# The Antiderivative Sequence and its Infinite Series: The Rediscovery of the Polylogarithm from fundamental calculus

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## Abstract

This study explores the Antiderivative Power Rule Sequence, demonstrating how its infinite series leads to the polylogarithm. By iteratively applying the power rule for antiderivatives to successive powers of x, we derive the sequence, which, when expressed as an infinite series, converges to -ln(1-x). Differentiating the resulting series recovers the geometric series, highlighting a profound inverse relationship between 1/(1-x) and -ln(1-x). Furthermore, this formulation establishes a natural connection to the polylogarithm function, generalizing the relationship for higher orders of integration. This work provides both pedagogical and theoretical insights, reconstructing a transcendental function from elementary calculus operations.


## Introduction

There are five types of sequences: arithmetic, geometric, harmonic, and Fibonacci. However, the mathematics of sequences and series is not limited to these four; other unique and special series include prime number sequences, factorial sequences, Mersenne numbers, perfect numbers, and even look-and-say sequences.

Sequences are important in mathematics because they provide a systematic approach to studying patterns, change, and progression. They serve as the foundation for many mathematical concepts, including limits, series, and functions, allowing us to examine how quantities change over time or according to specific rules.

It is critical to study these fundamental patterns because they have the potential to reveal deeper secrets in the mathematical field. And this paper will demonstrate why

### Looking Back to the Basics

We are familiar with how an indefinite integral works. It is the antiderivative of a function f'(x) written as:

$$\int f'(x)\,dx = f(x) + C$$

Where C is a constant. We can also write this as:

$$\int f(x)\,dx = F(x) + C$$

But either way, both notations operate the same way. Now the power rule states that the antiderivative of the function $f(x) = x^n$ is equal to:

$$\int f(x)\,dx = \frac{x^{n+1}}{n+1} + C$$

So if $F(x) = \frac{x^{n+1}}{n+1}$, then taking the derivative will give us:

$$F'(x) = \frac{1}{n+1}(n+1)x^{(n+1)-1} = x^n$$

**The Antiderivative sequence and Infinite series**

Using the power rule for the antiderivative, let's take the indefinite integral of $x^n$ using successive values of n (starting n=0, $x^n$ becomes one with respect to the laws of exponents. As long as x is not equal to zero):

$$\int 1\,dx = x + C$$

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x^3\,dx = \frac{x^4}{4} + C$$

And we can keep doing this until n approaches infinity. Now from the results that we just obtained, let's collect them into one sequence:

$$x, \frac{x^2}{2}, \frac{x^3}{3}, \frac{x^4}{4}, \dots$$

Simple and elegant. To escalate this further, we will turn this sequence into an infinite series.

First, define the sequence as:

$$S_n = \frac{x^n}{n}\ , n \geq 1$$

And now, we can define the infinite series as S(x):

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

We begin evaluating the infinite series by differentiating the series term by term:

$$S'(x) = \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{x^n}{n}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n}\right) = x^{n-1}$$

$$S'(x) = 1 + x + x^2 + x^3 + x^4 + \cdots$$

We recognize S'(x) as a geometric series with a common ratio of x. Since this is a geometric series, we can use the formula for an infinite geometric series:

$$S = \frac{a}{1 - r}$$

Plugging the values into our formula, the infinite series simplifies to:

$$\frac{1}{1 - x}, |x| < 1$$

$$\boxed{\therefore S'(x) = \frac{1}{1 - x}}$$

Now integrate both sides to find our infinite antiderivative series:

$$\int S'(x)\, dx = \int \frac{1}{1 - x}\, dx$$

$$\boxed{\therefore S(x) = -\ln|1 - x| + C}$$

But what should we do about the constant of integration C? We evaluate S(0) first before anything:

$$S(0) = \sum_{n=1}^{\infty} \frac{0^n}{n} = 0$$

Substitute $x = 0$ into the right-hand side:

$$0 = -\ln(1 - 0) + C$$

$$0 = 0 + C$$

$$\boxed{C = 0}$$

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$$\boxed{\therefore S(x) = -\ln(1 - x)}$$

## Discussion

The construction of the Antiderivative Power Rule Series reveals an important and profound connection between elementary calculus operations and transcendental functions. By successively applying the power rule for integration to x, the sequence you saw above earlier emerges naturally. Summing the sequence infinitely converges to a transcendental function which is $-\ln(1-x), |x| < 1$. This result demonstrates that through purely fundamental calculus principles, one can rediscover a function traditionally introduced through analytic continuation and complex analysis.

As Rhodes observed, "*The Polylogarithm is a very simple Taylor series, a generalization of the widely used logarithm function*" [1].

My series represents the **lowest-order case** of this generalization, corresponding to the polylogarithm order one, $Li_1(x) = -\ln(1-x)$. Hence, what begins as a simple looking exercise leads to a more vast conceptual territory such as a polylogarithm.

Wood (1992) similarly formalized this hierarchy by defining *"The Polylogarithm function, $Li_p(z)$, ... for its computation, valid in different ranges of its real parameter p and complex z" [2].*

And within this framework, my result shows that the iterative antiderivative process naturally reproduces the case p = 1, demonstrating how a fundamental rule in real calculus anticipates complex analytic generalizations.

This relationship also supports Gangl's later remark that *"polylogarithm functions, defined by the power series $Li_m(z) = \Sigma_{n\geq 1} \frac{z^n}{n^m}$, were long considered to be 'just another class' of special functions"* [3]

The work I did reinterprets Gangl's remark by showing that such 'special functions' may, in fact, arise organically from repetitive application of elementary calculus rules or mathematical rules, thus connecting the ordinary with the transcendental.

Altogether, these connections suggest that the **Antiderivative Power Rule Series** acts as a gateway from algebraic integration (indefinite integral) to transcendental analysis. Affirming that many transcendental functions can indeed be *rediscovered through fundamental calculus operations.*

## Conclusion

This study investigated the idea of how the basic pattern of antiderivatives would be able to encompass something much bigger in terms of the essence of the nature of calculus and how it is related to higher mathematics. By developing the sequence which adheres to the power rule of antiderivatives and then transitioning it into the infinite series, the research revealed that this method naturally leads to the identical form of the logarithmic and polylogarithmic functions previously expressed above. This indicates that what may be seen as the simple process of repeated integration is actually the route to the rediscovery of the more profound mathematical truths which existed within the core of the calculus all the time.

The significance of this work lies in showing that complex and transcendental functions do not always need to be introduced as advanced topics separate from basic principles. Instead, they can emerge directly from those principles through persistence and generalization. This realization connects elementary calculus with more abstract areas of mathematical analysis, providing a new way to appreciate how the simplest rules can evolve into sophisticated structures. It serves as a reminder that even the most familiar formulas have hidden layers of meaning that can be revealed through careful thought and pattern recognition.

For the future, this finding also creates potential for both education and research. Educationally, it offers the means to demonstrate to students how large mathematical concepts can develop out of the elementary material with which they already are familiar, closing the gap between elementary and advanced knowledge. Researchwise, it can stimulate new approaches to studying functions, series, and integrals by elementary operations that may give new interpretations to polylogarithmic and related transcendental behaviors. Truly, the present paper shows that rediscovering complexity within simplicity is one of the strongest methods in mathematics. By extending the reasoning behind the power rule of antiderivatives all the way to infinity, this paper shows how basic reasoning can lead back to some of the richest and most stunning structures in the realm of mathematical thought.

## References

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**Extra sources**

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