by AM-GM it is moreover bounded below by a geometric mean, so the product is the sharp, symmetry-respecting choice.

2. The Damping Factor α (*A Dissipative, Linear-Level Quantity*)

In contrast, the damping factor, α , does not represent stored energy. It serves as the coefficient of a dissipative "force" or attenuation mechanism, analogous to a frictional drag in mechanics. Such dissipative terms, as formalized in the Rayleigh dissipation function, are proportional to the square of a generalized velocity ($\mathcal{D} \propto \alpha \dot{q}^2$), making the dissipative force itself **linear** in the velocity or amplitude ($F_{diss} \propto \alpha \dot{q}$). This requires α to be a **linear-level (or amplitude norm-level)** coefficient. To construct α from our geometric building blocks, all components must be expressed at this same linear level.

- **R1 & R2:** The invariants \mathcal{N}_g and \mathcal{N}_Δ can be interpreted directly as norm-level factors as they define the metric and flux normalization.
- **R3:** The stiffness or elastance of the substrate is the inverse of its compliance, $\mathcal{N}_k = 1/\mathcal{N}_{self}$. This is a quadratic-level coefficient from the energy functional ($U \propto \mathcal{N}_k A^2$). To convert this quadratic-level stiffness into a linear-level damping coefficient, we must take its square root as a necessary step to transition from the energy domain to the force domain.

Therefore, the normalized stiffness at the amplitude-norm level is $\mathcal{N}_{lin} = \sqrt{\mathcal{N}_k} = \sqrt{1/\mathcal{N}_{self}}$. The damping factor α , as the product of the three linear-level admittances, is thus:

$$\alpha \equiv \mathcal{N}_g \cdot \mathcal{N}_\Delta \cdot \mathcal{N}_{lin} = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4\pi}\right) \cdot \sqrt{\frac{1}{\frac{3}{5} \cdot 4\pi}} \quad (\text{III.11})$$

Remark III.4 (Summary of the Structural Distinction). *The different treatment of the third invariant is*

not an arbitrary choice but a requirement for physical and structural consistency. This distinction ensures that we are consistently combining coefficients that describe the same level of physical response. A more extended derivation from the QEG Lagrangian can be found at Appendix E.

Interpretation of the Result.

The emergence of Y_0 and α as the product of the three geometric invariants has a natural physical meaning. Each factor encodes a distinct and irreducible way in which the substrate constrains any event: the gauge channel ensures conservation and consistency, the flux channel ensures correct propagation and locality, and the capacitive channel ensures stability through self-interaction. Each factor is an irreducible geometric gate: gauge consistency (communication), unit-flux propagation (locality), and spherical storage (stability). The product law encodes a bottleneck: if any gate nearly closes, the global response closes. Thus, both Y_0 and α are the universal *geometric bottleneck* of spacetime—the minimal restrictions imposed by symmetry and topology on all admissible events.

IV. THE ANALYTIC EXPANSION OF SECOND-ORDER TERMS

A. Methodology for First-Order Corrections in QEG

A central result of this work is the derivation of the first-order correction coefficients, $C_1^{(X)}$, for the principal physical constants. Before presenting these calculations, it is crucial to clarify the methodology, as it differs from the standard perturbative approach in Quantum Field Theory (QFT).

While we may use the term "first-order" by analogy to the first term in a perturbative expansion in powers of α_0 , the calculations presented herein are *not* canonical "one-loop" computations in the QFT sense. A standard one-loop calculation in QFT involves evaluating divergent momentum integrals derived from Feynman rules, followed by regularization and renormalization. In contrast, the QEG framework, in the infrared (IR) and static limit, yields finite leading-order corrections determined by the geometric and statistical properties of the elastic substrate itself.

Our procedure is not based on ad hoc rules, but on a well-defined mapping derived from the principles of the theory:

1. **Corrections from compatibility of the retarded kernel (gauge sector).** For quantities directly tied to the emergent gauge structure (e.g., the fine-structure coupling α), the first-order correction follows from the *compatibility-normalized* retarded kernel in the static, transverse limit. As shown for $C_1^{(\alpha)} = 1$, the finite zero-momentum term of the transverse polarization is proportional to the bare coupling under the fixed normalizations of Sec. III, fixing the

slope without any ultraviolet subtractions or regularization. This is a direct consequence of enforcing current normalization and background compatibility in the effective action.

2. **Corrections from geometric averaging (static storage sector).** For static elastic moduli such as the torsional rigidity μ_0 , isotropy and homogeneity reduce the first-order correction to $O(3)$ -projector averages on the sphere of directions (S^2), at fixed shell weight, together with the spherical closure of Sec. III. In this setting, $C_1^{(\mu)} = 3/5$ is the unique, finite result of coarse-graining the first-order tensor interaction in an isotropic medium.
3. **Thermal sector (phenomenology beyond minimal quadratic closure).** For thermodynamic quantities like the Boltzmann constant k_B , the minimal quadratic closure does not fix the first-order slope. A phenomenological estimate based on Poissonian micro-exchanges within a causal cell suggests $C_1^{(k_B)} \approx e - 1$; we present it as an *operational* estimate (Appendix J) and do not use it in core derivations.

In summary, the QEG framework replaces the regularization/renormalization machinery of divergent integrals with finite procedures rooted in symmetry, geometry, and statistical coarse-graining. The appearance of simple rational numbers (e.g., 1, 3/5) reflects $O(3)$ geometry and compatibility normalization, not arbitrary rules.

B. Analytic expansion of second-order terms

Quantum corrections generically induce a trace anomaly, $T^\mu{}_\mu \propto \sum_i \beta_i(\alpha) \mathcal{O}_i$, which governs the logarithmic running of dimensionless couplings. This does not reintroduce intrinsic dimensionful parameters at minimal order. Our closure principle concerns the anomaly-free minimal limit²; higher-order corrections are treated perturbatively and absorbed into renormalized couplings.

Proposition IV.1 (Analyticity of dimensionless corrections.). *Assume: (i) locality and covariance of the effective action Γ , (ii) passivity and absence of new relevant scales in the minimal closure limit, (iii) a single dimensionless bare parameter α_0 fixed at leading order, and (iv) standard background-field renormalization with counterterms organized by order. Then any renormalized dimensionless quantity X (coupling or constant extracted from Γ) admits, in a neighborhood of $\alpha_0 = 0$, an analytic expansion in powers of α_0 :*

$$X = X_{(0)} \left(1 + x_1 \alpha_0 + x_2 \alpha_0^2 + \dots \right),$$

² By "anomaly-free minimal limit," we refer to the static, infrared (IR) regime where (i) no new relevant mass scales are introduced, (ii) all corrections can be organized as an analytic power series in α_0 , and (iii) any logarithmic dependencies from renormalization are fully absorbed into the running of the coupling $\alpha(\mu)$, as described by the trace anomaly. Our analysis focuses on the fixed-point structure described by this power series.

With scheme-dependent coefficients x_n determined by angular averages, projector traces and local operator mixing. Logarithms in intermediate steps appear only through the renormalization scale and are resummed into the running $\alpha(\mu)$; no non-analytic dependence on new scales arises at the closure level.

Proof. By principles (i)-(iv), the effective action Γ is a local functional that depends analytically on the single dimensionless parameter α_0 in the IR limit. Standard power counting arguments ensure that all corrections can be organized as a formal power series in α_0 . Any dimensionless observable extracted from Γ will therefore also be analytic in α_0 near $\alpha_0 = 0$. Logarithms associated with scale dependence, $\ln(\mu/\mu_0)$, are fully absorbed into the running of $\alpha(\mu)$ via the beta function, preserving the analyticity of observables with respect to the bare parameter α_0 at the closure level. \square

As a consequence, once α_0 is fixed at leading order and no extra small parameters are introduced, *all* higher corrections must be expressible as a power series in α_0 . We adopt the following self-consistent parameterization:

$$\begin{aligned} \alpha &= \alpha_0 \left(1 + C_1 \alpha_0 + C_2 \alpha_0^2 + \dots \right), \\ Y_0 &= Y_{0,0} \left(1 + D_1 \alpha_0 + D_2 \alpha_0^2 + \dots \right), \\ \dots & \\ X &= X_{(0)} \left(1 + x_1 \alpha_0 + x_2 \alpha_0^2 + \dots \right) \end{aligned} \quad (\text{IV.1})$$

with C_i, D_i, \dots dimensionless geometric coefficients encoding higher-derivative, nonlinear, and curvature-induced corrections. The series is to be read as an asymptotic/analytic expansion around the minimal, scale-free solution.

Throughout the present work, first-order coefficients C_1 come out as exact rationals or simple geometric constants. This is expected: in an isotropic setting, projector traces and angular averages on S^2 produce rational weights, and the fixed normalizations of Sec. III pin down the finite constants in the retarded kernel. Several constants appear via independent routes; the consistency of the extracted C_1 values is therefore a non-trivial coherence check of the closure scheme (no hidden parameters are introduced).

As a concrete application of this analytic framework, we now turn to the fine-structure coupling α . Its *first-order geometric* determination provides a direct check of the universal expansion form and fixes the canonical slope $C_1^{(\alpha)}$.

First-order geometric determination of $C_1^{(\alpha)}$

Proposition IV.2. *Let α_0 be the dimensionless geometric coupling fixed at minimal order. Define the physical fine-structure coupling α operationally from the static transverse response of the torsional sector, i.e. from the $k \rightarrow 0$ coefficient of the $1/k^2$ interaction mediated by the transverse propagator D_T between two conserved probe sources. Then*

$$\alpha = \alpha_0 \left(1 + \alpha_0 + \mathcal{O}(\alpha_0^2) \right) \Rightarrow C_1^{(\alpha)} = 1.$$

Proof. Work at zero frequency and small momentum in background-field gauge (so that background compatibility and current normalization are preserved). The compatibility-normalized transverse kernel can be written as

$$D_T(0, \mathbf{k}) = \frac{1}{Z_s \mathbf{k}^2 - \Pi_T(0, \mathbf{k})} \mathbb{P}_T,$$

where Z_s is the spatial stiffness of the torsional mode at minimal order, Π_T is the transverse polarization kernel, and \mathbb{P}_T the transverse projector. By locality and analyticity, the finite constant term of the transverse polarization admits the small- k expansion

$$\Pi_T(0, \mathbf{k}) = Z_s \alpha_0 + \mathcal{O}(\alpha_0 \mathbf{k}^2) + \mathcal{O}(\alpha_0^2),$$

fixed by the same compatibility normalization that defines the current and the minimal coupling in Sec. III. Inserting this into D_T ,

$$\begin{aligned} D_T(0, \mathbf{k}) &= \frac{1}{Z_s} \frac{1}{\mathbf{k}^2 (1 - \alpha_0)} \mathbb{P}_T = \\ &= \frac{1}{Z_s} \frac{1}{\mathbf{k}^2} \left(1 + \alpha_0 + \mathcal{O}(\alpha_0^2) \right) \mathbb{P}_T, \end{aligned} \quad (\text{IV.2})$$

and reading off the static $1/\mathbf{k}^2$ coefficient yields $\alpha = \alpha_0(1 + \alpha_0 + \dots)$. \square

Remark IV.3 (Scope and scheme). *The result depends on (i) defining α from a physical static response (Kubotype), and (ii) fixing the finite piece of $\Pi_T(0, 0)$ by the same compatibility normalization that defines α_0 . Any scheme preserving these compatibilities yields the same leading slope; small deviations in fits are naturally attributed to $\mathcal{O}(\alpha_0^2)$ terms.*

V. EMERGENCE OF DYNAMICS FROM GEOMETRIC CONSTRAINTS

Having established the substrate's static invariants and the composite admittance Y_0 in Sec. III, we now derive its dynamical constants. The logic proceeds in a linear chain:

$$Y_0 \Rightarrow Z_0 = \frac{1}{Y_0} \Rightarrow \mu_0, c \Rightarrow \varepsilon_0.$$

A. Step I: From Y_0 to Z_0

By definition, the vacuum impedance can be expressed in two equivalent ways:

$$Z_0 = \mu_0 c = \frac{1}{Y_0}. \quad (\text{V.1})$$

Since Y_0 has already been computed as

$$Y_{0,0} = \mathcal{N}_g \mathcal{N}_\Delta \mathcal{N}_k = \left(\frac{1}{4} \right) \left(\frac{1}{4\pi} \right) \left(\frac{1}{\frac{3}{5}4\pi} \right),$$

we obtain directly

$$Z_{0,0} = \frac{1}{Y_{0,0}} = (4)(4\pi) \left(\frac{3}{5} 4\pi \right) = \alpha_0^{-1} \cdot \sqrt{\frac{3}{5}} 4\pi \quad (\text{V.2})$$

Thus, the product $\mu_0 c$ is fixed once and for all:

$$\mu_0(\alpha) c(\alpha) = \left(\sqrt{\frac{3}{5}} 4\pi \right) \alpha_0^{-1} (1 + \mathcal{O}(\alpha_0)). \quad (\text{V.3})$$

B. Step II: Separating μ_0 and c

Derivation of the causal speed c

a. Set-up (minimal transverse sector). Let A_i be the transverse 1-form obtained from the projector in Sec. III.4. In Fourier variables (ω, \mathbf{k}) , passivity and isotropy imply the retarded kernel has the form

$$\mathcal{K}_T(\omega, \mathbf{k}; \alpha) = \left[Z_t(\alpha) \omega^2 - Z_s(\alpha) \mathbf{k}^2 \right] - i \alpha \mathcal{D}_0(\omega, \mathbf{k}), \quad (\text{V.4})$$

with $Z_t, Z_s > 0$ (elastic), a positive dissipative functional $\mathcal{D}_0 \geq 0$ (FDT/Kubo), and *single* dimensionless dissipation strength α in the minimal closure (Sec. III). The dispersion relation follows from $\det \mathcal{K}_T = 0$, hence at leading order in α

$$c^2(\alpha) = \frac{\omega^2}{\mathbf{k}^2} = \frac{Z_s(\alpha)}{Z_t(\alpha)} + \mathcal{O}(\alpha) \quad (\text{V.5})$$

We now determine the *relative* scaling of Z_t, Z_s with α by two independent routes. Both routes rely only on (i) the fixed normalizations N_g, N_Δ, N_k and (ii) causality/analyticity. Combining them yields $c(\alpha) \propto \alpha^{-4}$.

b. Route I (impedance vs. permeability). Define the vacuum wave impedance at zero frequency as the norm-level ratio of conjugate response coefficients for plane waves,

$$Z_0(\alpha) := \frac{\|E\|}{\|H\|} \Big|_{\omega \rightarrow 0^+, \mathbf{k} = \omega \hat{\mathbf{k}}/c} = \sqrt{\frac{Z_t(\alpha)}{Z_s(\alpha)}} Z_{\text{geom}},$$

$$Z_{\text{geom}} := \frac{N_g^{-1/2}}{N_\Delta^{1/2}}, \quad (\text{V.6})$$

where Z_{geom} is the *parameter-free* geometric factor fixed once and for all by (N_g, N_Δ) .³ Next, define the static magnetic permeability $\mu_0(\alpha)$ from the quadratic storage

functional (Sec. III.C),

$$U_B(\alpha) := \frac{1}{2\mu_0(\alpha)} \int_{\mathbb{R}^3} d^3x \|B\|^2,$$

$$B := \nabla \times A, \quad B_i = \epsilon_{ijk} \partial_j A_k, \quad (\text{V.7})$$

so that μ_0^{-1} is *quadratic-level* and inherits its normalization from N_k (storage) together with N_g, N_Δ :

$$\mu_0(\alpha) = \mu_{\text{geom}} \alpha^{+3} \left(1 + \mathcal{O}(\alpha) \right),$$

$$\mu_{\text{geom}} := N_g^{-1} N_\Delta^{-1} N_k^{+3/2} \quad (\text{V.8})$$

Lemma V.0.1. *In a scale-free, passive medium with a single dimensionless dissipation strength α , the low-frequency wave impedance scales as*

$$Z_0(\alpha) = Z_{\text{geom}} \alpha^{-1} \left(1 + \mathcal{O}(\alpha) \right). \quad (\text{V.9})$$

Proof. Let $\sigma(\omega; \alpha)$ denote the transverse conductivity-like kernel entering the retarded response. By passivity and the fluctuation–dissipation theorem, $\text{Im} \sigma(\omega; \alpha) \propto \alpha \omega$ near $\omega \rightarrow 0^+$ (no other small scales exist). Kramers–Kronig then implies $\text{Re} \sigma(\omega; \alpha) = \sigma_1 \alpha + \mathcal{O}(\omega^2)$ with a geometric constant σ_1 . For plane waves the ratio $\|E\|/\|H\|$ at small ω reduces to $(\text{Re} \sigma)^{-1}$ up to the fixed geometric normalization Z_{geom} set by (N_g, N_Δ) , hence $Z_0 \propto \alpha^{-1}$. \square

Using the exact identity $Z_0(\alpha) = \mu_0(\alpha) c(\alpha)$ (definition of impedance), combine (V.9) and (V.8):

$$c(\alpha) = \frac{Z_0(\alpha)}{\mu_0(\alpha)} =$$

$$\frac{Z_{\text{geom}}}{\mu_{\text{geom}}} \alpha^{-1-3} \left(1 + \mathcal{O}(\alpha) \right) = K_c \alpha^{-4} \left(1 + \mathcal{O}(\alpha) \right), \quad (\text{V.10})$$

with the *dimensionless* geometric prefactor $K_c := Z_{\text{geom}}/\mu_{\text{geom}}$ fully determined by (N_g, N_Δ, N_k) .

c. Route II (Weyl covariance of the retarded kernel). Consider the Weyl rescaling $(t, \mathbf{x}) \mapsto (\lambda t, \lambda \mathbf{x})$ in the scale-free IR theory. The retarded kernel (V.4) must transform homogeneously so that the quadratic action retains Weyl weight zero after inserting the fixed normalizations (N_g, N_Δ, N_k) . Causality (Kramers–Kronig) and passivity imply the *same* dissipative prefactor multiplies ω^2 and \mathbf{k}^2 at small arguments:

$$\mathcal{D}_0(\omega, \mathbf{k}) = \gamma(\omega^2 + \mathbf{k}^2) + \mathcal{O}(\omega^4, \mathbf{k}^4, \omega^2 \mathbf{k}^2), \quad \gamma > 0, \quad (\text{V.11})$$

otherwise the light cone would tilt under rescaling (hyperbolicity loss). Hence

$$\mathcal{K}_T(\omega, \mathbf{k}; \alpha) = [Z_t(\alpha) - i\alpha\gamma] \omega^2 - [Z_s(\alpha) + i\alpha\gamma] \mathbf{k}^2 + \dots \quad (\text{V.12})$$

Weyl covariance with $(t, \mathbf{x}) \rightarrow (\lambda t, \lambda \mathbf{x})$ forces the *ratio* of elastic to dissipative coefficients to be *scale-*

³ This follows from the canonical normalization of the gauge sector (unit LSZ residue, Lemma A in §III) and the unit-flux Green kernel (Lemma B in §III), which fix the relative norms of the electric-like and magnetic-like quadratic forms; the ratio of those norms is Z_{geom} .

invariant:⁴

$$\frac{Z_t(\alpha)}{\alpha} = C_t, \quad \frac{Z_s(\alpha)}{\alpha} = C_s, \\ C_t, C_s \text{ constants set by } (N_g, N_\Delta, N_k) \quad (\text{V.13})$$

Analyticity around $\alpha = 0$ (Sec. III.D) then implies

$$Z_t(\alpha) = C_t \alpha^{+1} (1 + \mathcal{O}(\alpha)), \quad Z_s(\alpha) = C_s \alpha^{+1} (1 + \mathcal{O}(\alpha)) \quad (\text{V.14})$$

Insert (V.14) into (V.5) and match to the constitutive identities $Z_0 = \mu_0 c$ and $\mu_0 \sim \alpha^{+3}$ (quadratic \rightarrow linear passage uses $\sqrt{N_k}$ as in §III.D): solving the consistency system yields

$$c(\alpha) = \frac{Z_s(\alpha)}{Z_t(\alpha)} \cdot \frac{Z_0(\alpha)}{\mu_0(\alpha)} \propto \alpha^{+0} \cdot \alpha^{-4} = \alpha^{-4}, \quad (\text{V.15})$$

consistent with (V.10).

d. *Main result (unique exponent).*

Theorem V.1 (Causality, passivity, Weyl covariance \Rightarrow unique scaling of c). *In the minimal, scale-free closure of QEG with a single dimensionless dissipation strength α , the causal speed obeys*

$$c(\alpha) = K_c \alpha^{-4} \left(1 + C_1 \alpha + C_2 \alpha^2 + \dots \right), \\ K_c = \frac{Z_{\text{geom}}}{\mu_{\text{geom}}} > 0 \quad (\text{V.16})$$

where K_c is a pure geometric constant fixed by the previously derived invariants (N_g, N_Δ, N_k) . No alternative exponent $n \neq 4$ is compatible with the simultaneous requirements of (i) Kramers–Kronig dispersion with a single small parameter (passivity/analyticity), (ii) Weyl covariance of the quadratic action after fixing (N_g, N_Δ, N_k) , and (iii) absence of any extra dimensionless invariant in the IR.

Remark V.2 (On uniqueness). *If $Z_0(\alpha) \propto \alpha^{-p}$ and $\mu_0(\alpha) \propto \alpha^q$ in a single-parameter, scale-free regime, then Kubo (low-frequency dissipation) fixes $p = 1$, while the quadratic-to-linear passage in the storage channel (Sec. III.C–D) fixes $q = 3$. Hence $c = Z_0/\mu_0 \propto \alpha^{-1-3} = \alpha^{-4}$. Any different pair (p, q) would either (a) violate passivity/dispersion (if $p \neq 1$), or (b) inject an additional invariant in the storage channel (if $q \neq 3$), contradicting minimal completeness.*

Derivation of the Torsional Rigidity μ_0

Having established the substrate's causal speed, we now fix the torsional rigidity μ_0 from the same closure principles. The torsional (shear) sector is transverse by construction, so both the static quadratic energy and the static propagator are controlled by the transverse

projector $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ on the sphere of directions $\hat{\mathbf{k}} \in S^2$.

1. *Projector and angular averages on S^2 .* We work at $L = 1$ (natural units) within a causal cell and use the standard Fourier convention $f(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{k})$. Two identities hold in $d = 3$:

$$\text{tr } P(\hat{\mathbf{k}}) = 2, \quad \langle P_{ij}(\hat{\mathbf{k}}) \rangle_{S^2} = \frac{2}{3} \delta_{ij}, \quad (\text{V.17})$$

and, for quadratic contractions,

$$\langle P_{ij}(\hat{\mathbf{k}}) P_{ij}(\hat{\mathbf{k}}) \rangle_{S^2} = \langle \text{tr } P^2 \rangle_{S^2} = \langle \text{tr } P \rangle_{S^2} = 2, \\ \langle \hat{k}_i \hat{k}_j \rangle_{S^2} = \frac{1}{3} \delta_{ij}. \quad (\text{V.18})$$

The first average encodes that shear lives in a two-dimensional polarization subspace and the second that quadratic projections carry a 1/3 angular weight in $d = 3$.

2. *Static energy in Fourier space and spherical closure.* For a static transverse configuration ψ ,

$$E_{\text{shear}}[\psi] = \frac{\mu_0}{2} \int d^3x (\nabla_\perp \psi)^2 = \\ \frac{\mu_0}{2} \int \frac{d^3k}{(2\pi)^3} k^2 \psi_i(-\mathbf{k}) P_{ij}(\hat{\mathbf{k}}) \psi_j(\mathbf{k}) \quad (\text{V.19})$$

To fix the *global* normalization from the *local* quadratic density we adopt the same spherical closure used in the longitudinal/compressive sector: evaluate the energy on an isotropic, thin spherical shell in k -space, $|\mathbf{k}| = k_0$ with angular average on S^2 , and compare with the static interaction kernel extracted from the transverse propagator. For an isotropic packet normalized on the shell,

$$\int \frac{d^3k}{(2\pi)^3} \rightarrow \frac{1}{(2\pi)^3} k_0^2 \Delta k \int d\Omega, \\ \langle \psi_i(-\mathbf{k}) \psi_j(\mathbf{k}) \rangle_{S^2} = \frac{A_0}{k_0^2} \frac{\delta_{ij}}{3}, \quad (\text{V.20})$$

with A_0 a dimensionless shell weight fixed by the causal cell normalization and by the requirement that the same mode, when read from the static Green function, reproduces the canonical long-range $1/r$ behavior. Using (V.17)–(V.18) in (V.19) yields

$$E_{\text{shear}} = \frac{\mu_0}{2} \frac{k_0^2 \Delta k}{(2\pi)^3} \int d\Omega k_0^2 \frac{A_0}{k_0^2} \frac{\delta_{ij}}{3} P_{ij}(\hat{\mathbf{k}}) = \\ \frac{\mu_0}{2} \frac{k_0^2 \Delta k}{(2\pi)^3} A_0 \underbrace{\int d\Omega \frac{2}{3}}_{(4\pi) \cdot \frac{2}{3}} \quad (\text{V.21})$$

3. *Matching to the static propagator.* The static transverse propagator is

$$D_T(0, \mathbf{k}) = \frac{1}{\mu_0 k^2} P(\hat{\mathbf{k}}), \quad (\text{V.22})$$

so the induced interaction between two conserved probe sources carries the long-range kernel $V(r) \propto$

⁴ Because $(\omega, \mathbf{k}) \rightarrow (\omega/\lambda, \mathbf{k}/\lambda)$, the entire bracket in (V.12) must scale as λ^{-2} times a *constant* matrix to preserve the action's Weyl weight after inserting the fixed, dimensionless normalizations; any residual λ dependence would introduce a new invariant.

$\int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{P(\hat{\mathbf{k}})}{\mu_0 k^2} = \frac{1}{4\pi\mu_0} \frac{1}{r} \langle P(\hat{\mathbf{k}}) \rangle_{S^2}$. With the compatibility normalization used across channels (Appendix D.1), the causal/angular regularization contributes the universal factor $1/(2\pi)$ per quadratic projector average⁵, so that the *effective* isotropic coefficient of the $1/r$ kernel is

$$\frac{1}{4\pi\mu_0} \times \frac{2}{3} \times \frac{1}{2\pi} = \frac{1}{\mu_0} \frac{1}{6\pi^2}. \quad (\text{V.23})$$

Demanding spherical closure—i.e. matching the isotropic $1/r$ coefficient to the canonical unit normalization used to define the physical coupling (so that the torsional channel closes on the same $1/r$ unit as the longitudinal one)—fixes

$$\frac{1}{\mu_0} \frac{1}{6\pi^2} = 1 \quad \implies \quad \boxed{K_\mu = \frac{\pi^2}{3}}. \quad (\text{V.24})$$

Equivalently, inserting (V.24) back in (V.21) gives precisely the canonical spherical storage expected from the closure principle, showing that the same geometric/causal normalization controls both the energy and the interaction kernel.

Remark V.3 (Physical content and basis independence). *Equation (V.24) follows from: (i) transversality (projector P), (ii) isotropy (S^2 averages), (iii) the causal/angular regularization $1/(2\pi)$ used uniformly across channels, and (iv) matching the $1/r$ long-range normalization (spherical closure). None of these steps depends on a particular polarization basis; the only basis choice is encoded in P , whose traces are invariant. Hence K_μ is fixed canonically.*

4. *The Scaling Law:* $\mu_0 \propto \alpha^3$ A scaling of μ_0 with α follows the same causal-geometric logic as the derivation of $c(\alpha)$, but in a static setting. The crucial distinction is:

- $c(\alpha)$ is a *dynamic* property, determined in the full 4D action.
- $\mu_0(\alpha)$ is a *static* elastic modulus, defined on a 3D spatial slice at fixed time.

The conservative energy of a torsional deformation is

$$E_{\text{shear}}[\psi] \sim \frac{\mu_0(\alpha)}{2} \int_{\mathbb{R}^3} |\nabla_\perp \psi|^2 d^3x,$$

so the rigidity $\mu_0(\alpha)$ must compensate the Weyl weight of the 3D volume element ($d^3x \sim \lambda^3$). Since the dissipative kernel is linear in α in 4D, projecting it onto a 3D slice introduces three additional powers to balance the dimensional weight, giving

$$\mu_0(\alpha) \propto \alpha^3.$$

An easy cross-check for the above conclusion can be performed just noting that Eq. V.3, together with the

previous scaling $c \propto \alpha^{-4}$ (Eq. V.15), forces $\mu_0 \propto \alpha^3$ for consistency.

As a result, we finally get that

$$\boxed{\mu_0(\alpha) = \frac{\pi^2}{3} \alpha_0^3 \left(1 + D_1 \alpha_0 + D_2 \alpha_0^2 + \dots\right)} \quad (\text{V.25})$$

C. Closure with Z_0 and induced value of K_c .

Eq. V.3 enforces

$$K_\mu K_c = \sqrt{\frac{3}{5}} 4\pi$$

Using (V.24) we obtain the induced causal prefactor

$$\boxed{K_c = \frac{\sqrt{\frac{3}{5}} 4\pi}{K_\mu} = \frac{3}{\pi^2} \sqrt{\frac{12\pi}{5}} = \frac{2}{\pi} \sqrt{\frac{27}{5\pi}}} \quad (\text{V.26})$$

Thus K_μ is fixed purely geometrically by volumetric isotropy and angularization, and K_c follows uniquely from the closure constraint (V.3), with no additional freedom introduced.

Consistency check. The canonical value in (V.26) follows from spherical closure and is used throughout the main text. As a physically motivated check, if the vacuum is modeled—in the sense of the QEG principles recalled in the Introduction—as a network of coupled quantum harmonic oscillators, two independent ingredients naturally appear: (i) an angular/spectral factor $2/\pi$ (Wallis-type), and (ii) an amplitude modifier $\sqrt{e-1}$ from connected multi-mode contributions. Under that operational viewpoint one expects

$$K_c^{(\text{op})} \longrightarrow \frac{2}{\pi} \sqrt{e-1},$$

see Appendices H–I for a concise justification. This check is not used elsewhere and does not alter the canonical closure line.

D. Step III: Derivation of ε_0

Finally, the standard kinematic identities

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}, \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}},$$

together with $Z_0 = \mu_0 c$, uniquely determine ε_0 . Substituting the scalings of μ_0 and c , we obtain

$$\varepsilon_0(\alpha) = \frac{1}{\mu_0(\alpha) c(\alpha)^2} = \frac{\alpha_0^5}{K_\mu K_c^2} \left(1 + \mathcal{O}(\alpha_0)\right) \quad (\text{V.27})$$

With the calibrations fixed above,

$$\boxed{\varepsilon_0(\alpha) = \left(\frac{5\pi}{36}\right) \cdot \alpha_0^5 \left(1 + \mathcal{O}(\alpha_0)\right)} \quad (\text{V.28})$$

⁵ This accounts for the phase-space angular cell used in the closure construction and is the same factor appearing in the longitudinal sector; see Sec. IX. and Appendix F

Summary

The chain of deductions is therefore:

$$Y_0 \Rightarrow Z_0 \Rightarrow \mu_0 \propto \alpha^3, \quad c \propto \alpha^{-4} \Rightarrow \varepsilon_0 \propto \alpha^5.$$

All four constants $(\alpha, \mu_0, c, \varepsilon_0)$ are thus interlocked by geometric closure and causal scaling, with no additional freedom.

E. The analytic expansion of second-order terms

First-order geometric correction to the torsional rigidity: $C_1^{(\mu)}$

Proposition V.4 (Static torsional slope). *Let the transverse projector be $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ and define the static transverse BB kernel as $\Gamma_{BB}^T = \Gamma_{BB,0}^T (1 + \alpha_0 b_B + \dots)$. Extract the rigidity from the static IR limit, $\mu_0^{-1} = \lim_{\omega \rightarrow 0, k \rightarrow 0} \Gamma_{BB}^T / k^2$. Then*

$$\mu_0 = \mu_{0,0} (1 + \alpha_0 C_1^{(\mu)} + \dots), \quad C_1^{(\mu)} = -b_B,$$

and, at first-order geometric,

$$\boxed{C_1^{(\mu)} = \frac{3}{5}}$$

Proof. Work at $L = 1$. The static quadratic energy for a transverse deformation ψ reads

$$E_{\text{shear}} = \frac{\mu_0}{2} \int \frac{d^3 k}{(2\pi)^3} k^2 \psi_i(-\mathbf{k}) P_{ij}(\hat{\mathbf{k}}) \psi_j(\mathbf{k}).$$

Following the geometric averaging method, the first-order correction to μ_0^{-1} is controlled by the minimal two-derivative insertion in the quadratic energy functional. The angular content reduces to S^2 averages of $P_{ij} \hat{k}_\ell \hat{k}_m$ and their quadratic contractions. Using the standard identities in $d = 3$,

$$\begin{aligned} \langle \hat{k}_i \hat{k}_j \rangle_{S^2} &= \frac{1}{3} \delta_{ij}, \\ \langle \hat{k}_i \hat{k}_j \hat{k}_\ell \hat{k}_m \rangle_{S^2} &= \frac{1}{15} (\delta_{ij} \delta_{\ell m} + \delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell}), \end{aligned} \quad (\text{V.29})$$

and $\text{tr } P = 2$, $\langle P_{ij} \rangle_{S^2} = \frac{2}{3} \delta_{ij}$, $\langle P_{ij} P_{ij} \rangle_{S^2} = 2$, a minimal cubic vertex with two spatial derivatives yields an

isotropic contraction proportional to ⁶

$$\underbrace{\frac{2}{3}}_{\text{transverse rank}} \times \underbrace{\left(\frac{5}{6}\right)}_{\text{quartic angular average}} = \frac{5}{9},$$

while the corresponding longitudinal (reference) normalization carries $\frac{1}{3}$. The ratio of transverse to reference weights is therefore $\frac{5/9}{1/3} = \frac{5}{3}$. This relative enhancement of the kernel corresponds to a correction coefficient b_B . The conventions for comparing transverse and longitudinal channels in this framework, together with the algebraic inversion to get μ_0 from μ_0^{-1} , fix this coefficient as $b_B = -3/5$. Hence, $C_1^{(\mu)} = -b_B = 3/5$. Passivity (convexity of energy) fixes the overall sign. \square

Remark V.5 (Methodology). *The calculation relies only on identities on S^2 and projector traces, consistent with the geometric averaging principle outlined in Sec. IV A. The result is basis-independent and derived purely from the geometric rules of the theory.*

First-order geometric correction to the impedance: $C_1^{(Z_0)}$

Operational definition. The first-order correction to the impedance is determined by the geometric response weights a_E and a_B . Following the geometric averaging method (Sec. IV A), these weights are calculated from the isotropic averages of the relevant tensor components on S^2 .

We define the IR vacuum impedance as

$$\begin{aligned} Z_0 &\equiv \sqrt{\frac{\mathcal{Z}_E(0,0)}{\mathcal{Z}_B(0,0)}}, \\ \Gamma^{(2)}[\mathbf{A}_T] &= \frac{1}{2} \int \frac{d\omega d^3 k}{(2\pi)^4} \left[\mathcal{Z}_E(\omega, k) |\mathbf{E}|^2 - \mathcal{Z}_B(\omega, k) |\mathbf{B}|^2 \right] \end{aligned} \quad (\text{V.30})$$

At first-order geometric,

$$\begin{aligned} \mathcal{Z}_E(0,0) &= \mathcal{Z}_{E,0} (1 + \alpha_0 a_E), \\ \mathcal{Z}_B(0,0) &= \mathcal{Z}_{B,0} (1 + \alpha_0 a_B), \end{aligned} \quad (\text{V.31})$$

so

$$\begin{aligned} \frac{Z_0}{Z_{0,0}} &= \sqrt{\frac{1 + \alpha_0 a_E}{1 + \alpha_0 a_B}} = 1 + \frac{\alpha_0}{2} (a_E - a_B) + \mathcal{O}(\alpha_0^2), \\ \boxed{C_1^{(Z_0)} = \frac{1}{2} (a_E - a_B)} & \end{aligned} \quad (\text{V.32})$$

At this order, the electric response weight a_E has

⁶ Angular sketch for the 5/6 factor: the minimal two-derivative BB insertion brings a quadratic transverse weight $\langle \sin^2 \theta \rangle_{S^2} = 2/3$ and a quartic normalization $\langle 1 - \cos^4 \theta \rangle_{S^2} = 1 - 1/5 = 4/5$; their ratio is $(2/3)/(4/5) = 5/6$. This multiplies the transverse rank factor $\langle \text{tr } P \rangle / 3 = 2/3$ used in the proof.

two natural pieces:

$$a_E = a_E^{(\text{scalar})} + a_E^{(\text{mixed})},$$

with (i) a purely scalar/compressive quadratic contribution (dominant), and (ii) the lowest nontrivial scalar–transverse mixed correction (quartic):

a. Scalar (quadratic) piece: $a_E^{(\text{scalar})} = 1/3$. Let $\hat{\mathbf{k}}$ be the (longitudinal) direction and \mathbf{p} a fixed unit probe axis. Isotropy on S^2 gives the standard average

$$a_E^{(\text{scalar})} = \langle (\mathbf{p} \cdot \hat{\mathbf{k}})^2 \rangle_{S^2} = \langle \cos^2 \theta \rangle_{S^2} = \frac{1}{3}. \quad (\text{V.33})$$

b. Mixed (quartic) piece: $a_E^{(\text{mixed})} = 1/15$. The lowest isotropic scalar–transverse mixing is captured by the quartic correlator that ties the longitudinal axis to the transverse plane. In coordinates with $\mathbf{p} \parallel \hat{\mathbf{z}}$,

$$a_E^{(\text{mixed})} = \langle \hat{k}_x^2 \hat{k}_y^2 \rangle_{S^2}. \quad (\text{V.34})$$

The isotropic fourth–moment tensor on S^2 is

$$\langle \hat{k}_i \hat{k}_j \hat{k}_\ell \hat{k}_m \rangle = \frac{1}{15} (\delta_{ij} \delta_{\ell m} + \delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell}). \quad (\text{V.35})$$

Setting $(i, j, \ell, m) = (x, x, y, y)$ in (V.35) yields

$$\langle \hat{k}_x^2 \hat{k}_y^2 \rangle = \frac{1}{15} (\delta_{xx} \delta_{yy} + 0 + 0) = \frac{1}{15}. \quad (\text{V.36})$$

(Equivalently, the well–known identities $\langle \hat{k}_z^4 \rangle = 1/5$ and $\langle \hat{k}_x^2 \hat{k}_y^2 \rangle = B$ with $A + 2B = \langle \hat{k}_z^2 \rangle = 1/3$ and $A = 3B$ give $B = 1/15$.)

Electric weight and impedance slope. Combining (V.33) and (V.36) gives

$$a_E = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}. \quad (\text{V.37})$$

Geometric fixing of the transverse weight. The magnetic response weight a_B , like the electric one, follows from geometric averaging of projectors on S^2 . Let $\hat{\mathbf{k}} \in S^2$ be a unit direction and define $\langle f \rangle_{S^2} \equiv \frac{1}{4\pi} \int_{S^2} f(\hat{\mathbf{k}}) d\Omega$. For any fixed probe $\mathbf{u} \in \mathbb{R}^3$,

$$a_L \equiv \frac{\langle \|(\hat{\mathbf{k}} \hat{\mathbf{k}}^\top) \mathbf{u}\|^2 \rangle_{S^2}}{\|\mathbf{u}\|^2}, \quad a_T \equiv \frac{\langle \|(\mathbb{I} - \hat{\mathbf{k}} \hat{\mathbf{k}}^\top) \mathbf{u}\|^2 \rangle_{S^2}}{\|\mathbf{u}\|^2}.$$

Using the isotropic identities $\langle \hat{k}_i \hat{k}_j \rangle_{S^2} = \delta_{ij}/3$ and $\langle (\hat{\mathbf{k}} \cdot \mathbf{u})^2 \rangle_{S^2} = \|\mathbf{u}\|^2/3$, one gets

$$a_L = \frac{1}{3}, \quad a_T = 1 - \frac{1}{3} = \frac{2}{3} \Rightarrow \frac{a_T}{a_L} = 2.$$

Hence, in any scheme where the reference scalar (longitudinal) contribution is normalized to unity ($a_E \equiv 1$), the purely transverse magnetic response is fixed to

$$a_B \equiv \frac{a_T}{a_L} = 2$$

As a result, one finally gets

$$a_E - a_B = \frac{2}{5} - 2 = -\frac{8}{5} \quad \Rightarrow \quad C_1^{(Z_0)} = \frac{1}{2} \left(\frac{2}{5} - 2 \right) = -\frac{4}{5}. \quad (\text{V.38})$$

Remark V.6 (Methodology and Parsimony). *The decomposition $a_E = a_E^{(\text{scalar})} + a_E^{(\text{mixed})}$ is a direct application of the geometric averaging principle (Sec. IV A). It uses only standard S^2 moments (no basis-dependent artifacts) and the minimal quartic invariant that correlates the longitudinal direction with the transverse plane. The procedure is therefore both parsimonious and physically motivated, while remaining independent of any input about c or μ_0 .*

First-order linearization rules and propagation of slopes

Let

$$Z_0 = \mu_0 c, \quad Y_0 = \frac{1}{Z_0}, \quad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}. \quad (\text{V.39})$$

Linearizing at $\mathcal{O}(\alpha_0)$ we obtain

$$C_1^{(Z_0)} = C_1^{(\mu)} + C_1^{(c)}, \quad (\text{V.40})$$

$$C_1^{(Y_0)} = -C_1^{(Z_0)}, \quad (\text{V.41})$$

$$C_1^{(\epsilon)} = -C_1^{(\mu)} - 2C_1^{(c)}. \quad (\text{V.42})$$

Hence, the two independent inputs obtained ($C_1^{\mu_0}$ and $C_1^{Z_0}$) fix all other C_1 's. Using Prop. V.4 ($C_1^{(\mu)} = \frac{3}{5}$) and Sec. V E ($C_1^{(Z_0)} = -\frac{4}{5}$) we obtain

$$C_1^{(c)} = C_1^{(Z_0)} - C_1^{(\mu)} = -\frac{7}{5}$$

$$C_1^{(\epsilon)} = -\frac{3}{5} - 2\left(-\frac{7}{5}\right) = \frac{11}{5}$$

Remark V.7 (Final observation: the emergence of a simple rational arithmetic). *A striking outcome of the whole construction is that, despite starting from heterogeneous inputs (geometric closure for K_μ , causality and scale freedom for the exponents, and explicit first-order geometric angular algebra for the slopes), the first-order corrections of all observables coalesce into a neat rational pattern:*

$$C_1^{(\mu)} = \frac{3}{5}, \quad C_1^{(Z_0)} = -\frac{4}{5}, \quad C_1^{(c)} = -\frac{7}{5}, \quad C_1^{(\epsilon)} = \frac{11}{5}$$

All corrections are integer multiples of $1/5$. This ‘‘arithmetic closure’’ is nontrivial and acts as a powerful internal consistency check of the framework: the same constants recur across independent channels and derivations, yet conspire to a tightly organized set of rational slopes. In particular, the electric response choice $a_E = 2/5$ —derived from standard S^2 moments—is precisely what makes the global structure lock into the above rational pattern, suggesting that the foundational

principles are not just self-consistent but facets of a deeper unifying mathematical structure.

VI. CONSISTENCY CHECK AND PREDICTION: THE UNIFIED RESPONSE IDENTITY AND THE GRAVITATIONAL CONSTANT G

Having derived the fundamental constants of the substrate (α , c , μ_0 , ε_0) from geometric first principles, we now arrive at the theory's most unifying result. All these constants are tied together by a single, exact identity.

Corollary VI.1 (Unified Response Identity). *The compressive (gravitational), torsional (shear), and capacitive (storage) responses of the substrate are equivalent. They satisfy the triple equality:*

$$\boxed{\frac{1}{16\pi c} \equiv \mu_0 \alpha^2 \equiv \frac{3}{5} 4\pi \varepsilon_0} \quad (\text{VI.1})$$

This identity shows that the apparently distinct response channels of the substrate are in fact different manifestations of the same underlying structure. It is therefore a nontrivial check of the internal consistency of the QEG axioms

A. Verification

Using the calibrated relations

$$c = \frac{3}{\pi^2} \sqrt{\frac{12\pi}{5}} \alpha_0^{-4}, \quad \mu_0 = \frac{\pi^2}{3} \alpha_0^3, \quad \varepsilon_0 = \left(\frac{1}{\frac{3}{5} 4\pi}\right) \left(\frac{\pi^2}{3}\right) \alpha_0^5,$$

each term evaluates to the same quantity:

$$\frac{1}{16\pi c} = \frac{\pi^2}{3} \alpha_0^5, \quad \mu_0 \alpha^2 = \frac{\pi^2}{3} \alpha_0^5, \quad \frac{3}{5} 4\pi \varepsilon_0 = \frac{\pi^2}{3} \alpha_0^5$$

B. Interpretation: Modal Equivalence

Equation (VI.1) reveals that spacetime's stiffness is the same, regardless of which mode of deformation is probed:

- **Gravitational (compressive):** resistance to curvature, quantified by $1/(16\pi c)$.
- **Electromagnetic torsional (shear):** resistance to magnetic shear, via $\mu_0 \alpha^2$.
- **Electromagnetic capacitive (storage):** compliance to electric polarization, via $\frac{3}{5} 4\pi \varepsilon_0$.

Thus the substrate possesses a single fundamental stiffness, whose gravitational, torsional, and capacitive faces are different projections of one and the same geometric constant.

C. Identification with the Gravitational Constant G

The triple equality is proven, and we have interpreted its meaning as a unification of the substrate's different response modes. The final step is to give a physical name to the constant value to which all three terms are equal.

Within our framework, the factor

$$\frac{1}{16\pi c} = \left(\frac{1}{4}\right) \cdot \left(\frac{1}{4\pi}\right) \cdot \frac{1}{c}$$

arises as the unique geometric decomposition of the medium's large-scale, static *compressive response*. We have already stated that the gauge factor $\frac{1}{4}$ originates from the canonical normalization of the quadratic gauge action, $\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ (Restriction R1), encoding the irreducible contribution of gauge invariance to any elastic deformation of the substrate. Also, we have seen how the flux factor $\frac{1}{4\pi}$ comes from Gauss normalization of the Green function in \mathbb{R}^3 (Restriction R2), encoding the isotropic spreading of influence from a point source, encoded in the fundamental solution $G(r) = 1/(4\pi r)$, and expressing how any curvature or compression of the substrate must dilute geometrically in three spatial dimensions.

The third component, the causal factor $1/c$, comes from the causal speed c governing the dynamic response of the substrate. Its reciprocal, $1/c$, carries the imprint of dissipation (α^4) combined with the geometric normalization of the transverse causal channel. Explicitly,

$$\frac{1}{c} = \frac{\pi^2}{3} \frac{1}{\sqrt{\frac{3}{5} 4\pi}} \alpha^4$$

This factor is the dynamical bridge: it links static compressive elasticity to the substrate's causal propagation properties.

As a result, it becomes natural to interpret the expression $\frac{1}{16\pi c}$ as the substrate's intrinsic *tension of curvature or compression*. It is precisely this quantity that plays the role of the gravitational coupling constant in the effective field equations in the QEG framework. We therefore identify

$$\boxed{G_{\text{geom}} \equiv \frac{1}{16\pi c}},$$

meaning that Newton's constant is not an arbitrary dimensional parameter but the manifestation of geometric closure: the combined effect of gauge normalization, flux isotropy, and causal scaling.

Equivalent Forms of the Gravitational Coupling

Besides its compressive definition $G_{\text{geom}} = 1/(16\pi c)$, the obtained identity reveals how the gravitational constant emerges with two equivalent faces, linked to the torsional (shear) and capacitive (storage) channels of the substrate:

$$G \equiv \mu_0 \alpha^2 = \frac{3}{5} 4\pi \varepsilon_0.$$

a. (i) *The torsional form* $G = \mu_0 \alpha^2$. Here the key factors are:

- μ_0 : the torsional rigidity of the vacuum, expressing its resistance to transverse shear deformations. It scales as $\mu_0 = \frac{\pi^2}{3} \alpha^3$ after fixing the calibration constant.
- α^2 : the fundamental damping factor, appearing squared because gravity couples to quadratic energy densities.

Physically, gravity in this picture is a second-order (two-vertex) static effect of the torsional sector: the substrate's conservative stiffness supplies the scale, while the squared dissipative coupling encodes the bilinear sourcing by quadratic energy densities.

b. (ii) *The capacitive form* $G = \frac{3}{5} 4\pi \varepsilon_0$. Here the factors are:

- ε_0 : the permittivity of the vacuum, quantifying its ability to store energy under polarization by electric-type deformations.
- $\frac{3}{5} 4\pi$: the universal spherical capacitance factor (Restriction R3), which encodes the optimal self-storage geometry allowed by the substrate.

This identity shows how the gravitational constant G admits a dielectric-geometric reading: it is not an arbitrary coupling but the macroscopic measure of the substrate's polarization capacity under the most efficient (spherical) closure.

Thus, the triple identity,

$$G \equiv \frac{1}{16\pi c} \equiv \mu_0 \alpha^2 \equiv \frac{3}{5} 4\pi \varepsilon_0,$$

demonstrate that gravity is not a separate, independent channel of interaction. Rather, it is the **universal residual stiffness** of the quantum-elastic substrate, the single geometric stiffness underlying compression, torsion, and storage, revealed differently in each mode of deformation. No single channel is privileged: the equality demonstrates that the gravitational constant is the universal residual stiffness equally accessible through compression, torsion, or capacitive storage.

VII. THE MINIMAL ACTION AND ITS CAUSAL-CAPACITIVE ORIGIN

To connect our geometric framework with quantum mechanics, we must identify the minimal quantum of action, \hbar . We demonstrate that \hbar is not an independent parameter but emerges from a profound self-consistency between the causal (4D) and capacitive (3D) properties of the substrate.

A. The Causal Origin: Action as a 4D Geometric Restriction

The Minimal Causal Volume

Let L be an arbitrary characteristic length. As we have established, the causal speed c is a **dimensionless**

geometric constant in our framework. This allows us to define the fundamental "*causal length element*" as $dx_{\text{causal}} \sim L/c$. This represents the minimal length for a self-consistent causal event, as any spatial extent L is effectively scaled by the causal factor $\frac{1}{c}$ which governs the propagation of information within that region. This is the natural length scale where space and time are unified by the causal structure.

The minimal 4-volume of interaction is the isotropic hypercube constructed from this single, fundamental causal length:

$$d^4x_{\text{min}}(L) = (dx_{\text{causal}})^4 = \frac{L^4}{c^4}. \quad (\text{VII.1})$$

This is the irreducible "pixel" of spacetime in which a complete, self-consistent interaction can occur.

From Minimal Volume to Minimal Action

In any field theory, the action, S , is the fundamental quantity whose minimization yields the equations of motion. It is defined as the integral of a density, \mathcal{L} , over a four-dimensional volume of spacetime:

$$S = \int \mathcal{L} d^4x.$$

The action S must have units of [Energy] \times [Time], while the density \mathcal{L} has units of [Energy]/[Volume].

a. (ii) *The Curvature Density of Spacetime.* What is the most fundamental density available in a purely geometric theory? It is the *curvature* of spacetime itself. As described by the Einstein-Hilbert action, the Lagrangian density for pure geometry is proportional to the Ricci scalar curvature, R , which has units of [Length] $^{-2}$.

$$\mathcal{L}_{\text{geom}} \propto R \implies [\mathcal{L}_{\text{geom}}] = [L]^{-2}.$$

Therefore, for a characteristic four-dimensional volume of spacetime of size L (i.e., $d^4x = L^4$), the minimal action associated purely with its geometry scales as:

$$S_{\text{geom}} = \mathcal{L}_{\text{geom}} \cdot L^4 = [L]^{-2} \cdot [L]^4 = [L]^2.$$

Substituting our minimal causal volume from Eq. (VII.1):

$$S_{\text{min}}(L) \sim [L]^{-2} \cdot \frac{L^4}{c^4} = \frac{L^2}{c^4}.$$

This expression gives the minimal action for a causal cell of *any* arbitrary size L .

b. *The Quantum of Action, \hbar .* In a natural unit system ($L = 1$), the minimal quantum of action becomes:

$$\hbar = K_{\hbar} \cdot \frac{1^2}{c_0^4} = \frac{K_{\hbar}}{c_0^4} \quad (\text{VII.2})$$

where K_h is a dimensionless geometric constant of order unity. Thus, the identity $\hbar \propto c^{-4}$ is the direct consequence of defining the minimal action from the minimal causal volume of the substrate.

B. The Capacitive Origin: Action as a 3D Volumetric Bound

We now construct an independent expression for the minimal action from a purely spatial perspective, based on the substrate's capacity to store deformation. This serves as a powerful consistency check for the causal definition derived previously.

a. The Principle of Volumetric Composition. The permittivity ε_0 quantifies the compliance of the substrate in a single polarization channel. By the Principle of Channel Independence (see C.8), the total isotropic compliance of a 3D spatial cell cannot be a sum but must be the invariant scalar built from the three orthogonal channels. Geometrically, this corresponds to the determinant of the compliance tensor:

$$\varepsilon_{\text{vol}} \equiv \det(\varepsilon_{ij}) = \varepsilon_0^3.$$

The cubic power is thus the consequence of isotropy and rotational invariance, representing the full volumetric capacity of a 3D spatial region to store deformation.

b. From Compliance to Action. Action in our framework is fundamentally an area quantity, $[S] = [L]^2$. The volumetric permittivity ε_0^3 is dimensionless. To form an action from this scalar compliance, one must contract it with the fiducial area scale L^2 arising from the reduction of the three-channel tensor to a scalar. This yields the cyclic action of a spatial cell:

$$S_{\text{cyclic}} \equiv K_{\varepsilon_{\text{vol}}} \cdot \varepsilon_0^3 L^2. \quad (\text{VII.3})$$

where $K_{\varepsilon_{\text{vol}}}$ is a dimensionless geometric constant of order unity. This represents the total deformability of the spatial cell, expressed in units of action. As the construction is based on *static* (ε_0) and *spatially complete* (volumetric) properties, it represents the action of a complete, self-contained unit of deformation, which is conceptually analogous to the action of a full cycle, h .

c. Result. Altogether, and using the identity $\hbar = \frac{h}{2\pi}$, the capacitive derivation yields

$$\hbar_{\text{capacitive}} = \frac{S_{\text{cyclic}}}{2\pi} = \frac{K_{\varepsilon_{\text{vol}}} \cdot \varepsilon_0^3 L^2}{2\pi} \quad (\text{VII.4})$$

which stands as the isotropic invariant imposing the minimal geometric restrictions on any action. Any alternative construction (e.g. ε_0^n with $n \neq 3$) would explicitly violate isotropy or tensorial consistency. This construction introduces no additional free parameters: \hbar is fixed entirely by the volumetric permittivity, the fiducial geometric area, and the universal phase factor 2π .

Causal-Capacitive Closure: A Consistency Test

The principle of a unified substrate demands that these two independent derivations for the minimal action must coincide, having the identity:

$$\hbar_{\text{causal}} = \hbar_{\text{capacitive}} \implies \frac{K_h}{c^4} = \frac{K_{\varepsilon_{\text{vol}}} \cdot \varepsilon_{0,0}^3}{2\pi}.$$

d. Verification. From the minimal causal cell we have

$$\hbar = \frac{L^2}{c^4} \implies h = 2\pi \hbar = \frac{2\pi L^2}{c^4}.$$

Using the calibrated laws

$$c = K_c \alpha_0^{-4}, \quad \mu_0 = \frac{\pi^2}{3} \alpha_0^3, \quad \varepsilon_0 = \frac{1}{\mu_0 c^2} = \frac{5\pi}{36} \alpha_0^5,$$

we obtain

$$\varepsilon_0^3 = \left(\frac{5\pi}{36}\right)^3 \alpha_0^{15} = \frac{125\pi^3}{46656} \alpha_0^{15}. \quad (\text{VII.5})$$

On the other hand, with our normalization for K_c ,

$$K_c = \frac{3}{\pi^2} \sqrt{\frac{3}{5}} 4\pi$$

we have the factorization

$$\begin{aligned} \frac{1}{c^4} &= \frac{1}{K_c^4} \alpha_0^{16} = \alpha_0^{15} \frac{1}{K_c^4} \alpha_0 = \alpha_0^{15} \frac{1}{K_c^4} \frac{1}{16\pi \sqrt{\frac{3}{5}} 4\pi} \\ &= \alpha_0^{15} \left[\frac{25}{11664} \pi^6 \cdot \frac{1}{16\pi \sqrt{\frac{3}{5}} 4\pi} \right] = \alpha_0^{15} \frac{25}{186624} \frac{\pi^5}{\sqrt{\frac{3}{5}} 4\pi}, \end{aligned} \quad (\text{VII.6})$$

where we used $K_c^4 = \frac{11664}{25} \pi^{-6}$ and $\alpha_0 = (16\pi \sqrt{\frac{3}{5}} 4\pi)^{-1}$ in our geometric closure.

Combining (VII.6) and (VII.5) in the consistency equation, we obtain

$$\begin{aligned} K_h \cdot \frac{25}{186624} \frac{\pi^5}{\sqrt{\frac{3}{5}} 4\pi} \alpha_0^{15} &= K_{\varepsilon_{\text{vol}}} \cdot \frac{125\pi^3}{46656} \alpha_0^{15} \cdot \frac{1}{2\pi} \\ \implies K_h \frac{\pi^3}{10\sqrt{\frac{3}{5}} 4\pi} &= K_{\varepsilon_{\text{vol}}} \end{aligned} \quad (\text{VII.7})$$

Geometric Justification and calibration of K_h and $K_{\varepsilon_{\text{vol}}}$

Normalization lemma (volumetric route). Let ε_{ij} be diagonal in an orthonormal polarization frame with identical eigenvalues ε_0 (isotropy), and let the volumetric composition be the determinant $\varepsilon_{\text{vol}} \equiv \det(\varepsilon_{ij}) = \varepsilon_0^3$. Define the cyclic action of a spatial cell by $S_{\text{cyclic}} \equiv K_{\varepsilon_{\text{vol}}} \varepsilon_0^3 L^2$ and set the *fiducial area* by the same spherical closure used in Y_0/Z_0 , i.e. the channel weights that

render $Z_{0,0}Y_{0,0} = 1$. With this choice, the volumetric normalization is *fixed* to

$$\boxed{K_{\varepsilon_{\text{vol}}} = 1} \quad (\text{VII.8})$$

Sketch. Under isotropy, the 3D compliance tensor reduces to a scalar ε_0 per channel and its determinant collapses to ε_0^3 . The same spherical cell used to normalize Y_0 (angular closure and channel independence) fixes the global volumetric weight, so no extra geometric multiplier survives when composing orthogonal channels. Thus the volumetric construction is *scheme-matched* to the Y_0/Z_0 normalization, implying $K_{\varepsilon_{\text{vol}}} = 1$.

Order-zero closure with the causal route. With $K_{\varepsilon_{\text{vol}}} = 1$, Eq. (VII.7) gives

$$\boxed{K_h \frac{\pi^3}{10\sqrt{\frac{3}{5}}4\pi} = 1 \implies K_h = \frac{10}{\pi^3} \sqrt{\frac{3}{5}} 4\pi} \quad (\text{VII.9})$$

First-order slope matching (capacitive vs. causal). We expand \hbar *directly* at fixed geometric normalizations,

$$\begin{aligned} \hbar_{\text{causal}} &= \frac{K_h}{c_0^4} \left(1 + C_1^{(\hbar_{\text{causal}})} \alpha_0 + \mathcal{O}(\alpha_0^2) \right), \\ \hbar_{\text{capacitive}} &= \frac{K_{\varepsilon_{\text{vol}}} \varepsilon_{0,0}^3}{2\pi} \left(1 + C_1^{(\hbar_{\text{capacitive}})} \alpha_0 + \mathcal{O}(\alpha_0^2) \right), \end{aligned} \quad (\text{VII.10})$$

so that all accumulations from c and ε_0 are absorbed into the *order-zero* constants K_h and $K_{\varepsilon_{\text{vol}}} \varepsilon_{0,0}^3 / (2\pi)$. Imposing $\hbar_{\text{causal}} = \hbar_{\text{capacitive}}$ order by order yields the *route-independent* slope

$$\boxed{C_1^{(\hbar_{\text{causal}})} = C_1^{(\hbar_{\text{capacitive}})} \equiv C_1^{(\hbar)}} \quad (\text{VII.11})$$

First-order geometric correction to the quantum of action: $C_1^{(\hbar)}$

Following the geometric averaging principle (Sec. IV A), the first-order correction in the static, IR limit is determined by the simplest available isotropic scalar. The only candidate that is *linear* in α_0 , dimensionless, and can correct the normalization of a scalar observable constructed from a single preferred axis (the causal direction, or a single polarization axis) is the quadratic projection of that axis onto a random spatial direction $\hat{\mathbf{k}}$:

$$\Xi \equiv \langle (\mathbf{p} \cdot \hat{\mathbf{k}})^2 \rangle_{S^2} = \langle \cos^2 \theta \rangle_{S^2} = \frac{1}{3}$$

Here \mathbf{p} is any fixed unit axis (time-like foliation or chosen polarization); by isotropy, Ξ is basis-independent. In the causal route, the $\mathcal{O}(\alpha_0)$ insertion that dresses the minimal 4D cell contributes multiplicatively with weight Ξ (angular average of the quadratic projector that ties the causal axis to the spatial slice). In the capacitive route, the unique linear, isotropic correction to the volumetric scalar built from three identical channels reduces—again by isotropy and background-field Ward identities—to the same quadratic projection fac-

tor Ξ that multiplies the route's order-zero constant. Thus,

$$C_1^{(\hbar_{\text{causal}})} = \Xi, \quad C_1^{(\hbar_{\text{capacitive}})} = \Xi,$$

and enforcing route equality at $\mathcal{O}(\alpha_0)$ yields $C_1^{(\hbar)} = \Xi = 1/3$.

Remark VII.1 (Why no other number can appear at linear order). *Any alternative linear correction would require either a basis-dependent tensor (forbidden by isotropy), a quartic angular invariant (suppressed to $\mathcal{O}(\alpha_0^2)$ in our normalization, since it needs an extra projector contraction), or an independent dimensionless parameter (forbidden by scale freedom at this order). Hence the unique linear scalar is $\langle \cos^2 \theta \rangle_{S^2} = 1/3$.*

Therefore, choosing the empirical/theoretical benchmark $C_1^{(\hbar)} = \frac{1}{3}$ fixes *both* routes consistently:

$$\boxed{C_1^{(\hbar_{\text{capacitive}})} = \frac{1}{3}, \quad C_1^{(\hbar_{\text{causal}})} = \frac{1}{3},} \quad (\text{VII.12})$$

with $K_{\varepsilon_{\text{vol}}} = 1$ and K_h given by (VII.9). This realizes the causal-capacitive closure at order zero and at first order without importing additional slope contributions from c or ε_0 into \hbar .

Physical interpretation. Equation $h \simeq \varepsilon_0^3 L^2$ states that the *quantum of action* per causal cell (an ‘‘area’’ L^2/c^4) is equivalent to the substrate's *volumetric capacitive response* times the geometric area scale L^2 . In other words, the minimal phase increment $\exp\{iS/\hbar\}$ across a causal cell and the vacuum's capacity to polarize in three independent directions are two facets of the same geometric bottleneck.

Corollary VII.2 (The Origin of Action). *The minimal quantum of action, \hbar , is the geometric invariant that reconciles the causal restriction of 4D spacetime ($1/c^4$) with the volumetric capacitive bound of its 3D spatial subspace ($\varepsilon_0^3/2\pi$). The near-perfect agreement at bare order, and the controlled deviation explained by higher-order terms, demonstrate that \hbar is not postulated but emerges as the unique invariant ensuring self-consistency between causality and capacity within the substrate.*

VIII. THE ELEMENTARY CHARGE AS DAMPED CAUSAL LENGTH

A. Principle of Charge as a Stabilized Excitation

The minimal causal action cell defines the causal length

$$S_{\text{min}}(L) = \frac{L^2}{c^4} = [L^2] \implies \ell_c \equiv \frac{L}{c^2},$$

the linear dimension associated with the square root of the minimal action. This is the ‘ideal’ geometric reach of a fundamental excitation.

However, the substrate is dissipative. Its universal damping ratio is $\zeta = \alpha$ (App. D), hence its quality

factor is

$$Q_{\text{geom}} = \frac{1}{2\alpha}$$

A localized excitation that survives dissipation must therefore be shorter than the ideal causal length by this factor. We are thus led to the operational identification that the elementary charge is the stabilized causal length,

$$e = \frac{\ell_c}{Q_{\text{geom}}} = K_{e_l} \cdot 2\alpha \ell_c$$

where K_{e_l} is a dimensionless constant of order unity.

A consistency check of the above can be reasoned as follows. Consider two conserved transverse probes coupled to the torsional gauge-like mode. In background-field gauge, the amputated static exchange at $k \rightarrow 0$ reads

$$\begin{aligned} \mathcal{M}_{\text{IR}}(0, \mathbf{k}) &= \left[\alpha \mathcal{V}_E \right] D_T(0, \mathbf{k}) \left[\alpha \mathcal{V}_E \right], \\ D_T(0, \mathbf{k}) &= \frac{1}{\mu_0 k^2} P(\hat{\mathbf{k}}), \end{aligned} \quad (\text{VIII.1})$$

where \mathcal{V}_E is the unit electric-type vertex and P the transverse projector. Causality (Kramers-Kronig, see App. A 4) and the background-field Ward identity fix the real static susceptibility to be linear in α and to carry the same causal weight $1/c^2$. In the IR, the projector average and Gauss normalization produce the universal $1/(4\pi r)$ kernel, with $\langle P_{ij} \rangle_{S^2} = \frac{2}{3} \delta_{ij}$. Collecting the two identical vertex insertions (one per leg) yields a factor of 2 at the level of the *coupling*, so that

$$\begin{aligned} e &\equiv \lim_{k \rightarrow 0} k^2 \chi_T(0, k), \\ \chi_T(0, k) &= \frac{\delta^2 W[J]}{\delta J_i(-k) \delta J_i(k)} \Big|_{J=0}, \end{aligned} \quad (\text{VIII.2})$$

$$e \equiv \lim_{k \rightarrow 0} k^2 \chi_T(0, k) = \frac{2\alpha}{c^2} \quad (\text{VIII.3})$$

under the canonical Gauss/spherical normalization (the same used in Y_0/Z_0). Equivalently, matching the $1/r$ coefficient to the Y_0/Z_0 Gauss normalization fixes the canonical choice $K_{e_\alpha} = 1$ in the shorthand $e = K_{e_\alpha} 2\alpha/c^2$.

B. Consistency with Volumetric Rigidity

Independently, the static rigidity of the substrate yields a volumetric torsional invariant rigidity of the substrate. The volumetric torsional invariant is

$$\frac{\mu_0^3}{4\pi},$$

a cubic measure aggregating three orthogonal shear channels with azimuthal normalization $1/(4\pi)$. Consistency demands

$$K_{e_\alpha} \cdot 2\alpha \ell_c = K_{e_\mu} \cdot \frac{\mu_0^3}{4\pi} \quad (\text{VIII.4})$$

The above can be reasoned as follows. Let J_i be a conserved transverse probe that couples linearly to the static shear field ψ_i , and let e denote the residue of the static long-range interaction,

$$\begin{aligned} e &\equiv \lim_{k \rightarrow 0} k^2 \chi_T(0, k), \\ \chi_T(0, k) &= \frac{\delta^2 W[J]}{\delta J_i(-k) \delta J_i(k)} \Big|_{J=0}, \end{aligned} \quad (\text{VIII.5})$$

with W the generating functional and the transverse projector understood. In the static, isotropic limit the only building blocks for the two-probe kernel are δ_{ij} , the transverse projector P_{ij} on S^2 , and scalar moduli. Assuming channel independence implies three identical shear gates; the unique rotational scalar that survives after angular closure is the *volumetric* invariant formed by the three gate eigenvalues. Since each gate carries stiffness μ_0 at leading order, the invariant scales as μ_0^3 .

In addition, the static Green function contributes the universal $1/k^2$ pole; its Fourier transform produces the $1/(4\pi r)$ kernel. The S^2 average of P_{ij} collapses to $(2/3)\delta_{ij}$, and the Gauss normalization used throughout (Y_0, Z_0) fixes the angular weight to $1/(4\pi)$ for the residue.

By QEG scaling, $\mu_0 \propto \alpha_0^3$, so any static scalar built solely from the rigidity must scale as α_0^{3n} . The dynamic/Kubo route (App. A 2) yields $e \propto \alpha_0^n$ (Sec. VIII, Eq. (VIII.3)); hence $n = 3$ is *forced*. No μ_0^n with $n \neq 3$ can match the residue's scaling and symmetries.

Combining the above and matching the $1/r$ coefficient with the same Gauss cell used elsewhere yields

$$e = K_{e_\mu} \frac{\mu_0^3}{4\pi}. \quad (\text{VIII.6})$$

Consequence. Eq. (VIII.6) is *not* a definition but the unique isotropic scalar compatible with tensorial reduction, Gauss normalization, scale selection and the canonical $1/r$ normalization. Its equality to the Kubo/causal result $e = 2\alpha/c^2$ then fixes the relative normalizations as in Eqs. (VIII.7)–(VIII.8).

Consistency of the two definitions of e . We posit

$$e = K_{e_\mu} \frac{\mu_0^3}{4\pi} \quad \text{and} \quad e = K_{e_\alpha} \frac{2\alpha_0}{c^2},$$

and use the leading scalings in α_0 written as a constant K_i times a power:

$$\mu_0 = K_\mu \alpha_0^3, \quad c = K_c \alpha_0^{-4} \quad \implies \quad \frac{1}{c^2} = \frac{1}{K_c^2} \alpha_0^8.$$

Hence both routes give the same α_0 -weight (α_0^9):

$$\begin{aligned} e &= K_{e_\mu} \frac{(K_\mu \alpha_0^3)^3}{4\pi} = K_{e_\mu} \frac{K_\mu^3}{4\pi} \alpha_0^9, \\ e &= K_{e_\alpha} \frac{2\alpha_0}{c^2} = K_{e_\alpha} \frac{2}{K_c^2} \alpha_0^9. \end{aligned}$$

Equating coefficients of α_0^9 yields the general relation between the normalizations:

$$\boxed{K_{e_\mu} \frac{K_\mu^3}{4\pi} = K_{e_\alpha} \frac{2}{K_c^2}} \iff \boxed{K_{e_\alpha} = \frac{K_{e_\mu} K_\mu^3 K_c^2}{8\pi}} \quad (\text{VIII.7})$$

Canonical insertion. With the canonical calibrations

$$K_\mu = \frac{\pi^2}{3}, \quad K_c = \frac{3}{\pi^2} \sqrt{\frac{3}{5}} 4\pi,$$

we have

$$K_\mu^3 K_c^2 = \left(\frac{\pi^2}{3}\right)^3 \cdot \left(\frac{3}{\pi^2} \sqrt{\frac{12\pi}{5}}\right)^2 = \frac{4\pi^3}{5}.$$

Therefore (VIII.7) reduces to the simple proportionality

$$\boxed{K_{e_\alpha} = \frac{\pi^2}{10} K_{e_\mu}} \iff \boxed{K_{e_\mu} = \frac{10}{\pi^2} K_{e_\alpha}}. \quad (\text{VIII.8})$$

Canonical normalization of the α -route: $K_{e_\alpha} = 1$. We define e operationally from the static long-range kernel between two conserved probes in the transverse sector. In background-field gauge, the amputated $k \rightarrow 0$ exchange reads (up to the universal $1/k^2$ factor)

$$\mathcal{M}_{\text{IR}} \propto \frac{2\alpha}{c^2},$$

where the factor 2 accounts for the two identical source insertions (one per leg), and $1/c^2$ is fixed by the causal normalization. Matching the coefficient of the $1/r$ kernel to the same Gauss-type normalization used in Y_0/Z_0 leaves no further angular/volumetric weight. Hence the *canonical* choice is

$$\boxed{K_{e_\alpha} = 1}, \quad e = \frac{2\alpha}{c^2}. \quad (\text{VIII.9})$$

Implication for the μ -route. From the equality of the two definitions, $e = K_{e_\mu} \mu_0^3/(4\pi) = K_{e_\alpha} 2\alpha/c^2$, and using the canonical calibrations $K_\mu = \pi^2/3$ and $K_c = \frac{3}{\pi^2} \sqrt{\frac{3}{5}} 4\pi$, we previously found (Eq. (VIII.8))

$$K_{e_\alpha} = \frac{\pi^2}{10} K_{e_\mu}.$$

With (VIII.9) this fixes

$$\boxed{K_{e_\mu} = \frac{10}{\pi^2}} \quad (\text{VIII.10})$$

First-order geometric correction to the elementary charge: $C_1^{(e)}$

Proposition VIII.1. *Define e from the amputated two-probe exchange and expand at fixed geometric normalizations so that propagator dressings are absorbed into the order-zero constants (vertex-dominated scheme, as in the \hbar analysis). Then*

$$\boxed{C_1^{(e)} = 2}.$$

Proof. Following the first principle of our methodology (Sec. IV A), this correction is fixed by the symmetries of the effective action. Background-gauge invariance implies the linear Ward identity

$$q_\mu \Gamma_{J\psi\psi}^\mu(p+q, p) = \Gamma^{(2)}(p+q) - \Gamma^{(2)}(p), \quad (\text{VIII.11})$$

In background-field gauge (BFG), the amputated two-probe correlator factorizes in the IR as

$$\begin{aligned} \Gamma_{JJ}^{(2)}(\omega, \mathbf{k}) &\xrightarrow{\text{IR, BFG}} [Z_V \alpha] D_T(\omega, \mathbf{k}) [Z_V \alpha], \\ Z_V &= 1 + C_1^{(\alpha)} \alpha_0 + \mathcal{O}(\alpha_0^2), \end{aligned} \quad (\text{VIII.12})$$

with D_T the transverse static propagator.⁷ Taking the static limit and matching to the $1/r$ kernel under the canonical Gauss normalization, the effective coupling extracted from the two-leg exchange is

$$\begin{aligned} e &\equiv \lim_{k \rightarrow 0} k^2 \chi_T(0, k) = K_{e_\alpha} \frac{2}{c_0^2} [\alpha Z_V]^2 \\ &= K_{e_\alpha} \frac{2\alpha}{c_0^2} \left(1 + 2C_1^{(\alpha)} \alpha_0 + \mathcal{O}(\alpha_0^2)\right) \end{aligned} \quad (\text{VIII.13})$$

Therefore, at fixed geometric normalizations (vertex-dominated scheme),

$$\boxed{C_1^{(e)} = 2C_1^{(\alpha)} = 2} \quad (\text{VIII.14})$$

since $C_1^{(\alpha)} = 1$ by the Ward identities of App. A.3. \square

Remark VIII.2. *If, instead, one lets the causal factor contribute explicitly at $\mathcal{O}(\alpha_0)$, the α -route gives $C_1^{(e)} = C_1^{(\alpha)} - 2C_1^{(e)} = 1 + \frac{14}{5} = \frac{19}{5}$, while the μ -route with fixed K_{e_μ} yields $C_1^{(e)} = 3C_1^{(\mu)} = \frac{9}{5}$. The vertex-dominated scheme used above (absorbing the propagator dressing into the order-zero constants) restores route equality and yields the compact, physically transparent result $C_1^{(e)} = 2$.*

⁷ Notation and IR conventions as in App. A.1–A.2.

C. Physical Meaning

Identity (VIII.4) packs a clear message:

- *Minimal causal reach vs. dissipation.* The factor $2\alpha\ell_c$ ties the dissipation budget to the minimal causal linear size of an interaction cell. Stronger dissipation (larger α) squeezes the admissible causal reach ℓ_c so as to keep the product fixed.
- *Cubic shear stiffness.* The right-hand side, $\mu_0^3/(4\pi)$, is a *cubic* torsional measure: it aggregates three, mutually orthogonal, transverse shear channels (reflecting the three independent geometric gates at play) with the azimuthal normalization $1/(4\pi)$ inherited from the polarization circle. In short, a ‘volume’ of shear rigidity.
- *Geometric trade-off (closure).* Once $c(\alpha)$ and $\mu_0(\alpha)$ are inserted, Eq. (VIII.4) becomes α -independent. This means it is a genuine *closure constraint*: the substrate trades off *causal extent* (ℓ_c) against *dissipation* (α) to match a fixed *rigidity volume* set by μ_0 .

Operationally, Eq. (VIII.4) can be read in two equivalent ways: (i) given (α, μ_0) it fixes the minimal causal line element ℓ_c ; or (ii) given (α, ℓ_c) it fixes the cubic rigidity scale $\mu_0^3/(4\pi)$. Either way, dissipation, causal reach, and torsional rigidity are not independent but different projections of the same geometric constant. As a result, the elementary charge e is a rigidity–causality invariant, equivalently defined as:

1. a **damped causal length**, $e = K_{e_l} \cdot 2\alpha\ell_c (1 + \mathcal{O}(\alpha_0))$;
2. a **volumetric rigidity**, $e = \mu_0^3/(4\pi) (1 + \mathcal{O}(\alpha_0))$.

Thus, e is not an empirical input but the unique invariant that reconciles dissipation, causality, and cubic rigidity in the substrate.

Remark VIII.3 (From description to prediction). *The interlocking constraints derived in Secs. III–VIII elevate the framework from a descriptive model to a predictive one: the constants $(\mu_0, c, \varepsilon_0, e)$ and their first-order slopes are not independently adjustable. They are fixed by a single input α plus geometric–causal closure. Consequently, a precise measurement of any one of these constants determines the others within this scheme, providing a clear and falsifiable test.*

IX. THE BOLTZMANN CONSTANT FROM EQUIPARTITION OF MODAL ENERGY

We now derive the Boltzmann constant, k_B , demonstrating it is not an independent thermal parameter but is instead fixed by the equipartition of energy within the substrate’s fundamental causal modes.

a. The Causal Cell as a TEM Element. To derive the energy of a fundamental mode, we model the minimal causal cell of size L as a canonical TEM transmission-line element. For a cubic cell, this corresponds to a waveguide section of length L with square cross-section $A = L^2$. This model provides the canonical geometric realization of a one-dimensional propagation channel within the 3D substrate. For such

a structure, the capacitance and inductance per unit length are fixed by the substrate’s intrinsic properties:

$$C' = \varepsilon_0, \quad L'_m = \mu_0.$$

The total capacitance and inductance of the cell are thus $C = \varepsilon_0 L$ and $L_m = \mu_0 L$. A more detailed justification can be consulted in Appendix G.

b. Canonical Energy of the Fundamental Mode. The fundamental mode of this element has a causal frequency $\omega_L = c/L$. The time-averaged stored electric and magnetic energies in this mode are equal:

$$\begin{aligned} \langle U_E \rangle &= \frac{1}{2} C V^2 = \frac{1}{2} (\varepsilon_0 L) V^2, \\ \langle U_B \rangle &= \frac{1}{2} L_m I^2 = \frac{1}{2} (\mu_0 L) I^2, \end{aligned} \quad (\text{IX.1})$$

with $V/I = Z_0 = \mu_0 c$. The total energy of the mode is $U_{\text{mode}} = \langle U_E \rangle + \langle U_B \rangle$. Crucially, scale invariance requires that the mode energy be independent of the fiducial length L . With the same canonical normalization used throughout the framework (e.g. for \hbar), the amplitudes are fixed such that

$$\langle U_{\text{mode}} \rangle = \frac{1}{2} \frac{\mu_0}{c^2}. \quad (\text{IX.2})$$

c. Equipartition and the Geometric Boltzmann Constant. By the equipartition theorem, the average thermal energy per quadratic degree of freedom is $\langle U_{\text{thermal}} \rangle = \frac{1}{2} k_B T$. Thermal equilibrium demands equality of modal and thermal energies. In a natural unit system with $T = 1$,

$$\langle U_{\text{mode}} \rangle = \langle U_{\text{thermal}} \rangle \implies \frac{1}{2} \frac{\mu_0}{c^2} = \frac{1}{2} k_B.$$

This yields the geometric Boltzmann constant without free parameters:

$$k_B^{(\text{geom})} = K_{K_B} \cdot \frac{\mu_0}{c^2} \quad (\text{IX.3})$$

where K_{K_B} is a geometric constant of order unity.

d. Consistency and closure. This derivation is the thermodynamic counterpart of the dynamic closure principle used earlier. It employs the same geometric gates (ε_0, μ_0) and introduces no additional invariants. Using reciprocity $\mu_0 \varepsilon_0 = 1/c^2$, the result may be equivalently written $k_B = \mu_0^2 \varepsilon_0$, highlighting that the Boltzmann constant, too, is a geometric invariant rooted in the same causal-elastic substrate.

A. Closure from $\mu_0 \cdot e$ and the geometric Boltzmann constant

Starting from the causal expression for the elementary charge (VIII.3)

$$e = \frac{2\alpha}{c^2},$$

the product with the torsional rigidity reads

$$\mu_0 \cdot e = \mu_0 \frac{2\alpha}{c^2} = 2\alpha \frac{\mu_0}{c^2} \equiv 2\alpha K_B, \quad (\text{IX.4})$$

Comparing with (IX.3) we obtain the *geometric* value of the normalization:

$$\boxed{K_{K_B} = 1} \quad (\text{IX.5})$$

Consistency. With the canonical calibrations $K_\mu = \pi^2/3$ and $K_c = \frac{3}{\pi^2} \sqrt{\frac{3}{5}} 4\pi$, the thermal scale inherits the expected weight and contains no extra angular/volumetric factor beyond those already fixed in Z_0 and e :

$$k_B = \frac{\mu_0}{c^2} = \frac{K_\mu}{K_c^2} \alpha_0^{11}$$

Thus, the equality $\mu_0 e \equiv 2\alpha k_B$ is a direct manifestation of geometric closure: the same causal-elastic gates that fix μ_0 , c , and e also pin down k_B with $K_{K_B} = 1$.

Remark IX.1. *The first-order geometric correction to the Boltzmann constant, $C_1^{(k_B)}$, is not fixed within the minimal quadratic closure defined in this framework. For completeness, a phenomenological model suggesting $C_1^{(k_B)} \approx e - 1$ is discussed in Appendix J.*

X. THE ORIGIN OF ZERO-POINT ENERGY FROM GEOMETRIC ADMITTANCE

The foundations of quantum mechanics, laid by the Schrödinger equation, predict a non-zero ground-state energy for any stable quantum system: the Zero-Point Energy, $E_0 = \frac{1}{2} \hbar \omega$. In the QEG framework, where the vacuum is a quantized elastic substrate, this energy represents the fundamental quantum "tension" of spacetime. In this section, we derive this energy from the theory's first principles, showing it is not an axiom but an inevitable structural identity linking the substrate's geometric admittance Y_0 to the hierarchical structure of spacetime.

A. Algebraic Derivation from Constitutive Identities

First, we derive an algebraic consequence of the constitutive relations derived throughout this framework:

$$E_0(L) = \frac{\hbar c}{2L} = Y_0^{10} L \quad (\text{X.1})$$

From the expressions derived throughout this Paper, one can check that the Vacuum Constitutive Equation $\frac{\hbar c}{L} \equiv e \cdot \mu_0 \equiv k_B \cdot T \cdot 2\alpha$ derived in [15] holds (assuming $L \equiv T$ as established in [15]). Substituting $e \equiv \frac{\mu_0^3}{4\pi} \cdot L$ and operating, we get that

$$\hbar c = \frac{\mu_0^4}{4\pi} \cdot L^2$$

$$\frac{\hbar \cdot c}{2} = \frac{\mu_0^4}{(4\pi)^2} \cdot L^2$$

$$\sqrt{\frac{\hbar \cdot c}{2}} = \frac{\mu_0^2}{4\pi} \cdot L$$

Note that, as $e = \frac{\mu_0^3}{4\pi} \cdot L$, then we have that

$$e = \sqrt{\frac{\hbar \cdot c}{2}} \cdot \mu_0$$

And thus, we have that

$$e \cdot c = \sqrt{\frac{\hbar \cdot c}{2}} \cdot \mu_0 \cdot c$$

As $Z_0 = \mu_0 \cdot c$, we can state that

$$e \cdot c = \sqrt{\frac{\hbar \cdot c}{2}} \cdot Z_0$$

Other hand, as a fundamental consequence of the relations $\hbar \equiv \epsilon_0^3 \cdot 1 \text{ m}^2$ and $K_B \equiv \frac{\mu_0}{c^2} = \epsilon_0 \cdot \mu_0^2$, one has that

$$\frac{\hbar}{k_B} \equiv \frac{\epsilon_0^3 \cdot L^2}{\epsilon_0 \mu_0^2} = Y_0^4 \cdot L^2 \quad (\text{X.2})$$

From the vacuum constitutive equation one has $\frac{\hbar}{K_B} \equiv e \cdot c \cdot L$, we have $e \cdot c \equiv Y_0^4 \cdot L$. Then, we have

$$\sqrt{\frac{\hbar \cdot c}{2}} \cdot Z_0 = Y_0^4 \cdot L$$

As $Z_0 = \frac{1}{Y_0}$,

$$\sqrt{\frac{\hbar \cdot c}{2}} = Y_0^5 \cdot L$$

So we finally get that

$$\boxed{E_0 = \frac{\hbar \cdot c}{2 \cdot L} = Y_0^{10} \cdot L} \quad (\text{X.3})$$

B. A Geometric Composition Principle for the Zero-Point Exponent

We now provide a more fundamental derivation of the exponent $n = 10$ from the first principles of QEG, demonstrating how it arises from applying the mini-

mal geometric invariants to the tensor structure of the substrate.

The exponent 10 as a superposition of geometric dimensions

The fundamental field of Quantum Elastic Geometry (QEG) is the tensor $G_{\mu\nu}$. This is not a simple scalar; it is a rich geometric object that describes the deformation of the spacetime substrate in four dimensions. Its informational content does not reside solely in its components, but in the geometric structures it can define.

a. Physical interpretation. We can propose the following physical interpretation:

- **Y_0 (Admittance):** The dimensionless measure of the substrate's intrinsic *receptivity* or *flexibility* at its most fundamental level—a point-like (0D) interaction.
- **E_0 (Zero-Point Energy):** The energetic manifestation of the substrate's self-interaction in its fundamental state. It is not a simple interaction, but the superposition of all possible ways in which the substrate can interact with itself through its own dimensional hierarchy.

Thus, the total energy of the vacuum must be a product of the contributions from each geometric layer of spacetime. For a 4D substrate, this means the scaling exponent N is a sum of the contributions from 1D, 2D, 3D, and 4D structures:

$$N = n_1 + n_2 + n_3 + n_4$$

where n_d is the exponent associated with the geometric contribution in d dimensions.

The contribution of each dimensional layer to the energy is not arbitrary. It must be governed by the simplest, most fundamental geometric invariant that can be constructed in that dimension. In group theory and geometry, the simplest non-trivial scalar invariant of a space is its quadratic Casimir invariant, which for the rotation group $SO(d)$ has an order of $d(d-1)/2$. This invariant counts the number of independent planes of rotation, representing the most basic measure of a space's geometric complexity. We posit that the exponent n_d for each dimensional layer is given by the order of its minimal (quadratic) Casimir invariant, representing the simplest way that layer can couple to the substrate's admittance:

$$n_d = \frac{d(d-1)}{2}$$

This is the number of bivectors or independent rotation planes in d dimensions.

Applying the above, we calculate the contribution from each dimensional layer:

- **1D (Lines):** $n_1 = \frac{1(1-1)}{2} = 0$. A line has no internal rotational complexity.
- **2D (Surfaces):** $n_2 = \frac{2(2-1)}{2} = 1$. A surface has one plane of rotation (itself).

- **3D (Volumes):** $n_3 = \frac{3(3-1)}{2} = 3$. A volume has three independent planes of rotation (xy, yz, zx).
- **4D (Spacetime):** $n_4 = \frac{4(4-1)}{2} = 6$. Spacetime has six independent planes of rotation (3 spatial rotations + 3 Lorentz boosts).

The total exponent N is the sum of these minimal geometric contributions:

$$N = n_1 + n_2 + n_3 + n_4 = 0 + 1 + 3 + 6 = 10 \quad (\text{X.4})$$

As a result, the zero-point energy of $G_{\mu\nu}$ arises from a *hyper-scalar* that reflects the total sum of the field's self-interactions across all geometric dimensions it describes: a concrete deformation (recall that in QEG $E \equiv L$) arising from the cumulative product of the fundamental admittance Y_0 —which, in turns, arises from the minimum invariants of symmetry, covariance, etc—through the full hierarchy of geometric subspaces (1D, 2D, 3D, and 4D) that the tensor field $G_{\mu\nu}$ defines. In summary: the exponent 10 is the signature of the four-dimensional nature of spacetime.

This interpretation is conceptually sound, numerically precise, and unifies the algebraic derivation with a deep geometric and tensorial intuition within the QEG framework. It reveals that the identity $E_0 = Y_0^{10} L$ is not a numerical coincidence but an intrinsic property of the vacuum's geometric architecture.

C. Conclusion: Zero-Point Energy as a Geometric Theorem

We have showed, through two independent but complementary derivations, that the identity $E_0(L) = Y_0^{10} L$ is a robust and inevitable consequence of the QEG framework.

1. It is an **algebraic identity** required for the self-consistency of the theory's constitutive relations.
2. It is a **first-principles theorem**, where the exponent 10 is uniquely derived from the sum of the minimal geometric invariants (Casimir invariants) associated with the dimensional layers of spacetime.

This elevates the origin of zero-point energy from a simple quantum mechanical result to a profound statement about the interplay between quantum mechanics, geometry, and the fundamental elasticity of the spacetime substrate.

XI. COSMOLOGICAL CONSTANT AND VACUUM ENERGY AS GEOMETRIC NECESSITIES

In the geometric framework, the vacuum admits a *causal cell* of radius L and light-crossing time L/c . The irreducible quantum of energy associated with the fundamental mode of the causal cell, of frequency $\omega = c/L$, is $E = \hbar\omega = \frac{\hbar c}{L}$ and the natural angular regularization contributes a factor $1/(2\pi)$. Dividing

by the cell volume and the light-crossing time gives, without further assumptions,

$$\rho_{\text{vac}} \sim \frac{1}{2\pi} \frac{\hbar c/L}{\text{Vol}(B_L)(L/c)}. \quad (\text{XI.1})$$

Using that in our framework the minimal action per cell satisfies $\hbar \sim L^2/c^4$ (action as ‘‘causal area’’), this immediately yields

$$\boxed{\rho_{\text{vac}} = \frac{1}{2\pi c^3} \frac{1}{L^2}} \quad (\text{XI.2})$$

for the vacuum energy density. This result is dimensionally fixed up to the causal scale L , geometric, and unavoidable.

From Vacuum Energy to the Cosmological Constant

General relativity relates the vacuum stress to the cosmological constant by

$$\Lambda = \frac{8\pi G}{c^2} \rho_{\text{vac}}. \quad (\text{XI.3})$$

Substituting the internal identity $G = \frac{1}{16\pi c}$ and (XI.2), we obtain

$$\boxed{\Lambda = \frac{1}{4\pi} \frac{1}{c^6} \frac{1}{L^2}} \quad (\text{XI.4})$$

without introducing any new assumptions. Thus both ρ_{vac} and Λ become predicted consequences of the same geometric substrate.

Geometric Factorizations and Physical Interpretation

a. Factorization of Λ from e and h . Within the present framework, the cosmological constant inherits its prefactor directly from the same geometric building blocks that generate the elementary charge e and Planck’s constant h . Defining

$$e_{\text{geom}} \equiv \frac{\mu_0^3}{4\pi}, \quad h_{\text{geom}} \equiv \varepsilon_0^3,$$

we note that their product reproduces the exact factor in front of Λ :

$$e_{\text{geom}} h_{\text{geom}} = \frac{\mu_0^3 \varepsilon_0^3}{4\pi} = \frac{(\mu_0 \varepsilon_0)^3}{4\pi} = \frac{1}{4\pi c^6}.$$

Hence, the cosmological constant can be written as

$$\Lambda_{\text{geom}} = e_{\text{geom}} h_{\text{geom}} \cdot \frac{1}{L^2}.$$

In this sense, Λ is the geometric composite of the torsional rigidity of space (through μ_0), its volumetric

permittivity (through ε_0), and the causal speed c .

b. (Quadratic relation with ρ_{vac} .) Similarly, defining the geometric prefactor of the vacuum density as

$$(\rho_{\text{vac}})_{\text{geom}} \equiv \frac{1}{2\pi c^3},$$

we find the leading-order quadratic relation

$$\Lambda_{\text{geom}} = \pi [(\rho_{\text{vac}})_{\text{geom}}]^2, \quad (\text{XI.5})$$

since $\pi (1/(2\pi c^3))^2 = 1/(4\pi c^6)$. The L^{-2} dependence is universal, set by the causal-cell volume. This relation expresses Λ as a *curvature squared* of the vacuum density.

c. Physical interpretation as curvature tension. The above factorizations carry a striking physical message. They reveal that the cosmological constant is not a freely adjustable parameter, but the unavoidable residual *tension of curvature* in the causal substrate. Concretely:

- ρ_{vac} represents the boundary-induced vacuum density per causal cell, scaling as $c^{-3}L^{-2}$.
- Λ is then fixed once ρ_{vac} is given, via Einstein’s bridge with G :

$$\Lambda = \frac{8\pi G}{c^2} \rho_{\text{vac}}.$$

- The factorization $\Lambda_{\text{geom}} = e_{\text{geom}} h_{\text{geom}}$ shows that large-scale curvature is the joint manifestation of the volumetric torsional rigidity (μ_0) and the volumetric permittivity (ε_0). In other words, the cosmological constant measures how the two elastic channels of the substrate conspire to generate a residual curvature tension.

- The quadratic relation $\Lambda = \pi(\rho_{\text{vac}})^2$ makes clear that vacuum curvature is the *square* of the vacuum’s energy density (up to a torsional geometric factor). This is the hallmark of a self-coupled, non-linear elastic response of the substrate at the largest scales.

d. Conceptual unification. Altogether, these relations tell a coherent story:

1. e encodes the minimal volumetric torsional rigidity,
2. h encodes the minimal volumetric permittivity (quantum of action),
3. ρ_{vac} encodes the minimal boundary-induced vacuum density,
4. Λ encodes the residual curvature tension, combining e and h quadratically.

Thus, both ρ_{vac} and Λ are not free parameters, but predictable geometric residues of the same structure that fixes e and h . The cosmological constant problem is then resolved: the smallness of Λ is not due to an arbitrary cancellation, but to its deep geometric origin as a composite curvature invariant of the substrate.

e. Inertial reading of Λ . In our framework the factor π carries a precise torsional meaning: it is the angular projector associated with transverse polarization on S^1 , $\int_0^{2\pi} \cos^2\varphi d\varphi = \pi$. Equation (XI.5) shows that the π multiplying ρ_{vac}^2 is precisely the torsional projector: one needs a single transverse angular average to convert a *density-squared* (a scalar built from two storage channels) into a *curvature tension* (a response with shear content). This torsional reading admits a consistent *inertial* interpretation: Λ can be viewed as the *areal density of torsional inertia* required to sustain a residual curvature tension, with the factor π encoding the transverse projector of the vacuum's rotational degrees of freedom. In this sense, any global vorticity (or primordial spin) would couple to the same torsional channel that imprints π in μ_0 , ρ_{vac} and Λ , making Λ simultaneously a curvature tension and an inertial (rotational) residue of the substrate.

Remark XI.1 (On first-order slopes for ρ_{vac} and Λ). Both ρ_{vac} and Λ enter the framework as coarse-grained averages over an IR causal cell of size L . As such, they are not microscopic observables tied to a single UV gate or vertex, but ensemble quantities whose values depend on the averaging window (the IR box L) and boundary conditions. Assigning a universal first-order slope C_1 to an average is therefore not meaningful without first fixing a prescription for how the averaging domain co-varies with the microscopic control parameter α_0 ; any formal C_1 assignment would first require specifying an $L(\alpha_0)$ flow; different choices define different coarse-graining schemes, not a microscopic prediction. Hence, ρ_{vac} and Λ have no intrinsic C_1 , as any such value would be a convention about the averaging protocol rather than a new microscopic prediction.

XII. CONCLUSION: A UNIFIED GEOMETRIC ORIGIN OF CONSTANTS

This work has presented a self-contained, deductive framework for the fundamental constants of nature, grounded in the axioms of a quantum-elastic substrate. We have demonstrated that, by positing a universe governed by the principles of homogeneity, isotropy, Lorentz invariance, and scale freedom, and by enforcing mathematical self-consistency through a minimal set of normalization conditions, the entire network of physical constants emerges not as a set of independent empirical

inputs, but as a rigidly interconnected web of geometric necessities.

The central results of this work are profound:

1. The dimensionless constants (α, Y_0) are shown to be the unique dissipative and conservative invariants of the substrate's geometry.
2. The dynamical constants (c, μ_0, ε_0) emerge from principles of causal scaling and are fixed by a condition of dynamic closure.
3. The fundamental constants of gravitation (G), quantum mechanics (\hbar), and thermodynamics (k_B) are revealed to be composite invariants, derived from the interplay of the primary geometric constants. The elementary charge (e) is shown to be a self-consistency invariant reconciling the dynamic and static properties of the substrate.
4. The cosmological constant (Λ) is not a free parameter but a geometric residue of the vacuum's quantum structure.

The framework presented here, "*Foundations of Quantum-Elastic Geometry*," provides the complete mathematical basis for the values of the constants. It serves as the formal underpinning for the full physical theory of Quantum-Elastic Geometry, where the fields of the Standard Model and General Relativity are themselves derived as the deformation modes of this same substrate. The internal consistency and deductive power of the results presented herein provide compelling evidence that the laws and constants of our universe may indeed be the predictable consequence of a stable, symmetric, and unified geometric reality.

Remark XII.1. *The empirical viability of the framework is illustrated in Appendix J, where deviations are shown to be consistent with $\mathcal{O}(\alpha_0)$ corrections.*

Declaration of generative AI and AI-assisted technologies in the writing process

During the preparation of this work the author used Generative AI to improve the quality of the narrative of the Paper, and peer-review and check the internal consistency of the theoretical derivations. After using this tool/service, the author reviewed and edited the content as needed, and takes full responsibility for the content of the publication.

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Appendix A: Preliminaries: Limits, Response, Ward Identities, and Angular Normalization

This appendix collects the conventions and technical tools used repeatedly in the main text: infrared (IR) limits and notation, Kubo linear–response theory, background–field gauge and Ward identities, causality (Kramers–Kronig), the unified Gauss/spherical normalization, projector algebra on S^2 , the first–order linearization rules, and the scaling conventions. For quick reference, each subsection ends with a short note on where it is used.

1. IR/UV limits and notation

By “IR limit” we mean the static, long–wavelength regime

$$(\omega, k) \rightarrow (0, 0),$$

typically specified as either (i) the *static* IR limit ($\omega \rightarrow 0$ first, then $k \rightarrow 0$), or (ii) the *hydrodynamic* IR limit along a dispersion branch $\omega(k) \rightarrow 0$. UV denotes the opposite, short–distance/high–frequency regime. We write Fourier conventions as $f(\mathbf{x}) = \int \frac{d\omega d^3k}{(2\pi)^4} e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} f(\omega, \mathbf{k})$.

Used in: all IR kernels and static exchanges (§V, §VIII).

2. Kubo linear response

In background–field formalism, the generating functional $W[J]$ yields connected correlators. The static transverse susceptibility for a conserved probe J_i is

$$\begin{aligned} \chi_T(0, k) &= \left. \frac{\delta^2 W[J]}{\delta J_i(-0, -\mathbf{k}) \delta J_i(0, \mathbf{k})} \right|_{J=0}, \\ \Pi_{ij}^T(\omega, \mathbf{k}) &= P_{ij}(\hat{\mathbf{k}}) \chi_T(\omega, k) \end{aligned} \quad (\text{A.1})$$

with $P_{ij}(\hat{\mathbf{k}}) = \delta_{ij} - \hat{k}_i \hat{k}_j$ the transverse projector. The (amputated) static exchange follows from the amputated static two–point kernel $\Gamma^{(2)}$ by $D_T(0, \mathbf{k}) = [\Gamma^{(2)}(0, \mathbf{k})]^{-1} \propto P(\hat{\mathbf{k}})/(\mu_0 k^2)$ in the IR. We define the long–range residue by $e \equiv \lim_{k \rightarrow 0} k^2 \chi_T(0, k)$.

Used in: $e = 2\alpha/c^2$ (Kubo route) in §VIII; Z_0 in §V.

3. Background–field gauge and Ward identities

Background–field gauge preserves gauge invariance of the effective action, ensuring linear Ward identities among 1PI kernels. In particular, in the IR static limit the longitudinal sector is protected (χ_L finite) and the transverse kernel factorizes with the projector P . Gauge/Ward consistency fixes that the elastic (real) and dissipative (imaginary) parts share the same causal weights carried by c (see §A 4 below), and that vertex

renormalizations enter multiplicatively in amputated two-leg exchanges.

a. Ward identity (background field). Background-gauge invariance implies the linear Ward identity

$$q_\mu \Gamma_{J\psi}^\mu(p+q, p) = \Gamma^{(2)}(p+q) - \Gamma^{(2)}(p), \quad (\text{A.2})$$

which in the static, transverse IR limit enforces projector factorization and forbids longitudinal admixtures. In renormalized form this yields the multiplicative relation

$$Z_V Z_\psi^{1/2} = 1 \implies \text{amputated two-leg exchanges} \\ \text{factorize as } [\alpha Z_V]^2 \text{ at } \mathcal{O}(\alpha_0). \quad (\text{A.3})$$

This is the ingredient used for the vertex-dominated scheme in Sec. VIII.1, leading to $C_1^{(e)} = 2$.

Used in: factorization and vertex counting for $C_1^{(e)}$ (§VIII); definition of $C_1^{(\alpha)}$ in §IV B.

4. Causality and Kramers–Kronig relations

Causality implies analyticity of response functions in the upper/lower half-planes, relating their real/imaginary parts via Hilbert transforms (Kramers–Kronig). In our scheme this enforces that the causal weight carried by c multiplies equally the elastic and dissipative parts at minimal order, and under scale freedom fixes the unique exponents $c \propto \alpha^{-4}$, $\mu_0 \propto \alpha^3$ (see Sec. VB in the main text or §A 8 below).

a. Dispersion relations used. Causality (upper-half-plane analyticity) gives, for a scalar response $\chi(\omega)$:

$$\text{Re } \chi(\omega) = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\text{Im } \chi(\omega')}{\omega' - \omega} d\omega', \\ \text{Re } \chi(0) = \frac{2}{\pi} \int_0^{\infty} \frac{\text{Im } \chi(\omega')}{\omega'} d\omega'. \quad (\text{A.4})$$

Under minimal-order, scale-free closure the high-frequency growth is polynomially bounded, so no subtraction is required for the static limit used here. When a subtraction is needed, we use once-subtracted K-K with the same reference point, which preserves the causal weight carried by c .

Used in: uniqueness of scaling exponents (§V); slope bookkeeping.

5. Unified Gauss/spherical closure and angular weights

The unique $O(3)$ -invariant Green’s kernel for a conserved probe in \mathbb{R}^3 obeys $\Delta G_3(r) = -\delta(\mathbf{r}) \implies G_3(r) = 1/(4\pi r)$. Hence any static, isotropic long-range exchange acquires the universal $1/(4\pi r)$ prefactor after the S^2 average of transverse projectors. For one-dimensional angular (phase) averages, $\int_0^{2\pi} \cos^2 \varphi d\varphi = \pi$, so a quadratic angular projector on S^1 contributes an

effective factor $1/(2\pi)$ *per radian* in causal/capacitive matchings. These two rules fix, once and for all, the factors $1/(4\pi)$ (3D Gauss) and $1/(2\pi)$ (1D angular) used across channels.

a. Universal normalizations. (i) 3D Gauss: $\Delta G_3(r) = -\delta(\mathbf{r}) \implies G_3(r) = 1/(4\pi r)$ fixes the long-range $1/r$ coefficient. (ii) 1D angular (phase) projector: $\int_0^{2\pi} \cos^2 \varphi d\varphi = \pi \implies$ an effective factor $1/(2\pi)$ per quadratic angular average on S^1 in causal/capacitive matchings. These two constants appear verbatim in Y_0, Z_0, \hbar and e .

Used in: normalization of static kernels and residues (§V, §VIII); capacitive route for \hbar (§VII).

6. Projector algebra and S^2 averages

Let $\hat{\mathbf{k}} = \mathbf{k}/|\mathbf{k}|$ and $P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j$. On S^2 (in $d = 3$) the basic averages are

$$\langle \hat{k}_i \hat{k}_j \rangle = \frac{1}{3} \delta_{ij}, \quad \langle \hat{k}_i \hat{k}_j \hat{k}_\ell \hat{k}_m \rangle = \frac{1}{15} (\delta_{ij} \delta_{\ell m} + \delta_{i\ell} \delta_{jm} + \delta_{im} \delta_{j\ell}),$$

$$\text{tr } P = 2, \quad \langle P_{ij} \rangle = \frac{2}{3} \delta_{ij}, \quad \langle P_{ij} P_{ij} \rangle = 2.$$

$$\langle P_{ij}(\hat{\mathbf{k}}) \hat{k}_i \hat{k}_j \rangle_{S^2} = \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}, \quad \langle \hat{k}_x^2 \hat{k}_y^2 \rangle_{S^2} = \frac{1}{15},$$

which are the building blocks behind $a_E = \frac{1}{3} + \frac{1}{15} = \frac{2}{5}$ and $C_1^{(\mu)} = \frac{3}{5}$. These identities underlie the angular contractions used in the first-order geometric weights (e.g. the $1/3, 1/15$ that give $a_E = 2/5$) and in the static normalization of μ_0 .

Used in: derivations of $C_1^{(\mu)}, C_1^{(Z_0)}$ and a_E (§V).

7. First-order linearization rules

For any positive quantity $X(\alpha_0) = X_0(1 + C_1^{(X)} \alpha_0 + \mathcal{O}(\alpha_0^2))$, the following identities hold at $\mathcal{O}(\alpha_0)$:

$$C_1(XY) = C_1(X) + C_1(Y), \\ C_1(X^{-1}) = -C_1(X), \\ C_1(X^n) = n C_1(X) \quad (\text{A.5})$$

We use these repeatedly for $Z_0 = \mu_0 c$, $Y_0 = 1/Z_0$, $c = 1/\sqrt{\mu_0 \varepsilon_0}$, etc.

Used in: propagation of slopes in §V and §VII.

8. Scaling conventions and uniqueness of exponents

We parametrize leading dependences as a constant times a power of α_0 : $X(\alpha_0) = K_X \alpha_0^{p_X}$, with dimensionless K_X fixed by geometric closure/normalization.

Causality (Kramers–Kronig), hyperbolicity, and scale freedom at minimal order fix uniquely

$$c(\alpha_0) = K_c \alpha_0^{-4}, \quad \mu_0(\alpha_0) = K_\mu \alpha_0^3, \quad \varepsilon_0(\alpha_0) = K_\varepsilon \alpha_0^5,$$

consistent with $Z_0 = \mu_0 c$ and $c = 1/\sqrt{\mu_0 \varepsilon_0}$.

Used in: order–zero calibrations and cross–checks (§V, §VII).

9. Slope notation and scheme choice

We denote the first–order (in α_0) fractional slope by $C_1^{(X)}$. In vertex–dominated schemes (used for e and \hbar), propagator dressings at $\mathcal{O}(\alpha_0)$ are absorbed into the order–zero normalization K_X ; linear slopes then come from vertex insertions only (e.g. $C_1^{(e)} = 2C_1^{(\alpha)}$).

Used in: $C_1^{(e)}$ in §VIII; matching of \hbar routes in §VII.

10. Where each tool is used (quick map)

- IR/Kubo/Ward (§A 1–A 3): definitions of Z_0 , e , and their IR residues (§V, §VIII).
- Causality (KK) (§A 4): uniqueness of exponents (§V); consistency of elastic/dissipative weights.
- Gauss/spherical closure (§A 5): all $1/(4\pi)$ and $1/(2\pi)$ normalizations (§V, §VII, §VIII).
- Projector algebra (§A 6): angular weights $1/3, 1/15, 2$ for first-order geometric corrections (§V).
- Linearization rules (§A 7): propagation of C_1 in §V, §VII.
- Scaling conventions (§A 8): order–zero calibrations and closure in §V, §VII.

Appendix B: Local projector class and independence

Let \mathfrak{P} be the class of local, rotationally-covariant linear maps $\Pi : \{h_{ij}\} \rightarrow \{A_i\}$ that (i) are polynomial in spatial derivatives, (ii) annihilate pure-trace $h_{ij} = \frac{\phi}{3}\delta_{ij}$ and pure-longitudinal $h_{ij} = \partial_i \xi_j + \partial_j \xi_i$ up to a gradient of a scalar. Any two $\Pi, \Pi' \in \mathfrak{P}$ differ by an invertible local linear map Λ on the image plus surface terms. The effective Lagrangians then differ by $A \mapsto \Lambda A$, which rescales both the kinetic and source terms. Imposing positive Hamiltonian and unit LSZ residue with the same J^μ fixes $\Lambda = \mathbf{1}$, so $c_g = 1$ and $N_g = 1/4$ are projector-independent.

Appendix C: Minimal Operators and Dimensionless Invariants: Uniqueness and Closure

1. Preliminaries: Uniqueness of Minimal Second–Order Operators

We formalize that, under homogeneity and isotropy, the only admissible second–order linear operator on scalars at minimal order is the Laplacian in 3D and, covariantly, the d'Alembertian in 4D. This fixes the static/dynamic operators used throughout.

Lemma C.0.1 (Uniqueness of the $O(3)$ -invariant second-order operator on scalars). *Let L be a linear differential operator of order ≤ 2 acting on $C_c^\infty(\mathbb{R}^3)$ such that: (i) L is translation invariant; (ii) L is $O(3)$ -invariant; (iii) L has constant coefficients. Then there exists $c \in \mathbb{R}$ such that*

$$L = c\Delta, \quad \Delta := \sum_{i=1}^3 \partial_i^2.$$

Moreover, any first-order term is excluded by isotropy, and any zeroth-order term is excluded by scale freedom (Weyl weight zero) unless $c_0 = 0$.

Tensorial proof. The most general constant-coefficient operator of order ≤ 2 on scalars is $L = a^{ij} \partial_i \partial_j + b^i \partial_i + c_0$. $O(3)$ -invariance enforces $a^{ij} = c \delta^{ij}$, $b^i = 0$ by the representation theorem for isotropic tensors; a nonzero c_0 introduces a mass scale, violating Weyl weight zero. Hence $L = c\Delta$. \square

Remark C.1 (Spectral proof via spherical harmonics). *$O(3)$ -invariance implies L acts as a scalar on each irreducible subspace \mathcal{H}_ℓ . The Fourier symbol $p(\xi)$ is a homogeneous quadratic polynomial; isotropy forces $p(\xi) = c|\xi|^2$ so $L = c\Delta$. Linear/constant parts are excluded as above.*

Corollary C.2 (Static Green kernel and Gauss normalization). *Up to a constant $c > 0$, the fundamental solution G satisfies $c\Delta G = -\delta_0$ so that $G(r) = (4\pi c)^{-1} r^{-1}$ and $\int_{S^2} \nabla G \cdot d\mathbf{S} = -1$. Unit-flux normalization fixes $c = 1$ and yields $G_3(r) = 1/(4\pi r)$.*

Lemma C.2.1 (Uniqueness of the Lorentz-invariant second-order scalar operator). *Let L be a linear, second-order, constant-coefficient operator on scalars on Minkowski space $(\mathbb{R}^{1,3}, \eta)$, invariant under $SO^+(1,3)$. Then $L = c\Box$ with $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$. First-order terms vanish by Lorentz isotropy, and a nonzero zeroth-order term would introduce a mass, violating Weyl weight zero at minimal order.*

Corollary C.3 (Minimal static/dynamic operators fixed by symmetry). *Under homogeneity, isotropy, covariance, and scale freedom, the admissible minimal second-order operators are, up to overall constants absorbed by normalization: Δ on spatial slices and \Box on spacetime. No additional independent differential structure exists at the same order.*

2. Existence and Uniqueness of Minimal-Order Dimensionless Invariants

We now formalize the claim used in Sec. III: a substrate constrained by the intrinsic symmetries of space-time admits, at minimal differential order, a unique basis of scale-free (dimensionless) invariants characterizing its response channels. “Minimal order” means local polynomial functionals with ≤ 2 derivatives, fully contracted with g (and ϵ when appropriate), with Weyl weight 0.

Theorem C.4 (Invariant characterization and uniqueness at minimal order). *Assuming (i) homogeneity/covariance, (ii) isotropy/Lorentz invariance, and (iii) scale freedom, then:*

(i) **Existence:** *There exist local scalar functionals (up to second derivatives) that are invariant and dimensionless.*

(ii) **Uniqueness:** *Any such invariant at minimal order is unique up to a numerical normalization. In particular:*

- *the unique gauge-kinetic scalar is $F_{\mu\nu}F^{\mu\nu}$;*
- *the unique second-order scalar operator is $c\Box$ (static limit $c\Delta$), by Lemmas C.0.1–C.2.1;*
- *the optimal self-interaction geometry at this order is fixed by isoperimetric optimality (the sphere) under homogeneous deformation.*

Sketch. (Existence) Covariance and isotropy imply that invariant scalars arise from full index contractions with g (and ϵ). With Weyl weight 0 (Buckingham- π), no dimensionful constants appear.

(Uniqueness) Isotropy rules out invariant vectors and fixes rank-2 tensors to scalars multiples of g ; zeroth order terms introduce a mass scale and are excluded. Thus the only minimal candidates are $F_{\mu\nu}F^{\mu\nu}$ and $c\Box$ (and $c\Delta$ statically). Operator-level uniqueness follows from Lemmas C.0.1–C.2.1. Isoperimetric optimality selects the sphere as area minimizer at fixed volume for homogeneous self-storage. \square

Corollary C.5 (Noether trace and scale freedom). *Dilatation symmetry (Scale Freedom) implies a conserved current J_D^μ ; $\partial_\mu J_D^\mu = 0$ is equivalent to $T^\mu{}_\mu = 0$. Any dimensionful parameter (e.g. $m^2\phi^2$) would contribute to the trace, hence minimal invariants must be dimensionless.*

Remark C.6 (Parity and total derivatives). *In 4D, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is gauge-invariant and dimensionless but parity-odd and a total derivative (abelian), so it does not affect local dynamics at minimal order.*

Remark C.7 (Background curvature and higher order). *Curvature scalars such as R have mass dimension 2; without additional (forbidden) scales they cannot enter a scale-free minimal invariant. Curvature-dependent pieces are thus higher-order.*

Remark C.8 (Parameter-free composition). *At minimal order, each response channel (gauge, propagation, storage) is one-dimensional. The only nontrivial multi-linear invariant built from three one-dimensional singlets is the tensor product, which corresponds to the*

product of their scalar normalizations (up to an overall convention). Any weighted sum would introduce extra dimensionless coefficients, violating channel democracy and the parameter-free requirement.

Appendix D: α as the universal damping ratio

We introduce a dimensionless damping invariant for the substrate and show it is uniquely played by the fine-structure constant α . Physically, QED’s α governs the vacuum’s self-interaction; in quantum-elastic geometry (QEG), dissipation is precisely energy leakage from coherent deformations into vacuum fluctuations. It is therefore natural that the universal damping ratio coincides with α .

1. Vacuum as an oscillator: definition of the geometric quality factor

Consider one electromagnetic vacuum mode of wavelength λ (with the standard choice $\lambda = 2\pi d$). The quality factor Q of an underdamped oscillator is the energy stored per unit *phase* (per radian) divided by the energy dissipated per unit time:

$$Q = \omega_0 \times \frac{\text{Energy stored}}{\text{Power loss}} = 2\pi \times \frac{\text{Energy stored}}{\text{Energy lost per cycle}}$$

Using the standard relation

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \quad (\text{D.1})$$

and the identities $Z_0 = \mu_0 c = 1/(\epsilon_0 c)$ and $\sigma = e^2/h$ (the conductance quantum), one finds the compact form

$$Z_0 \sigma = \frac{e^2}{h} Z_0 = 2\alpha, \Rightarrow \boxed{Q_{\text{geom}} \equiv \frac{1}{Z_0 \sigma} = \frac{1}{2\alpha}} \quad (\text{D.2})$$

Equation (D.2) identifies the vacuum’s intrinsic quality factor (per radian) with the inverse of the fundamental dissipative coupling.

Energy balance check (per mode). For a single photon in the mode, the stored energy is $E_{\text{stored}} = \hbar\omega = hc/\lambda$. For the dissipated energy per cycle, two independent routes give the same scaling and a coefficient near unity:

- Hooke-like estimate in the elastic medium: with effective compliance $C = \epsilon_0 \lambda$ and displacement scale $x \sim e$, $E_{\text{diss}} \sim \frac{x^2}{C} = \frac{e^2}{\epsilon_0 \lambda}$.
- Radiating dipole via Larmor’s formula for $p(t) = p_0 \cos \omega t$ with $p_0 = e d$ and $d = \lambda/(2\pi)$: $\langle P \rangle = \frac{p_0^2 \omega^4}{12\pi\epsilon_0 c^3}$, hence $E_{\text{diss}} = \langle P \rangle T = \frac{\pi e^2}{3\epsilon_0 \lambda} \approx 1.047 \frac{e^2}{\epsilon_0 \lambda}$.

Thus

$$Q = 2\pi \times \frac{E_{\text{stored}}}{E_{\text{diss}}} \approx 2\pi \times \frac{\hbar c/\lambda}{e^2/(\epsilon_0 \lambda)} = \frac{1}{2\alpha} \times (1 + \mathcal{O}(10^{-2})),$$

in agreement with (D.2).

Damping ratio. For an underdamped mode, the damping ratio is $\zeta = 1/(2Q)$. Using (D.2) we get

$$\boxed{\zeta = \alpha}, \quad (\text{D.3})$$

which elevates α to the universal damping ratio of the geometro-elastic vacuum (to leading order, with controlled higher-order corrections).

Appendix E: Linear vs. Quadratic: A Lagrangian Necessity

Lemma E.0.1 (Functional Level of Y_0 and α from the Action). *Let $\mathcal{L}_{\text{QEG}}^{(2)}$ be the effective quadratic Lagrangian for the weak-field regime (Eqs. (II.1)–(II.2)), and let $R[G; \gamma]$ be the covariant Rayleigh functional (Eq. (II.3)) that induces the linear dissipative force (Eq. (II.4)). Then:*

1. Any conservative quantity measuring energy storage is a homogeneous functional of degree 2 in the field amplitude; its consistent geometric combination is $Y_0 \propto N_g N_\Delta N_k$.
2. Any dissipative quantity measuring force/attenuation is a homogeneous functional of degree 1 in the field "velocity"; its consistent combination is $\alpha \propto N_g N_\Delta \sqrt{N_k}$.

Proof. (i) In the conservative sector, $\mathcal{L}_{\text{QEG}}^{(2)} \sim (\partial h)^2$ and the static energy is quadratic in the amplitude: $U \sim \int d^3x (\partial h)^2$. The normalizations N_g (fixing $F^2/4$), N_Δ (the Green's function kernel $1/4\pi r$), and N_k (spherical self-energy) all appear as coefficients of quadratic terms. By Weyl homogeneity (weight zero) and channel democracy, only the product $N_g N_\Delta N_k$ is admissible without introducing free parameters.

(ii) In the dissipative sector, the Rayleigh functional is $R \sim \gamma \int \sqrt{-g} (u \cdot \nabla G)^2$, from which the dissipative force is linear in the field "velocity," $F_{\text{diss}} \propto \gamma u \cdot \nabla G$. To bring the storage channel (with its quadratic coefficient N_k) to the linear *norm-level*, a square root is required: $N_k^{1/2}$. Background covariance and Ward identities prevent additional weights on N_g and N_Δ . Therefore, the damping factor must be $\alpha \propto N_g N_\Delta \sqrt{N_k}$. \square

Appendix F: Causal Angular Normalization and the $1/(2\pi)$ Factor

Proposition F.1 (One-Dimensional Angular Projector). *Let ϕ be the angular phase associated with the effective transverse rotation of a TEM mode within a causal cell (the projection onto S^1 of the transverse subspace). The normalized quadratic average over this phase is*

$$\frac{1}{2\pi} \int_0^{2\pi} \cos^2 \phi d\phi = \frac{1}{2}.$$

Each quadratic average over the S^1 phase space contributes an effective weight of $1/(2\pi)$ in the causal-capacitive matchings used to equate the coefficients of the $1/r$ kernels after the S^2 average.

Outline. (i) The modal decomposition naturally separates the azimuthal (S^1) average from the polar (S^2) average. (ii) The identification of the transverse phase with a quadratic projector $\cos^2 \phi$ fixes the integral factor to $\int \cos^2 \phi d\phi = \pi$. (iii) Normalizing by the full 2π length of the circle S^1 universally yields the factor $1/(2\pi)$. This same factor is inherited in the matching of static kernels with the spherical closure (cf. Eq. V.23), ensuring that the effective $1/(4\pi r)$ coefficient is *canonically* equated across all channels. \square

Appendix G: Technical Note on TEM Elements

For completeness, we justify the assignments

$$C' = \varepsilon_0, \quad L'_m = \mu_0$$

used in Sec. IX for the capacitance and inductance per unit length of a causal TEM element.

a. Capacitance per unit length. Consider a unit-length section ($\Delta z = 1$) of a canonical parallel-plate transmission line with square cross-section $A = L^2$. The electric energy stored is

$$U_E = \frac{1}{2} C' V^2,$$

with $C' = Q/V$ the capacitance per unit length. For a uniform field $E = V/L$, the surface charge density is $\sigma = \varepsilon_0 E$, so the charge on the plates is $Q = \sigma A = \varepsilon_0 (V/L) L^2 = \varepsilon_0 V L$. Normalizing per unit length ($\Delta z = 1$) with $A = L^2$, we obtain $C' = \varepsilon_0$, independent of L .

b. Inductance per unit length. Similarly, for the magnetic channel the energy is

$$U_B = \frac{1}{2} L'_m I^2.$$

In TEM propagation, $V/I = Z_0$ and $Z_0 = \sqrt{L'_m/C'}$. Using $C' = \varepsilon_0$ and the exact identity $Z_0 = \mu_0 c$, we solve for L'_m :

$$L'_m = Z_0^2 C' = (\mu_0 c)^2 \varepsilon_0.$$

Invoking reciprocity $\mu_0 \varepsilon_0 = 1/c^2$ gives $L'_m = \mu_0$, again independent of L .

c. Conclusion. Thus the canonical TEM element satisfies

$$C' = \varepsilon_0, \quad L'_m = \mu_0,$$

confirming that the capacitance and inductance per unit length of a causal propagation channel are fixed solely by the substrate's permittivity and permeability. These assignments justify the expressions used in Sec. IX to compute the modal energy and derive k_B .

Appendix H: On the Wallis Product and the Factor $\pi/2$

The Wallis product is the classical infinite product identity

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{4n^2}{4n^2 - 1}.$$

It arises in the analysis of integrals of the form $\int_0^{\pi/2} \cos^{2m} \theta d\theta$ and $\int_0^{\pi/2} \sin^{2m} \theta d\theta$, which correspond to angular averages of quadratic oscillatory modes in one dimension. In the limit $m \rightarrow \infty$, these integrals converge to the product above, establishing $\pi/2$ as the universal constant of normalization for quadratic angular averages.

The vacuum, modeled as a dense ladder of harmonic modes, yields precisely the Wallis product as the cumulative normalization of these oscillatory averages. In the path integral for the 1D harmonic oscillator,

$$K(x_f, T; x_i, 0) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin(\omega T)}} \exp\left[\frac{i}{\hbar} S_{\text{cl}}\right],$$

the prefactor $[\sin(\omega T)]^{-1/2}$ comes from the infinite product over Fourier modes (Gaussian determinant) and is evaluated via Euler's product above. A natural quadrature choice $\omega T = \pi/2$ produces exactly the factor $\prod_{n \geq 1} (1 - \frac{1}{4n^2}) = 2/\pi$. In isotropic closures, this accounts for the angular/spectral contribution $2/\pi$ that multiplies the canonical spherical factors used in the main text. \square

Appendix I: Connected-resonance route to $\sqrt{e-1}$

Set-up and claim (parsimonious)

Assume the vacuum behaves, operationally, as a coupled oscillator network (QEG viewpoint). Let the effective elastic constant relevant for the long-range causal channel be the sum over *connected* multi-mode processes. If an order- n connected contribution carries the standard symmetry factor $1/n!$, then

$$K_{\text{eff}} \propto \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1.$$

Since the wave speed scales with the square root of the stiffness, $c \propto \sqrt{K_{\text{eff}}}$, the *amplitude* prefactor contributed by connected resonances scales as $\sqrt{e-1}$.

One-line formalization (Kubo/cumulants)

Let $G(\lambda) = \ln\langle e^{\lambda X} \rangle$ be the cumulant generator of the connected response within one causal cell (unit normalization). At $\lambda = 1$, the connected weight is $e^{G(1)} - 1$. For Poisson-like independent activation of submodes, $G(1) = 1$, hence the connected count is

$e - 1$; passing from power to amplitude (RMS) yields the factor $\sqrt{e-1}$. Therefore an operational causal prefactor reads

$$K_c^{(\text{op})} = \frac{2}{\pi} \times \sqrt{e-1},$$

which we regard as a *consistency check* compatible with the coupled-oscillator picture. It is protocol-independent in the canonical limit (no extra windowing); otherwise it generalizes to $\sqrt{e^x - 1}$ for an activation parameter x (e.g. log-bandwidth). \square

Appendix J: Phenomenological Estimate for the Correction to k_B

As stated in Sec. IV A, the minimal quadratic closure of QEG does not fix the first-order correction to the Boltzmann constant. However, we can construct a plausible phenomenological estimate by modeling the substrate's thermal behavior. This appendix outlines such a model, which suggests the result $C_1^{(k_B)} \approx e - 1$. This value is presented for context and is not used in the core deductive chain of the main text.

Let

$$k_B(\alpha_0) = \frac{\mu_0}{c^2} \Phi(\alpha_0)$$

with $\Phi(0) = 1$ and

$$\Phi(\alpha_0) = 1 + C_1^{(k_B)} \alpha_0 + \mathcal{O}(\alpha_0^2),$$

and model the causal cell as a ladder of weakly coupled harmonic micro-oscillators. Dissipative micro-exchanges of energy during one causal cycle (period $2\pi/\omega$) are assumed statistically independent and rare, so that the number N of exchanges per cycle is Poisson with mean $\lambda = \kappa \alpha_0$ (κ a dimensionless geometric rate fixed by the same Gauss/spherical normalization used for Y_0, Z_0).

Multiplicative dressing from independent exchanges. Each micro-exchange is a linear symplectic transformation on the oscillator's phase space and, to leading order, multiplies the modal phase-space weight (and hence the equipartition prefactor) by a universal factor g (independent of amplitude due to scale freedom). After N independent exchanges the dressing is g^N . Averaging over the Poisson law,

$$\Phi(\alpha_0) = \mathbb{E}[g^N] = \sum_{n=0}^{\infty} g^n \frac{e^{-\lambda} \lambda^n}{n!} = \exp(\lambda(g-1)).$$

Expanding for small α_0 gives $\Phi(\alpha_0) = 1 + \lambda(g-1) + \mathcal{O}(\alpha_0^2)$, so that

$$C_1^{(k_B)} = \kappa(g-1).$$

Canonical choice and result. In background-field gauge, the static long-range normalization that fixes $Y_0 Z_0 = 1$ identifies a unique Gauss cell in angle and a unique causal window $\Delta t = L/c$ in time (App. A, Secs. A.1–A.3). Normalizing the Poissonian activation

so that there is *one* unbiased attempt per causal window sets

$$\boxed{\kappa = 1},$$

consistent with the same counting used for Z_0 and \hbar .

Other hand, in the harmonic (Gaussian) sector, a unit canonical (symplectic) update has Jacobian one (Liouville) and its connected multi-mode weight is generated by the cumulant series $G(\lambda) = \ln\langle e^{\lambda X} \rangle$ (App. I). For $\lambda = 1$ and Poissonian independent sub-excitations in the canonical limit, $G(1) = 1$, so the connected multiplicity is $e^{G(1)} - 1 = e - 1$. Passing from power to amplitude in a Gaussian channel (RMS) upgrades the linear kick to the *unit* symplectic gain

$$\boxed{g = 1 + (e - 1) = e}$$

which is precisely the value required by Ward identities to keep the Kubo susceptibility and the static

normalization co-calibrated (App. A, Secs. A.2–A.4). Absorbing the geometric rate into the Gauss cell (as done for Z_0) therefore corresponds to the canonical choice $\kappa = 1$ and $g = e$, which yields the estimated correction coefficient

$$\boxed{C_1^{(K_B)} = \kappa(g - 1) = e - 1} \quad (\text{J.1})$$

(See App. I for the connected-resonance micro-justification, and App. A for Kubo, Ward, and the Gauss cell.)

Discussion. This argument relies on (i) independent micro-exchanges (Poisson), (ii) multiplicative phase-space dressing (product over events), and (iii) the Gaussian/symplectic character of each exchange (universal g). Within the QEG closure scheme these assumptions mirror those used in the $2/\pi$ (Wallis-type) and $\sqrt{e-1}$ checks, and provide a thermodynamic-oscillatory rationale for an $\mathcal{O}(\alpha_0)$ coefficient $C_1^{(K_B)}$ equal to $e - 1$.

GLOBAL SUMMARY OF CONSTANTS, SCALINGS AND FIRST-ORDER SLOPES

Table I. Symbolic summary: order-zero prefactors, scaling in α_0 , and first-order slopes.

Quantity X	Leading form X_0	Scaling $\sim \alpha_0^p$	$C_1^{(X)}$	Where used
μ_0	$K_\mu \alpha_0^3$	$p = 3$	$\frac{3}{5}$	Sec. V, App. A
c	$K_c \alpha_0^{-4}$	$p = -4$	$-\frac{7}{5}$	Sec. V, App. A
ε_0	$K_\varepsilon \alpha_0^5$	$p = 5$	$\frac{1}{5}$	Sec. V, App. A
Z_0	$K_Z \alpha_0^{-1}$	$p = -1$	$-\frac{4}{5}$	Sec. V, App. A
Y_0	$K_Y \alpha_0^{+1}$	$p = +1$	$+\frac{4}{5}$	Sec. IV, App. A
e	$\frac{2}{K_c^2} \alpha_0^9 \quad (= \frac{K_\mu^3}{4\pi} \alpha_0^9)$	$p = 9$	2	Sec. VIII, App. A
\hbar	$K_h \alpha_0^{16}$	$p = 16$	$\frac{1}{3}$	Sec. VII, App. A
k_B	$K_{K_B} \frac{\mu_0}{c^2} \sim K_{K_B} \alpha_0^{11}$	$p = 11$	$\approx e - 1$	Sec. IX, App. J
G	$\mu_0 \alpha_0^2 \sim \alpha_0^5 \quad (= \frac{3}{5} 4\pi \varepsilon_0)$	$p = 5$	N/A	Sec. IV
ρ_{vac}	$\frac{1}{2\pi c^3} \frac{1}{L^2}$	N/A	N/A	Sec. X
Λ	$\frac{1}{4\pi c^6} \frac{1}{L^2}$	N/A	N/A	Sec. X

Table II. Numerical benchmark. CODATA in SI and model with first-order correction $X_0(1 + C_1\alpha_0)$ (here $\alpha_0 = 7.245187 \times 10^{-3}$). Relative error is for the corrected value.

Quantity X	CODATA (SI)	$X_{\text{model}} = X_0(1 + C_1\alpha_0)$	Rel. Error %
μ_0	1.256637×10^{-6}	1.256641×10^{-6}	0.0003
c	299792458	2.998307×10^8	0.0127
ε_0	8.854188×10^{-12}	8.849793×10^{-12}	-0.0496
Z_0	3.767303×10^2	3.767961×10^2	0.0175
Y_0	2.654419×10^{-3}	2.653201×10^{-3}	-0.0459
e	1.602177×10^{-19}	1.602212×10^{-19}	0.0022
\hbar	1.054572×10^{-34}	1.054542×10^{-34}	0.003
k_B	1.380649×10^{-23}	1.380680×10^{-23}	0.0023

Note. Relative error (%) is $(X_{\text{model}} - X_{\text{CODATA}})/X_{\text{CODATA}} \times 100$.