Refutation of consmological models of General Relativity

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Abstract: Section 1 of the article shows that the Schwarzschild metric and cosmological models with similar metrics are invalid because the spatial part of the metric is not a valid Riemannian metric in local Cartesian coordinates, as it should. Theorem 1 proves that a metric for the spatial part given in the spherical coordinates of $\mathbf{R}^3$ with only dr^2, $d\theta^2$ and $d\phi^2$ defines a valid metric in local Cartesian coordinates only if the spatial part of the metric is a scalar metric, i.e., a metric induced by a scalar field. Section 2 has some solutions for a scalar metric in the situation of a point mass in an otherwise empty space. Section 3 and 4 look at the Friedmann's cosmological model from Chapter 5 of Einstein's book combined from his lectures in Princeton. The findings are that each of Einstein's equation can be solved for a model that only depends on t and r, but the Einstein equations do not have a solution that solves them all and gives a valid metric. Additionally the Friedmann model does not give the cosmological solution that Einstein's book says.

Keywords: Schwarzschild metric, scalar gravitation field, Einstein's equations, Friedmann's universe.

1. The basic error in cosmological models of General Relativity

Cosmological models based on General Relativity are time dependent metrics used to investigate the large scale behavior of the universe. They are given in spherical coordinates of $\mathbf{R}^3$ and usually are of the type of the Schwarszhild metric:

$$ds^2 = Ac^2dt^2 - Bdr^2 - r^2d\theta^2 - r^2\sin^2(\theta)d\phi^2 \tag{1}$$

where the spatial part

$$d\rho^2 = Bdr^2 + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 \tag{2}$$

is defined in $\mathbf{R}^3$ in spherical coordinates.

In the Schwarzschild metric the functions A and B depend only on r, but in cosmological models there is also time dependency. This article explains the error in this approach.

Spherical coordinates relate to Cartesian coordinates of $\mathbf{R}^3$ as

$$x = r\sin(\theta)\cos(\phi) \quad y = r\sin(\theta)\sin(\phi) \quad z = r\cos(\theta). \tag{3}$$

The vector:

$$\vec{r} = x\vec{e_x} + y\vec{e_y} + z\vec{e_z} \qquad d\vec{r} = dx\vec{e_x} + dy\vec{e_y} + dz\vec{e_z} \tag{4}$$

has the norm

$$r^2 = x^2 + y^2 + z^2 \qquad (d\vec{r})^2 = dx^2 + dy^2 + dz^2 \tag{5}$$

and the vectors $d\vec{\theta} = 0$ and $d\vec{\phi} = 0$ as they are orthogonal to r, but the differentials dr, $d\theta$ and $d\phi$ of r, θ and ϕ are not zero. Differentiating

$$d(r^2) = d(x^2 + y^2 + z^2) \tag{6}$$

gives

$$dr = \frac{x}{r}dx + \frac{y}{r}dy + \frac{z}{r}dz \tag{7}$$

$$dr^2 = \frac{x^2}{r^2}dx^2 + \frac{y^2}{r^2}dy^2 + \frac{z^2}{r^2}dz + 2\frac{xy}{r^2}dxdy + 2\frac{xz}{r^2}dxdz + 2\frac{yz}{r^2}dydz. \tag{8}$$

Differentiating

$$\frac{\sin(\phi)}{\cos(\phi)} = \frac{y}{x} \tag{9}$$

gives

$$d\left(\frac{\sin(\phi)}{\cos(\phi)}\right) = \frac{\cos^2(\phi) + \sin^2(\phi)}{\cos^2(\phi)}d\phi = \left(1 + \frac{y^2}{x^2}\right)d\phi \tag{10}$$

$$d\left(\frac{y}{x}\right) = \frac{1}{x^2}\left(-ydx + xdy\right). \tag{11}$$

We get

$$d\phi = \frac{1}{x^2 + y^2}\left(-ydx + xdy\right). \tag{12}$$

From

$$x^2 + y^2 = r^2\sin^2(\theta) \tag{13}$$

$$r^2\sin^2(\theta)d\phi^2 = \frac{1}{x^2 + y^2}\left(-ydx + xdy\right)^2. \tag{14}$$

Differentiating

$$d\left(\sin^2(\theta)\right) = d\left(\frac{x^2 + y^2}{r^2}\right) \tag{15}$$

we get

$$2\sin(\theta)\cos(\theta)d\theta = \frac{2xz^2}{r^4} + \frac{2yz^2}{r^4} - \frac{2(x^2 + y^2)z}{r^4} \tag{16}$$

$$rd\theta = \frac{1}{r^2\sin(\theta}\frac{z}{r\cos(\theta)}\left(xzdz + yzdy - (x^2 + y^2)dz\right) \tag{17}$$

$$r^2d\theta^2 = \frac{1}{r^2(x^2 + y^2)}\left(xzdz + yzdy - (x^2 + y^2)dz\right)^2. \tag{18}$$

Summing (9), (15) and (19)

$$Adr^2 + Br^2d\theta^2 + Cr^2\sin^2(\theta)d\phi^2 \tag{19}$$

$$= dx^2 \left(A\frac{x^2}{r^2} + B\frac{x^2z^2}{x^2+y^2}\frac{1}{r^2} + C\frac{y^2}{x^2+y^2} \right) \tag{20}$$

$$+ dy^2 \left(A\frac{y^2}{r^2} + B\frac{y^2z^2}{x^2+y^2}\frac{1}{r^2} + C\frac{x^2}{x^2+y^2} \right) \tag{21}$$

$$+ dz^2 \left(A\frac{z^2}{r^2} + B\frac{x^2+y^2}{r^2} \right) \tag{22}$$

$$+ 2dxdy \left(A\frac{xy}{r^2} + B\frac{xyz^2}{(x^2+y^2)r^2}q - C\frac{xy}{x^2+y^2} \right) \tag{23}$$

$$+ 2dxdz \left(A\frac{xz}{r^2} - B\frac{xz}{r^2} \right) \tag{24}$$

$$+ 2dydz \left(A\frac{yz}{r^2} - B\frac{yz}{r^2} \right). \tag{25}$$

If $A = B = C = 1$ the result is the familiar formula

$$dx^2 + dy^2 + dz^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2. \tag{26}$$

Equation (2) is converted to Cartesian coordinates of $\mathbf{R}^3$ with (9) and (20)

$$d\rho^2 = (B-1)dr^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 d\phi^2 = (B-1)dr^2 + dx^2 + dy^2 + dz^2 \tag{27}$$

$$d\rho^2 = \left((B-1)\frac{x^2}{r^2} + 1 \right) dx^2 + \left((B-1)\frac{y^2}{r^2} + 1 \right) dy^2 \tag{28}$$

$$- \left((B-1)\frac{z^2}{r^2} + 1 \right) dz^2 - (B-1)2dxdy - (B-1)2dxdz - (B-1)2dydx.$$

We select a point P with $x_0 = y_0 = z_0 = \frac{r_0}{\sqrt{3}}$ for fixed value r_0. At any point a manifold has local Cartesian coordinates. We can choose the local coordinates as

$$x_1 = x - \frac{r_0}{\sqrt{3}} \quad x_2 = y - \frac{r_0}{\sqrt{3}} \quad x_3 = z - \frac{r_0}{\sqrt{3}} \tag{29}$$

$$dx_1 = dx \quad dx_2 = dy \quad dx_3 = dz. \tag{30}$$

Writing $C = (B(P)-1)\frac{1}{3} + 1$, $D = 2(B(P)-1)$, the metric in local coordinates at the point P is

$$d\rho^2 = C(dx_1^2 + dx_2^2 + dx_3^2) + D(dx_1dx_2 + dx_1dx_3 + dx_2dx_3). \tag{31}$$

On the coordinate axis x_i holds $dx_j = 0$ if $j \neq i$. On the coordinate axis x_i the metric is

$$d\rho^2 = C dx_i^2 \quad i = 1, 2, 3. \tag{32}$$

The local coordinates are orthogonal and Riemanninan metric is induced by an inner product. The metric at the point P can only be of the form

$$d\rho^2 = C_1 dx_i^2 + C_2 dx_2^2 + C_3 dx_3^2. \tag{33}$$

As (32) gives the value of ds^2 on each coordinate axis x_i, we can identify $C_i = C$ for $i = 1, 2, 3$. Expressing the space part in the Cartesian coordinates of $\mathbf{R}^3$ gives

$$d\rho^2 = C(dx_i^2 + dx_2^2 + dx_3^2) = C(dx^2 + dy^2 + dz^2) \tag{34}$$

which can be transformed to spherical coordinates of $\mathbf{R}^3$

$$d\rho^2 = C(dx^2 + dy^2 + dz^2) = Cdr^2 + Cr^2 d\theta^2 + Cr^2 \sin^2(\theta) d\phi^2. \tag{35}$$

The question is why (35) is not metric (2) multiplied by some constant in a small open environment of P expressed in the spherical coordinates of $\mathbf{R}^3$? The problem can only be the step from (32) to (33), but that means that the metric (2) is not a valid Riemannian metric. At point P, if a Riemannian metric satisfies (32) with orthogonal coordinates, the metric must be (33) with $C_i = C$ for all $i = 1, 2, 3$ in a small open environment of the point P.

In order to understand what goes wrong, consider making the same calculation with the two-form $d\rho^2 = ((B(r) - 1)\frac{1}{3} + 1)dr^2$ instead of (2). Clearly, this is not a metric for a three-dimensional space, but we can do all steps up to (32). Then comes step (33), which gives a three-dimensional Riemannian metric. It cannot be the same as this one-dimensional invalid metric (30). The part $(B(r) - 1)dr^2$ in (21) has exactly the same issue. The two-form (30) is not a valid Riemannian metric, this is why it does not equal (33) though it equals it at the coordinate axes in the local Cartesian coordinates at a small open environment of P.

Theorem 1. *A metric $d\rho$ in the spherical coordinates of* $\mathbf{R}^3$

$$d\rho^2 = A(t, r, \theta, \phi)dr^2 + B(t, r, \theta, \phi)r^2 d\theta^2 + C(t, r, \theta, \phi)r^2 \sin^2(\theta) d\phi^2 \tag{36}$$

must have

$$A(t, r, \theta, \phi) = B(t, r, \theta, \phi) = C(t, r, \theta, \phi) \tag{37}$$

This condition means that the metric is induced by a scalar field ψ where

$$\psi^2 = A(t, r, \theta, \phi) = B(t, r, \theta, \phi) = C(t, r, \theta, \phi). \tag{38}$$

Proof: Converting the metric into Cartesian coordinates of $\mathbf{R}^3$ as in (19)-(25) gives terms $dxdy$, $dxdz$ and $dydz$. These terms have zero coeffifients only if $A(t, r, \theta, \phi) = B(t, r, \theta, \phi) = C(t, r, \theta, \phi)$. Choose a point P in $\mathbf{R}^3$. Express $P = (p_1, p_2, p_3)$ in Cartesian coordinates of $\mathbf{R}^3$ and choose local Cartesian coordinates in a small open neighborhood of P as $x_1 = x - p_1$, $x_2 = y - p_2$, $x_3 = z - p_3$. Express the metric $d\rho^2$ in these local coordinates. On the coordinate axis x_i holds $dx_j = 0$ if $j \neq i$. On the coordinate axis x_i the metric is

$$d\rho^2 = C_i dx_i^2 \quad i = 1, 2, 3, \tag{39}$$

for some numbers C_i. They are numbers here, not functions, they are values of the corresponding functions at the point P. The metric at the point P can only be of the form

$$d\rho^2 = C_1 dx_i^2 + C_2 dx_2^2 + C_3 dx_3^2. \tag{40}$$

Notice what happens if any of the $dx_i dx_j$ has a nonzero coefficient in $d\rho^2$ in the local coordinates. It is not included in (40) and therefore the valid local metric (40) is different from (36). This means that (36) is not valid as a metric. If any of the terms $dxdy$, $dxdz$ and $dydz$ of the Cartesian coordinates of $\mathbf{R}^3$ has a nonzero coefficient at any point, then at that point a valid metric in local Cartesian coordinates cannot equal metric (36). EOP.

Notice that the form (36) does not give (valid) metrics that in the Cartesian coordinates of $\mathbf{R}^3$ have the form

$$d\rho^2 = Adx^2 + Bdy^2 + Cdz^2 \tag{41}$$

for any functions A, B, C. This is because form (41) when converted to spherical coordinates of $\mathbf{R}^3$ has nonzero terms $drd\theta$, $drd\phi$, $d\theta d\phi$. The correct way to formulate metrics for the space-time is not to give them in spherical coordinates of $\mathbf{R}^3$. Giving them in spherical coordinates assumes that the space dimensions are embedded in $\mathbf{R}^3$ and it also strongly restricts what metrics can be described: in spherical coordinates of $\mathbf{R}^3$ a metric that only includes dr^2, $d\theta^2$ and $d\phi^2$ terms can only describe one type of a Riemannian metric in local Cartesian coordinates: that of a metric induced by a scalar field. By defining a metric in local Cartesian coordinates one can describe other valid Riemannian metrics, or by including terms $drd\theta$, $drd\phi$, $d\theta d\phi$ in the metric in the spherical coordinates of $\mathbf{R}^3$, but doing so does not lead to a metric that is easily solved.

The Schwarzschild metric and metrics in cosmological models do not have constant speed of light at every point and they do not give the Minkowski metric at the tangent space at each point. The point P was selected especially to give the same speed of light to each space direction by requiring $x = y = z$. If this condition is not fulfilled, the metric (1) gives a different speed of light to different directions.

2. A solution to the Einstein equations for a scalar metric

Cosmological models in General Relativity seem to be expressed in spherical coordinates of $\mathbf{R}^3$ and Theorem 1 applies. This means that all these cosmological models have an invalid metric because they are not metrics that are induced by a scalar field. In this section we find a solution to Einstein equations for metric induced by a scalar field for the situation in the Schwarzschild solution: a point mass in otherwise empty space. We will work with Cartesian coordinates, it is easier than working in spherical coordinates, though one initially may think the opposite. There is much to say in favor of Cartesian coordinates in case the spatial manifold is embedded in $\mathbf{R}^3$ (which is not the general situation), one advantage being that it is easy to move to local Cartesian coordinates.

A scalar metric is metric induced by a scalar field ψ

$$ds^2 = \psi^2 c^2 dt^2 - \psi^2 dx^2 - \psi^2 dy^2 - \psi^2 dz^2. \tag{42}$$

In Cartesian coordinates the Ricci tensor entries for a scalar field are as follows. In the equations $x^0 = ct$ and j in x^j is an upper index.

$$R_{00} = -\psi^{-1}\Box\psi + \psi^{-2}\sum_{j=1}^{3}\left(\frac{\partial\psi}{\partial x^j}\right)^2 + 3\psi^{-2}\left(\frac{\partial\psi}{\partial x^0}\right)^2 - 2\psi^{-1}\frac{\partial^2\psi}{\partial {x^0}^2} \tag{43}$$

$$R_{ii} = \psi^{-1}\Box\psi - \psi^{-2}\sum_{j=1}^{3}\left(\frac{\partial\psi}{\partial x^j}\right)^2 + \psi^{-2}\left(\frac{\partial\psi}{\partial x^0}\right)^2 \tag{44}$$

$$-2\psi^{-1}\frac{\partial^2\psi}{\partial {x^i}^2} + 4\psi^{-2}\left(\frac{\partial\psi}{\partial x^i}\right)^2$$

for $i = 1, 2, 3$. These formulas are a convenient form for verifying

$$R = g^{\mu\mu}R_{\mu\mu} = -6\psi^{-3}\Box\psi \tag{45}$$

but not so convenient for solving the equations. We can write R_{00} as

$$R_{00} = \psi^{-1}\sum_{i=1}^{3}\frac{\partial^2\psi}{\partial {x^j}^2} + \psi^{-2}\sum_{i=1}^{3}\left(\frac{\partial\psi}{\partial x^j}\right)^2 \tag{46}$$

$$-\psi^{-1}\frac{\partial^2\psi}{\partial {x^0}^2} - 2\psi^{-1}\frac{\partial^2\psi}{\partial {x^0}^2} + 3\psi^{-2}\left(\frac{\partial\psi}{\partial x^0}\psi^{-1}\right)^2$$

$$R_{00} = \psi^{-1}\sum_{i=1}^{3}\left(\frac{\partial^2\psi}{\partial {x^j}^2} + \psi^{-1}\left(\frac{\partial\psi}{\partial x^j}\right)^2\right) \tag{47}$$

$$-3\psi^{-1}\sum_{i=1}^{3}\left(\frac{\partial^2\psi}{\partial {x^0}^2} - \psi^{-1}\left(\frac{\partial\psi}{\partial x^0}\right)^2\right)$$

$$R_{00} = \psi^{-1}\sum_{i=1}^{3}\psi^{-1}\frac{\partial}{\partial x^j}\left(\frac{\partial\psi}{\partial x^j}\psi\right) - 3\psi^{-1}\psi\frac{\partial}{\partial x^0}\left(\frac{\partial\psi}{\partial x^0}\psi^{-1}\right) \tag{48}$$

and finally as

$$R_{00} = \psi^{-2}\sum_{i=1}^{3}\frac{\partial}{\partial x^j}\left(\frac{\partial\psi}{\partial x^j}\psi\right) - 3\frac{\partial}{\partial x^0}\left(\frac{\partial\psi}{\partial x^0}\psi^{-1}\right). \tag{49}$$

If $R_{00} = 0$ we can write the equation $R_{ii} = 0$ by replacing the first two terms of (44) by the first two terms of (43) as

$$R_{11} = 3\psi^{-2}\left(\frac{\partial\psi}{\partial x^0}\right)^2 - 2\psi^{-1}\frac{\partial^2\psi}{(\partial x^0)^2} \tag{50}$$

$$+\psi^{-2}\left(\frac{\partial\psi}{\partial x^0}\right)^2 - 2\psi^{-1}\frac{\partial^2\psi}{(\partial x^i)^2} + 4\psi^{-2}\left(\frac{\partial\psi}{\partial x^i}\right)^2$$

which gives

$$R_{ii} = -2\psi^{-1}\left(\frac{\partial^2\psi}{\partial {x^0}^2} - 2\psi^{-1}\left(\frac{\partial\psi}{\partial x^0}\right)^2\right) - 2\psi^{-1}\left(\frac{\partial^2\psi}{\partial {x^i}^2} - 2\psi^{-1}\left(\frac{\partial\psi}{\partial x^i}\right)^2\right) \quad (51)$$

$$R_{ii} = -2\psi^{-1}\psi^2\frac{\partial}{\partial x^0}\left(\frac{\partial\psi}{\partial x^0}\psi^{-2}\right) - 2\psi^{-1}\psi^2\frac{\partial}{\partial x^i}\left(\frac{\partial\psi}{\partial x^i}\psi^{-2}\right) = 0. \quad (52)$$

and finally

$$R_{ii} = -2\psi\frac{\partial}{\partial x^0}\left(\frac{\partial\psi}{\partial x^0}\psi^{-2}\right) - 2\psi\frac{\partial}{\partial x^i}\left(\frac{\partial\psi}{\partial x^i}\psi^{-2}\right) = 0. \quad (53)$$

Notice that R_{ii} has this expression only if $R_{00} = 0$.

Equation (53) is solved by a trial

$$\frac{\partial\psi}{\partial x^a} = c_a\psi^2 \quad a = 0, 1, 2, 3. \quad (54)$$

Inserting (54) to (49) shows that (54) is solves (49) when

$$c_0^2 = \sum_{i=1}^{3} c_i^2. \quad (55)$$

$$R_{00} = \psi^{-2}\sum_{i=1}^{3}\frac{\partial}{\partial x^j}c_j\psi^2\psi - 3\frac{\partial}{\partial x^0}c_0\psi^2\psi^{-1}. \quad (56)$$

$$R_{00} = \psi^{-2}\sum_{i=1}^{3}3c_j\psi^2\frac{\partial\psi}{\partial x^j} - 3c_0\frac{\partial\psi}{\partial x^0}. \quad (57)$$

$$R_{00} = \psi^{-2}\sum_{i=1}^{3}3c_j^2\psi^4 - 3c_0^2\psi^2 = 0. \quad (58)$$

The solution for $R_{aa} = 0$ for all $a = 0, 1, 2, 3$ is therefore

$$\psi = \left(\sum_{a=0}^{4} c_a x^a + b\right)^{-1} \qquad c_0^2 = \sum_{i=1}^{3} c_i^2. \quad (59)$$

Let us look for more solutions of the type

$$\frac{\partial\psi}{\partial x^a} = c_a\psi^\beta \quad a = 0, 1, 2, 3. \quad (60)$$

If $\beta \neq 2$, then assuming that $R_{00} = 0$, the equation (53) for $R_{ii} = 0$ gives

$$c_i^2 = -c_0^2 \quad c_i = b, c_0 = \pm ib. \tag{61}$$

Inserting to (49) shows that for $\beta = \pm 1$ there are no solutions, but there is a solution for $\beta = 0$. This solution is

$$\psi = b \sum_{j=1}^{3} x^j \pm ibx^0 \tag{62}$$

where b is a constant and i is the imaginary unit.

We found two solutions, (59) and (62). There are no other solutions of the type (60). Whether there are more solutions of some other type will not be investigated as the Einstein equations are in any case incorrect: they cannot give any model for the static and spherical gravitational field of the Sun and the Earth.

Theorem 2. *In the situation of a point mass at the origin of otherwise empty space and with λ in Einstein's equations set to zero,*

a) two solution of the Einstein equations are

$$\psi = \left(\sum_{a=0}^{4} c_a x^a + b \right)^{-1} \qquad c_0^2 = \sum_{i=1}^{3} c_i^2. \tag{63}$$

$$\psi = b \sum_{j=1}^{3} x^j \pm ibx^0$$

b) There is no static nontrivial solution.

Proof: In this situation the Einstein equations reduce to $R_{aa} = 0$, $a = 0, 1, 2, 3$. The solutions in claim a) are (59) and (62). Claim b) is obtained from (53). If the solution is static, $c_0 = 0$ and if it is nontrivial, some $c_i \neq 0$. Then R_{00} in (49) is not zero. □

Israel's theorem claims that a static solution that approaches flat metric in infinity is the Schwarzschild metric. The Schwarzschild metric is not a valid solution. There is no static solution as Theorem 2 says.

Birkhoff's theorem says that if the metric is spherically symmetric and approaches flat metric in infinity, then it is static. If the metric is spherically symmetric, then the space part of it is as in (36) with $A = A(t, r)$, $B = B(t, r)$, $C = C(t, r)$. By Theorem 1 $A = B = C = \psi^2$ for a scalar field ψ. By Theorem 2 there is no static solution.

Einstein's book [1] is not a popular science book, though it is sold as one. It is based on Einstein's lectures in Princeton and he edited the book very close to the end of his life. This book represents best what Einstein himself thought about the General Relativity Theory rather than the theoretical papers [2][3] that are much older. Section 3 looks at Chapter 5 in this book because it deals with a cosmological model.

3. A product form scalar metric in spherical coordinates

Chapter 5 in [1] states that Einstein presents Friedmann's results. Friedmann's metric is not a scalar metric. It should be a scalar metric because every metric where the space part is given in spherical coordinates of $\mathbf{R}^3$ must satisfy Theorem 1. It must be induced by a scalar field because if it is not, the space part is not a valid Riemannian metric. Therefore an effort is made in this section to see what happens to this metric when it is properly formulated as a metric induced by a scalar field.

Friedmann considered a product form solution, but his metric was not a scalar metric. Let us consider a scalalar field of the product type

$$\psi = A(r)G(t) \tag{64}$$

and create from it a scalar metric. The Ricci tensor entries for this scalar metric in spherical coordinates (r, θ, ϕ) are:

$$R_{00} = \frac{A''}{A} + \left(\frac{A'}{A}\right)^2 + \frac{2}{r}\frac{A'}{A} - \frac{3}{c^2}\frac{G''}{G} + \frac{3}{c^2}\left(\frac{G'}{G}\right)^2 \tag{65}$$

$$R_{11} = -\left(3\frac{A''}{A} - 3\left(\frac{A'}{A}\right)^2 + \frac{2}{r}\frac{A'}{A} - \frac{1}{c^2}\frac{G''}{G} - \frac{1}{c^2}\left(\frac{G'}{G}\right)^2\right) \tag{66}$$

$$R_{22} = -r^2\left(\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 + \frac{4}{r}\frac{A'}{A} - \frac{1}{c^2}\frac{G''}{G} - \frac{1}{c^2}\left(\frac{G'}{G}\right)^2\right) \tag{67}$$

$$R_{33} = -r^2\sin^2(\theta)\left(\frac{A''}{A} + \left(\frac{A'}{A}\right)^2 + \frac{4}{r}\frac{A'}{A} - \frac{1}{c^2}\frac{G''}{G} - \frac{1}{c^2}\left(\frac{G'}{G}\right)^2.\right) \tag{68}$$

The Ricci scalar is

$$R = g^{ab}R_{ab} = A^{-2}G^{-2}\left(6\frac{A''}{A} + \frac{12}{r}\frac{A'}{A} - 6\frac{1}{c^2}\frac{G''}{G}.\right) \tag{69}$$

For easier verification of these formulas, the nonzero elements of the metric tensor are $g_{00} = c^2(AG)^2$, $g_{11} = -(AG)^2$, $g_{22} = -r^2(AG)^2$, $g_{33} = -r^2\sin^2(\theta)(AG)^2$, the sixteen nonzero Christoffel symbols are: $\Gamma^0_{00} = G'G^{-1}$, $\Gamma^0_{10} = A'A^{-1}$, $\Gamma^0_{11} = c^{-2}G'G^{-1}$, $\Gamma^0_{22} = c^{-2}r^2G'G^{-1}$, $\Gamma^0_{33} = c^{-2}r^2\sin^2(\theta)G'G^{-1}$, $\Gamma^1_{00} = c^{-2}A'A^{-1}$, $\Gamma^1_{10} = G'G^{-1}$, $\Gamma^1_{11} = A'A^{-1}$, $\Gamma^1_{22} = -(r + r^2A'A^{-1})$, $\Gamma^1_{33} = -sin^2(\theta)(r + r^2A'A^{-1})$, $\Gamma^2_{20} = G'G^{-1}$, $\Gamma^2_{21} = r^{-1} + A'A^{-1}$, $\Gamma^2_{33} = -\sin(\theta)\cos(\theta)$, $\Gamma^3_{30} = G'G^{-1}$, $\Gamma^3_{31} = r^{-1} + A'A^{-1}$, $\Gamma^3_{32} = \cot(\theta)$.

The flat metric expressed in spherical coordinates has the following nonzero elements as $\gamma_{00} = c^2$, $\gamma_{11} = -1$, $\gamma_{22} = -r^2$, $\gamma_{33} = -r^2 \sin^2(\theta)$.

We notice that every R_{bb}, $b = 0, 1, 2, 3$, gives $\gamma^{bb} R_{bb} = R_{bb}/\gamma_{bb}$ (no summation) of the type

$$\frac{R_{bb}}{\gamma_{bb}} = a_1 \frac{A''}{A} + a_2 \left(\frac{A'}{A} \right)^2 + \frac{a_3}{r} \frac{A'}{A} + \frac{a_4}{c^2} \frac{G''}{G} + \frac{a_5}{c^2} \left(\frac{G'}{G} \right)^2 \tag{70}$$

where a_i, $i = 1, , , 5$, are real numbers. What is important in (70) is that r and t separate

$$\frac{R_{bb}}{\gamma_{bb}} = f_b(r) + g_b(t). \tag{71}$$

From (69) we see that the term $\gamma^{bb} R_{bb} = R\psi^2$ is of the same type (71) and r and t separate.

Thus

$$R_{ab} - \frac{1}{2} R g_{ab} = \gamma_{ab} \left(\frac{R_{ab}}{\gamma_{ab}} - \frac{1}{2} R \psi^2 \right) \tag{72}$$

is of the type (71) and r and t separate.

We sum the Einstein equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \kappa_0 T_{ab} + \lambda g_{ab} \tag{73}$$

setting $\lambda = 0$ as Einstein does in his Chapter 5 of [1]. The result is

$$g^{ab} R_{ab} - g^{ab} \frac{1}{2} R g_{ab} = \kappa_0 g^{ab} T_{ab} \tag{74}$$

$$R - 2R = \kappa_0 T \tag{75}$$

$$R = -\kappa_0 T = -\frac{8\pi G}{c^4} T \tag{76}$$

where $T = g^{ab} T_{ab}$.

First let us make some observations. For all scalar fields ψ holds

$$R = -6\psi^{-3} \Box \psi. \tag{77}$$

Here $\psi = A(r)G(t)$. Inserting (77) to (76) gives

$$\psi^{-1} \Box \psi = -4\pi G \frac{1}{3c^4} T \psi^2. \tag{78}$$

The Ricci number R is not of the type (71), but $\psi^2 R$ is of the type (71). Thus, the left side in (78) is of the type (71). Therefore the right side of (78), $T\psi^2$, must also be of the type (71), the temporal and spatial parts separate as summands.

Einstein's Chapter 5 in [1] states that the density is constant in Friedmann's model (i.e, in the model in [1], not the one we have here, which is the scalar field model) and the equations in Chapter 5 in [1] show that T_{ab} only appears as fixed constants. Therefore the model must have set the tensor T_{ab} in the following way. The nonzero tensor values are

$$T_{00} = c^2 b_0 \qquad T_{11} = -b_1, \qquad T_{22} = -r^2 b_2 \tag{79}$$

$$T_{33} = -r^2 \sin^2(\theta) b_3 \qquad \text{with} \qquad \mathrm{b}_3 = \mathrm{b}_2 \tag{80}$$

where b_i, $i = 0, 1, 2$, are numbers. Notice that T_{33} has the same value $b_3 = b_2$ as T_{22}. It is because the equation

$$\frac{R_{ii}}{\gamma_{ii}} - \frac{1}{2} R\psi^2 = \frac{T_{ii}}{\gamma_{ii}} = b_i \tag{81}$$

is the same for $i = 2$ and $i = 3$. We are interested in what density can be set a constant in Friedmann's model in order to make the solution of the equations easier.

In classical equations the mass density is ρ in the Poisson equation (i.e. the Gauss equation)

$$\Delta\psi = -4\pi G\rho. \tag{82}$$

In Einstein's equations $\psi^{-1}\Box\psi$ separates into functions of t and r, so the property that can be constant is $\gamma^{ab}T_{ab} = T\psi^2$. It is of the form (71), i.e., the temporal and spatial parts separate. The expression $T\psi^2$ is not the density ρ, from (82) we see that it corresponds to

$$\rho\psi^{-1} = \frac{1}{3c^4} T\psi^2. \tag{83}$$

This is the expression that should be set to constant for an easy solution.

Let us also comment on λ. Einstein sets $\lambda = 0$ in Chapter 5 in [1]. He must do so because the term λg_{ii} is not of type (71), the temporal and spatial parts do not separate. It is of type $F(r)H(t)$ because $g_{ii}/\gamma_{ii} = A(r)^2 G(t)^2$. The problem is that if λ is not zero, we get an equation of the type

$$f(r) + g(t) = F(r)G(t). \tag{84}$$

Equation (81) implies that either $F(r)$ and $f(r)$ are constants or $G(t)$ and $g(t)$ are constants. In order to see that, select two values t_1 and t_2, then

$$g(t_1) - g(t_2) = F(r)(G(t_1) - G(t_2)) \tag{85}$$

holds for any chosen r. If $F(r)$ is not constant, $G(t)$ must be constant implying that $g(t)$ is constant, while if $F(r)$ is constant, then $f(r)$ must also be constant. So, λ must be set to zero to avoid this problem. The term λg_{ab} in the Einstein equations is incorrect. The term could only be $\lambda\gamma_{ab}$.

The Einstein equations in the situation of Chapter 5 in [1] for a scalar metric are

$$2\frac{A''}{A}+\frac{4}{r}\frac{A'}{A}-\left(\frac{A'}{A}\right)^2=3c^{-2}\left(\frac{G'}{G}\right)^2-b_0 \tag{86}$$

$$\frac{4}{r}\frac{A'}{A}+3\left(\frac{A'}{A}\right)^2=2c^{-2}\frac{G''}{G}-c^{-2}\left(\frac{G'}{G}\right)^2-b_1 \tag{87}$$

$$2\frac{A''}{A}+\frac{2}{r}\frac{A'}{A}-\left(\frac{A'}{A}\right)^2=2c^{-2}\frac{G''}{G}-c^{-2}\left(\frac{G'}{G}\right)^2-b_2 \tag{88}$$

The last equation from R_{33} is the same as (88) from R_{22}.

Taking the helpful choices of setting λ to zero and keeping $\rho\psi^{-1}$ as a constant, we proceed to solve the Einstein equations. With these helpful choices the temporal and spatial parts in every $R_{aa}=0$ separate as summands like in (71). As a consequence of these helpful choices, all equations $R_{aa}=0$ are easily solved.

4. Solution of the Einstein equations in this simple case

Nordström generalized the classical equation (82) into the form

$$\psi^{-1}\Box\psi=-4\pi T \tag{89}$$

Some sources give $\psi\Box\psi=-4\pi T$ but that only means a different definition of T, the formula is always the same and it corresponds to (82), [4] for Nordström's theory.

Einstein expanded this equation to several independent equations (in this case to three independent equations) of the type

$$R_{ii}-\frac{1}{2}g_{ii}=\kappa_0 T_{ii} \tag{90}$$

which can be summed into (76), but this division into three separate equations does not work, shown clearly by the functions R_{ii} not being close to zero for $\psi=r^{-1}$, the method is not working even for the Newtonian potential.

Especially the product form field $\psi=A(r)G(t)$ cannot give a solution to $R_{aa}=0$. The equations can be solved and the solutions show that their behavior is not at all similar to what Chapter 5 in [1] claims it is. We do not get the cosmological results that Einstein's Chapter 5 claims.

Let us start from the first equation R_{00}/γ_{00} is constant, equation (86). We can solve separately the radial and the temporal parts as the equations separate like in (71). The spatial part is some constant C:

$$2\frac{A''}{A}+\frac{4}{r}\frac{A'}{A}-\left(\frac{A'}{A}\right)^2=C. \tag{91}$$

First by the equality

$$\frac{d}{dr}\left(A'A^{\alpha}\right)=A''A^{\alpha}+\alpha\left(A'\right)^{2}A^{\alpha-1} \tag{92}$$

equation (88) is changed into

$$y'+\frac{2}{r}y=\frac{C}{4}f \tag{93}$$

where

$$y=A''A^{-\frac{1}{2}},\qquad f=\int ydr=2A^{\frac{1}{2}}. \tag{94}$$

This is a second order differential equation

$$rf''+2f'-\frac{C}{4}rf \tag{95}$$

which is solved by the Laplace transform

$$-\frac{d}{ds}\left(s^{2}F-sf(0)-f'(0)\right)+2(sF-f(0))+\frac{C}{4}F'=0 \tag{96}$$

$$F'=-\frac{f(0)}{s^{2}-\frac{C}{4}} \tag{97}$$

$$F'=-\frac{f(0)}{\sqrt{C}}\left(\frac{1}{s-\sqrt{C}/2}-\frac{1}{s+\sqrt{C}/2}\right) \tag{98}$$

$$F=-\frac{f(0)}{\sqrt{C}}\left(\ln(s-\sqrt{C}/2)-\ln(s+\sqrt{C}/2)\right). \tag{99}$$

Using the rule: if $\mathcal{L}\{\mathrm{f(t)}\}=\mathrm{F(s)}$ then $\mathcal{L}\{\mathrm{tf(t)}\}=-\mathrm{F'(s)}$ we get

$$f(r)=\frac{f(0)}{\sqrt{C}}\frac{1}{r}\left(\exp\left(\frac{1}{2}\sqrt{C}r\right)-\exp\left(-\frac{1}{2}\sqrt{C}r\right)\right) \tag{100}$$

and $A=f^{2}/4$

$$A(r)=\frac{f(0)^{2}}{4C}\frac{1}{r^{2}}\left(\exp\left(\sqrt{C}r\right)-2+\exp\left(-\sqrt{C}r\right)\right). \tag{101}$$

Depending on what C is, this function either grows exponentially or is a wave.

The temporal part of the equation (86) is simply (C here is different than in (91))

$$\frac{G'}{G}=C. \tag{102}$$

The solution is $G(t)=C_{1}\exp(Ct)$. Depending on C the growth is either exponential or the function is a wave.

The spatial part of R_{11}/γ_{11}=constant, equation (87), is a second order equation for $y = A'A^{-1}$

$$\frac{4}{r}\frac{A'}{A} + 3\left(\frac{A'}{A}\right)^2 = C \tag{103}$$

$$y^2 + \frac{4}{3r}y - \frac{C}{3} = 0 \tag{104}$$

$$y = -\frac{2}{3r} \pm \sqrt{\frac{4}{9r^2} + \frac{C}{3}} \tag{105}$$

$$\ln A = \int y dr \tag{106}$$

$$A = C_1 r^{-\frac{2}{3}} \exp\left(\pm \int \sqrt{\frac{4}{9r^2} + \frac{C}{3}} dr\right). \tag{107}$$

The growth is exponential, either growing or decreasing.

The temporal part of (87) is (C is different than before)

$$2\frac{G''}{G} - \left(\frac{G'}{G}\right)^2 = C \tag{108}$$

$$y = G'G^{-\frac{1}{2}} \qquad f = \int y dt = 2G^{\frac{1}{2}} \tag{109}$$

$$f'' = \frac{C}{4}f \tag{110}$$

$$f = C_1 \exp\left(\frac{\sqrt{C}}{2}t\right) + C_2 \exp\left(-t\frac{\sqrt{C}}{2}t\right) \tag{111}$$

and $G(t) = f^2/4$. Depending on C the growth is either exponential or the function is a wave.

The spatial part of R_{22}/γ_{22}=constant, equation (88), is a bit more difficult to calculate. The equation is

$$2\frac{A''}{A} + \frac{2}{r}\frac{A'}{A} - \left(\frac{A'}{A}\right)^2 = C. \tag{112}$$

Laplace transforming

$$F'\left(-s^2 + \frac{C}{4}\right) - sF = 0 \tag{113}$$

$$\frac{F'}{F} = -\frac{s}{s^2 + C/4} = -\frac{\sqrt{C}}{4}\left(\frac{1}{s - \sqrt{C}/4} + \frac{1}{s + \sqrt{C}/4}\right) \tag{114}$$

$$F = \left(s^2 - \frac{C}{4}\right)^{-\sqrt{C}/4}. \tag{115}$$

It can be difficult to make the inverse Laplace transform analytically (though it can be made numerically) for a freely chosen C, but for instance for $C = 16$ it can be made

$$F = \frac{1}{s^2 - C/4} = \frac{1}{\sqrt{C}} \left(\frac{1}{s - \sqrt{C}/2} - \frac{1}{s + \sqrt{C}/2} \right) \tag{116}$$

$$f(r) = \frac{1}{\sqrt{C}} \left(e^{t\sqrt{C}/2} - e^{-t\sqrt{C}/2} \right) \tag{117}$$

and $A(r) = f^2/4$. Depending on C the growth is either exponential or the function is a wave.

The temporal part of (88) is

$$2\frac{G''}{G} - \left(\frac{G'}{G}\right)^2 = C \tag{118}$$

This is calculated in the same way as (92). Depending on C the growth is either exponential or the function is a wave. The equation R_{33}/γ_{33}=constant is the same as R_{22}/γ_{22}=constant, equation (88).

The equation for the Ricci scalar in (76)

$$R\psi^2 = -\kappa_0 T\psi^2 = \text{constant} \tag{119}$$

gives

$$6\frac{A''}{A} + \frac{12}{r}\frac{A'}{A} - 6\frac{G''}{G} = b_0 + b_1 + 2b_2 \tag{120}$$

$$\frac{A''}{A} + \frac{2}{r}\frac{A'}{A} = \frac{G''}{G} + \frac{b_0 + b_1 + 2b_2}{6} \tag{121}$$

The spatial part is

$$rA'' + 2A' - CrA = 0 \tag{122}$$

Laplace transforming we get

$$A(r) = \frac{A(0)}{\sqrt{4C}}\frac{1}{r} \left(\exp(\frac{1}{2}\sqrt{4C}r - \exp(-\frac{1}{2}\sqrt{4C}r \right) \tag{123}$$

Depending on C the growth is either exponential or the function is a wave.

The temporal part is

$$\frac{G''}{G} = C \tag{124}$$

with again the same comments: either exponential or a wave.

As a conclusion, the equations for each R_{aa}/γ_a=constant in spherical coordinates can be solved for a metric that only depends on t and r. The equations have a certain form and the solutions to them are exponential both in r and in t.

Depending on if the constant C is chosen real or imaginary, the behavior can be exponential or a wave. The equation for the Ricci scalar has a similar behavior.

However, combining these solutions for R_{aa}/γ_a=constant to a full solution to the Einstein equations fails: the solutions for different R_{aa} give different functions $A(r)$ and $G(t)$. They should be the same from the different equations in order to give a solution for all Einstein equations. This confirms the findings in Section 2 made by a much more simple way of using Cartesian coordinates: there are no spherically symmetric solutions that give a valid metric - a valid metric in the local Cartesian coordinates is only obtained if the metric is a scalar metric.

Calculating in spherical coordinates does not make the task of solving the Einstein equations any easier. While it initially may seem like a good idea to have only r and t as unknowns, using spherical coordinates of $\mathbf{R}^3$ obscures the fact that the spatial part of the solution must be a Riemannian metric in local Cartesian coordinates. Metrics in a Riemannian manifold should be defined in local Cartesian coordinates, not in global spherical coordinates of $\mathbf{R}^3$ even in a lucky case that the 3-dimensional spatial submanifold would embed into $\mathbf{R}^3$, and if the spatial submanifold does embed in $\mathbf{R}^3$, it is still easier to calculate with Cartesian coordinates of $\mathbf{R}^3$. They show well that in order to get a function of r and t, the terms x, y and z that necessarily come from partial differentiation with respect to the coordinates must sum as $r^2 = x^2 + y^2 + z^2$. They do so in the Ricci scalar. They do not do so in R_{aa}.

5. Einstein's solution in Chapter 5 of [1]

Einstein does not solve the equations. He states that the equations are (equation (3) in chapter 5 in [1])

$$-\frac{1}{r}\left(\frac{A'}{A}\right)' + \left(\frac{A'}{Ar}\right)^2 = 0 \tag{125}$$

$$-\frac{2A'}{Ar} - \left(\frac{A'}{A}\right)^2 - BA^2 = 0 \tag{126}$$

and then he gets the solution (equations (3a) and (3b) in chapter 5 of [1])

$$A = \frac{c_1}{c_2 + c_3 r^2} \qquad B = 4\frac{c_2 c_3}{c_1^2} \tag{127}$$

Compare these equations with (86),(87) and (88) for a scalar metric in spherical coordinates. Friedmann did not use a scalar metric, though he should have. Yet the placement of r in (125)-(126) is not explained by a different metric.

Einstein has the term r^{-1} in the second derivative of A in (125) and can calculate

$$\frac{1}{r}\left(\frac{A'}{A}\right)' = \frac{1}{r}\frac{d}{dr}\left(\frac{A'}{A}\right) \tag{128}$$

$$= \frac{1}{r}\frac{A''}{A} - \frac{1}{r}\left(\frac{A'}{A}\right)^2 \tag{129}$$

In equations (86) and (88) there is the second derivative of A, but the term r^{-1} is only a coefficient in the term $A'A^{-1}$, see the form of the expression in equation (70). Einstein got the result (127) only because r is in the place it is in (125).

Einstein implies in Chapter 5 of [1] that Friedmann's model gives an expanding universe, but if the solutions grow (the constant C is real, not imaginary) then the growth is exponential, not what Chapter 5 says.

Einstein must have known that the speed of light is constant c in vacuum only if ψ is a scalar field because the speed of light to direction x_i, $i = 1, 2, 3$ in Cartesian local coordinates can be read from the space element. Light travels along light like world paths in relativity theory, thus $ds = 0$ in

$$ds^2 = c^2 g_{00} dx_0^2 - g_{11} dx_1^2 - g_{22} dx_2^2 - g_{33} dx_3^2 \tag{130}$$

The speed of light to the direction of x_i is obtained by setting $dx_j = 0$, $j \neq i$, $j \in \{1, 2, 3\}$

$$ds^2 = 0 = c^2 g_{00} dx_0^2 - g_{ii} dx_i^2 \tag{131}$$

$$c^2 = \frac{g_{ii}}{g_{00}}\frac{dx_i^2}{dx_0^2} = \frac{g_{ii}}{g_{00}} \tag{132}$$

as the differentials dx_i are Euclidean. Thus

$$g_{ii} = c^2 g_{00} \tag{133}$$

and we can define $\psi = c^{-1}\sqrt{g_{00}}$ and the field ψ is a scalar field.

In 1952 when Chapter 5 of [1] was written, Einstein had spent decades with the Einstein equations and must have known that they cannot be solved with $\psi = A(r)G(t)$ if the metric is a scalar metric. He also must have known that the Einstein equations do not have any scalar metric solutions that are close to the Newtonian gravitational field in the situation of a point mass in empty space even if we allow the field to depend on (r, θ, ϕ, t). That is the reason why there is the Schwarzschild solution. But the Schwarzschild metric is not possible. The spatial part of the metric is not a valid Riemannian metric. The Schwarzschild metric also does not have a constant speed of light every point to every direction and the metric is not Minkowskian at the tangent space of each point. The only conclusion is that the Einstein equations are incorrect.

6. Conclusions

Maybe one can bend the requirement that the speed of light is constant and that the metric in the tangent space is not Minkowskian, but one cannot bend the condition that the space part of the space-time is a Riemannian manifold. Sections 1 and 2 show that the Schwarzschild metric and cosmological models with similar metrics are invalid because the space part is not a valid Riemannian metric.

Section 3 and 4 look at an early cosmological model, Friedmann's model, from Einstein's book [1]. The findings are that each of Einstein's equation for a simple case of a point mass in otherwise empty space can be easily solved for a model that only depends on t and r, but the Einstein equations do not have a solution that solves them all and gives a valid metric. Using spherical coordinates is not any good idea, it hides the essential issue that the solutions that this method gives need not give a valid Riemannian metric for the space part. Simple calculations with local Cartesian coordinates, as in section 2, are the preferred method.

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