Spice–grains in kneaded dough are able to move similarly to masses free floating in space–time of earth's interior.

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A Abstract.

The following discourse describes a strange-looking similarity in movement-patterns of 2 processes (which at first glance) seemed to have little to do with each other. This similarity became obvious, when 2 masses simultaneously but independently oscillate on a straight path through earth's interior, and spice-grains embedded in dough get displaced against each other during kneading of the dough.

1 *The idea for the 1st process originates from a thought-experiment, where 2 earth-ships are considered starting time-shifted from a western point on earth's surface, passing in a shaft straight through the midpoint of earth to the opposite point on eastern part of earth's surface, where they will come to rest again. Just after a ship reaches its 1 rest-point, it immediately will float back to its opposite, and this procedure maybe repeated as often as required.*

1 *Both ships will move in accordance with the 2−dimensional space-time insight the shaft. The world-lines of the ships revealed themselves as geodesics on a sphere, starting close together and parallel but time-shifted. During each of their courses along the shaft the ships' geodesics first deviate from each other up to a maximum, then approach again and finally cross over. Thus, each ship finishes a period of its trip through the shaft by covering the pathlength between 2 consecutive rest-points. The periods of both ships are equal. Because of the sphere on which both geodesics ran, surface's curvature can relatively easy be calculated upon the movement's characteristics.*

2 *Regarding the 2cd of the mentioned processes, mathematicians agree that for special cases of chaos-applications, where iteration of transformations are considered, 3 common characteristics of chaos become visible: sensitive dependence on initial conditions, mixing of intervals along a path and dense periodic-points. Dough-kneading, a chaotic-process whose features shall be compared to those of the thought-experiment, is one of those just mentioned chaos-applications.The characteristics of dense periodic-points is of particular interest in this context, its consequences will be important in comparing the 2 physical-processes mentioned above.*

1 *Its started from model-presentation of 2 mutually suporting concepts. For both models kneadings are considered 1st by uniform stretching of the dough to double length and then either folding it over at its centre, or by cutting it at its centre and paste the 2 halfs on top of each other. The steps are continued as often as required. While both kneading-processes worked cooperatively their chaos-properties were demonstrably preserved.*

2 *Unfortunately, these steps were not enough to compare with outcomes of the thought-experiment. But a transition to a non-uniform stretching in form of the quadratic-iterator* $y \leftarrow x_{j+1} = 4x_j \cdot (1-x_j),\ j = 0,1,2,\dots$ *was finally able help here further. Various tests eventually proved that the consistency with regard to the chaos-characteristics were maintained for all 3 models.*

3 *Iterator* $y \leftarrow x_{j+1} = 4x_j(1-x_j)$ *projects point-sequences on the generic-parabola* $4x(1-x)$ *when executing transformation-steps. Thus, starting from 2 closely-adjacent periodic-points* $[0 \leq x_0, x'_0 \leq 1]$ *the iterator generates 2 cyclic-sequences each with its own period and their points positioned on generic-parabola. These cycles first deviate from and then approach towards each other, after had reached a maximum somewhere in between. Eventually, when each cycle finishes numbers of passes equal to smallest common multiple of the individual periods, they will meet with their starting-conditions again. Because corresponding points of the cyclic-sequences are shifted against each other, they can be looked-up as positively curved geodesics on a curved* $x,y-$*surface. Due to relation* $c = x/t$, *with* $c = 3 \cdot 10^8 [m/s]$ *vacuum light-speed and* t,x *as time- and space-unit, one can migrate from* $(x,y)-$ *to* $(x,t)-$*coordinate-system. Thus, appropriate curved surface and the geodesics on it can likewise be looked-at in space-time as well.*

4 *Differences in movement-patterns between thought-experiment and dough-kneading, are mainly limited to the fact that curvature once is referreded to a sphere, the other time to a parabolic-surface, the dynamics on both surfaces, however, takes place in similar manner.*

B Thought−Experiment.

In order to get some inside into gravity as a result of curved space-time, following thought-experiment may be considered. It's an old idea to have a tunnel right through the earth by which one can directly go from west to east, and no better term will exist in connection with this as boomeranging, to name an imaginary journey of someone who falls this way through the earth without ever encountering an obstacle and therefore will float (after having reach the opposite end of the shaft) easily back to his starting-point. Both inspirations have been brought together by this thought-experiment in order to acquire insights about gravity based on curvature of a 2 dimensional space-time and phenomena resulting from this fact.

1 *A tunnel could be conceived, extending from west (Atlantis, an imaginary island in the Atlantic Ocean) to east (Taralga, a place in Australia, in neighbourhood of Sydney and Canberra) straight through the midpoint of earth. It is assumed, there will be vacuum throughout the interior of the shaft. Therefore, other than gravity, earth will not exert any other influence (e.g. due to its rotation or temperature) on floating test-masses in the shaft. It means, earth-ships moving between Atlantis and Taralga (or vice-versa) will never encounter any obstacle on their imaginary journey, and thus will complete each of both routes (forth and back) within 42 min's constantly.*

2 *Earth–Ship–A starts moving through the Tunnel.*

Earth-ship-A will rush (free-floating) from Atlantis to Taralga forth and back through the shaft, that is without own propulsion and any feeling of weight inside and outside the ship.

1 *Unfortunately, nothing could be further experienced by this trip of ship–A about space-time inside earth. Thus, 2 earth-ships (ship–A and ship–B) moving simultaneously through the tunnel are required to ensure further progress herewith.*

3 *Ships–A and –B start Boomeranging through the Shaft.*

Two earth-ships (ship–A and ship–B not influencing each other in any kind, but which in turn are influenced by the gravity of earth, will boomerang simultaneously through the shaft, starting from Atlantis floating to Taralga and vice-versa, periodically in both directions.

1 *Precisely at 11.18 o'clock ship–A may start from platform Atlantis and rushes along the marked spots of earth's interior in direction Taralga. Ship–B sets off on the same journey 2 sec's later from the same platform.*

2 *Ship–A reaches the platform Taralga for the 1st time at 12.00 o'clock. Immediately it starts from Taralga for its return back to Atlantis. At the same time ship–B (still on its way upward to Taralga) is 20 m below the platform. Ship–B recognizes that its upward-speed has halved 1sec before 12.00 o'clock. Ship–B is in its final phase of 1st ascent to Taralga. In this position (only 5 m below the platform) the upwards-flying Ship–B passes downwards-rushing ship–A (which has already covered the first 5 m of return to Atlantis).*

1 *In order to calculate space for both ships gliding past each other when they meet in shaft, one actually should take into consideration separate starting-points for both ships. But due to dimensions of the shaft and a relatively low speeds of the ships, this effect is negligible.*

3 *2 sec's after 12 o'clock ship–B is at the peak of its flight. Ship–A rushes downwards with doubled speed it had 1 sec ago and it has already flown downwards 20 m.*

1 *After ship–B had finished its 1st turn to Taralga, 3 sec's after 12 o'clock it has reached a position 5 m below the Taralga-platform.*

4 *Both boomerangers passed each other next at Atlantis 12.42 o'clock. Ship–B gets slower in ascending, ship–A increases its speed starting downwards for another run to Taralga, where both ships will meet again 42 min's later.*

1 *This minuet will be repeated again and again and keeps its 42–min's–tact even if both masses in their state of free-floating will rush with 28440 km/h through the midpoint of earth.*

4 *Curvature of Space–Time in Boomerang–Shaft.*

The 2 ships in state of suspension had been observed up to now. Precisely at 12.00 o'clock their orbits came close together 5 m below Taralga. Afterwards they floated-away into darkness of the shaft. Their mutual distance increased every sec by 20 m, but this did not last forever.

1 *Just before the metronome struck for the 42–min's-tact both ships approach Atlantis. Important in this context is, ship–B comes closer to ship–A. The distance between both ships will not*

increase by 20 m anymore, but decreases by this amount.

1 *The distance between both ships quickly increased during 1st phase of the journey from Taralga to Atlantis and decreased quickly in its 2cd phase. Obviously exists 1 position between Taralga and Atlantis where slow increase of distance will change into a slow decreese, a point where distance between ships does not change any-more and reaches its maximum. This happens at midpoint of earth and the maxi-mum between ships at this point gets 16 km. Distance of 16 km is only a small part of earth-diameter therefore one may speak here about local-physics.*

2 *What can be learned from this behaviour of both ships, about the nature of space-time in a local-area?*

1 *Geometry of space-time prevented that both ships will fly-apart here for all time.*

2 *In the beginning observers in each of both ships respectively can see the other ship flying-away with uniform retreat-speed. As mutual distance between ships began to increase, retreat-speed starts to decrease from one moment to the next. An actual decrease of retreat-speed per time-unit is proportional to average-distance between both ships within appropriate time-unit.*

3 *While both ships boomerang through the shaft, their mutual distance (even at its maximum) never gets bigger than a small part of the earth-diameter. A reason why both ships always meet again cannot be found by flat geometry. Otherwise, ships will deviate from each other permanently. Geometry must be curved, only this way can be understood from space-time-geometry, why 2 world-lines (originally deviating from each other) gradually come closer together again finally and will cross eventually. Not the world-lines themselves are curved, it's the space-time where ships are freely-floating in.*

3 *The movement of the 2 boomerangers not only teaches about the existence of a curved space-time, it also provides a measure for this. The ratio between change of speed (by which both ships escape from each other) to the distance they finally will reach, is the distinct measure of curvature: $Curvature-of-space-time = (change-of-retreat-speed)/distance$.*

1 *Because initial distance between both ships is small (only 20 m) the initial retreat-speed of 20 m/s decreases only slowly. 1 sec later, when distance between both ships increased to 40 m, distance still grow further, but afterwards speed of distance-increase fell twice as fast as it did 1 moment before.*

2 *If distance between the ships came close to its maximum of 16 km, retreat-speed of the ships decreases most quickly. It falls to 0 and becomes negative thereafter. This means, retreat—speed changed into approach—speed.*

3 *3 different distances and 3 very different values of velocity-change but one size will keep constant: the ratio between change-of-retreat-speed and distance always has same value. This value is very small (only $1.73 \cdot 10^{-23}\ m^{-2}$). The reason for this is a small space-time-curvature caused by a diluted matter of earth.*

5 *Grip of the Mass onto Space—Time.*

Many properties of curvature reveal themselves on the 2—dimensional surface in space-time. The coordinates in space-time of the 2 boomerangers on their ways through the shaft are their distances and the time displayed in 1 of the ships (for example in ship—A).

1 *On surface's curvature, their positive or negative character is particular worth mentioning. No example will better illustrated, what positive curvature means than the sphere of a ball, nothing will give a better idea about a negative curvature than the surface of a saddle and nothing describes vanishing curvature (flatness) simpler than a sheet of paper. Thus, what will bring 2 ants started in parallel on a ball's-surface finally together again? It's positive space-time-curvature.*

2 *What brings 2 boomerangers, who free-float in space-time inside earth, finally together again? It's called positive space-time-curvature. This curved space-time has revealed itself through the menuet of the neighboured world-lines of the 2 earth-ships which sailed like boomerangs free through the earth and whose orbits passed close to each other every 42 min's.*

3 *The space-time where both earth-ships passed-through describes a positive curvature on a*

sphere. This can be seen not only from long-term-behaviour, when their orbits interveave each other, but more directly in short-time-behaviour when their worlds-lines curved towards each other.

1 *If only 1 is taken into acount, no space-time-curvature of the earth-interior will become visible. This is due to the fact that each ship follows the most-straight path through space-time during the state of foat. No single world-line alone, but the local bend of a world-line relative to another will indicate space-time-curvature.*

4 *While both earth-ships rushed through the midpoint of earth, each one described a free-floating world-line. At the midpoint of earth their mutual distance grow up to 16 km. Both ships now finished their drifting-apart from each other, but did not yet start to come closer to each other. The world-lines of the ships (now 16 km apart) describe parallel world-lines in space-time. Nevertheless, the ships began now to approach each other in oder to finally meet again.*

1 *Both world-lines were obviously bent relative to each other. This was not a matter of bending individual world-lines, but the bend of one line as seen from the other world-line perceived on account of distance-changing. This change of approach-speed from 0 to a positive value signals positive curvature of space-time.*

2 *The bend of a free-floating world-line towards or away from a neighboured world-line, which is passed in similar way, proves the local existence of space-time-curva-ture and thus the local existence of gravity.*

6 A free–floating World–Line is a Geodesic.

A potato-peel may serve briefly as model for a curved surface. On this surface a line is drawn with ballpoint-pen or by the path of an ant. This line is definitely curved, but is it straight too?

1 *An ant will crawl straight ahead on the potato-surface as best it can, without deviating to the left or to the right. This line cannot be called straight, but there is a clear difference between the line describe by the ant and other lines which do not have a strong sense of direction and meander-forward. Since a long time, such a line running either on a flat or curved surface, which deviates neither left nor right, is called a geodesic.*

2 *There is still an alternate way to distinguish a geodesic from other lines: A geodesic is the line of shortest length connecting 2 points in space. In space-time, among time-like world-lines (either squared or smooth) connecting a start- with a target-event, a geodesic is distinguished by the largest elapsed proper-time. Proper-time means, the amount of elapsed aging, displayed by the clock of reference-system. A geodesic-world-line describes a power-less connection of one event to another, it is the world-line of free-floating between these events.*

7 A Possibility of Curvature–Measurement.

To measure curvature of space-time means, to assign each elementary area of space-time its responsebility for bending of geodesics.

a. *Space means, the spatial distance between the 2 earth-ships.*

b. *Time means, change in approach-speed of both ships between a certain point-in-time and the moment after, measured within each ship separately.*

c. *Bend means, tendency of both world-lines to approach to or separate from each other, expressed by the change of relative-speed between both ships.*

d. *Cause of the bend means, 2 factors have to be taken into consideration, the spatial-distance and the time-interval between both measurements of the approach-speed. It's neither the space between the ships alone that counts, nor the time between both measurements of the approach-speed separately. Change of approach-speed from beginning to a subsequent moment is proportional to the product of spatial- and temporal-factor, thus is proportional to the space-time-area specified based on both measurements.*

1 *1 sec after the 2 earth-ships have reached their maximal-distance from each other at mid-point of earth, they approached each other with a speed of $2.4 \cdot 10^{-2}$ m/s (or related to the speed-of-light with $0.8 \cdot 10^{-10}$). 1 sec before the ships had reached their maximal-distance, they were deviating from each other by $-0.8 \cdot 10^{-10}$. Thus, in the interval of 2 sec during which both ships were next to the midpoint of earth, the change of their approach-speed thus was $1.6 \cdot 10^{-10}$ (relative to speed-of-light). This value is to be considered as measure for the bend of world-lines*

from both ships towards each other within 2 elapsed sec's.

2 The space-time-area responsible for the bend (just mentioned) is bordered by the time-lines between both ships at moments of 1^{st} and 2^{cd} speed-measurement and both world-lines (flowed-through between both measurements).

1 Near midpoint of earth both ships float with a speed of $7.9 \cdot 10^3 \mathrm{m/s}$ or nearly $1.6 \cdot 10^4$ m within 2 sec's. The time of 2 sec's expressed in geometric-units of space-time corresponds to $2\mathrm{s} \cdot (3 \cdot 10^8)\ \mathrm{m/s} = (6 \cdot 10^8)\ \mathrm{m}$. The product of width $(1.6 \cdot 10^4)$ m in space multiplied by the length $(6 \cdot 10^8)$ m of time will result in the spanned space-time-area $(9.6 \cdot 10^{12})\ \mathrm{m}^2$.

3 Which part of curvature every m^2 of the spanned space-time-area is responsible for, is given:

1 By dividing the measure of curvature $1.6 \cdot 10^{-10}$ by the spanned space-time-area of $(9.6 \cdot 10^{12})\ \mathrm{m}^2$. One will obtain the result $(17 \cdot 10^{-24})/\mathrm{m}^2$. The question comes up now, how such a small curvature can have such a big impact on the world-lines of the 2 ships on their way through the earth? The answer on this question cannot be found in the smallness of curvature, but in the slowness (compared to the speed-of-light) both boomerangers will gain on their journey from Atlantis to Taralga (or vice versa) if geometric measure of light-flow is taken into calculation.

C Uniform Kneading of Dough in Time– and Space–Dimension.

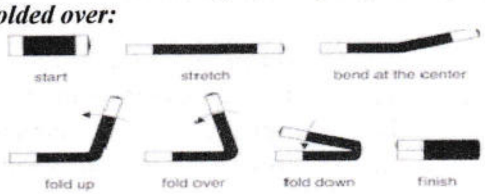
Mathematicians agree, in special cases of iteration of transformations there are 3 common characteristics of chaos:

a. Sensitive dependence on initial conditions.
b. Mixing of intervals along a path.
c. Dense periodic points.

Kneading of dough provides an intuitive access to all these mathematical problems of chaos.

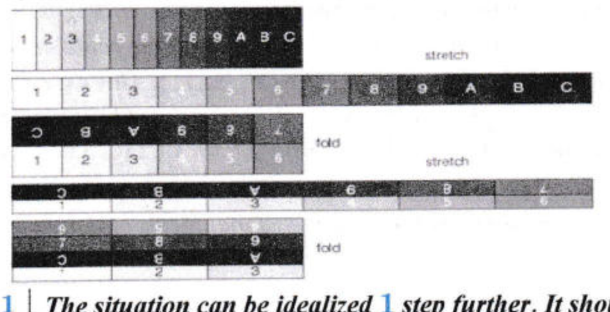
1 Kneading may be realized by stretching dough and folding it over, repeated many times. The result has many in common with randomness.

1 The dough now is considered schematically in side-view position in following stretch-and-fold-operations. It's homogeneously stretched to twice its length, afterwards bent at the centre and folded over:



Uniform kneading by stretch-and-fold.

2 To make the situation more clearer, the dough may be divided into 12 blocks which are then subjected to 2 stretch-and-fold-operations:

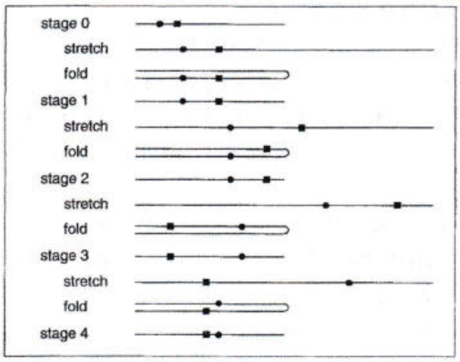


2 operations of stretch-and-fold-kneading on 12 blocks of dough.

1 The situation can be idealized 1 step further. It should be imagined working with infinitely-thin layers of dough. Folding these layers will not change the thickness and so one can represent the dough by a line-segment.

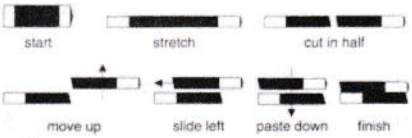
2 Two grains of spice may be marked insight dough and their paths will be followed. The 2 grains are rather close together initially. But after a few kneadings one will find them in very different places. This is a consequence of mixing-properties originated by a kneading-operation. Thus, kneading destroys neighbourhood-relations, grains being very close ini-

tially will likely not be close after a while. This is the ef-fect of sensitive-dependence-on-initial-conditions. Small deviations in beginning may lead to large devia-tions in course of the process.



2 grains, symbolized by a dot and a square, are subjected to 4 stages of stretch-and-fold. They are mixed through the dough.

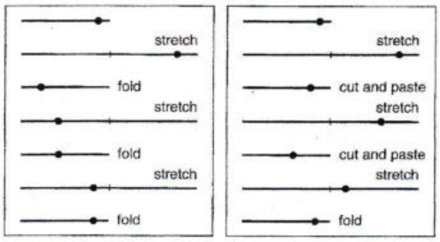
2 *A 2^{cd} variant of kneading shall be considered too. Here the dough is stretched again uniformly to twice its length, but then it is cut at its centre into 2 parts which are pasted on top of each other after-wards:*



Uniform stretch-cut-and-paste-kneading.

1 *When comparing the stretch-cut-and-paste-operation with the stretch-and-fold-operation, on 1^{st} glance it seems that both kneadings apparently mix particles around, but in very different man-ner, generating quite distinct iterating-behaviours. However, suprisingly both kneadungs are essentially the same. Again the dough is divided into 12 blocks. Then the stretch-cut-and-fold-operation is applied followed by 1 strech-and-fold-operation. The result is compared with 2 suc-ceeding stretch-and-fold-operations. It can be observed that they are identical.*

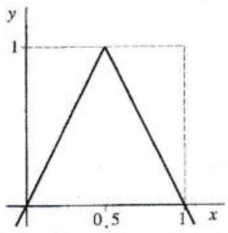
1 *By taking the interval $[0,1]$ as the original line-segment modelling the dough, one can check how both kneading-operations proceed. Symbol T is used for a stretch-and-fold-operation, the symbol S for a stretch-cut-and-paste-operation.*



Tracing a particle in interval $[0,1]$ for the $\mathrm{T}-$operation (left) and $\mathrm{S}-$operation (right). The particle starts in both experiments from the same initial-position $\mathrm{x_0}$.

1 *As be seen in both experiments, the particle arrives according to $\mathrm{T(\ T(T(x_0)))}$ and $\mathrm{T(S(S(x_0)))}$ exactly at same position though the route in between is diffe-rent. This means $\mathrm{T(T(T(x_0))) = T(S(S(\ x_0)))}$. This experience along with result of figure $\lll \mathrm{C_{2.1}} \ggg$ motivates to conjecture the substitution-property for 2 kneading-operations of $\mathrm{T^N = TS^{N-1}}$ with $\mathrm{T^N = T(T(\ldots(N-2)_{times}))}$ and $\mathrm{S^{N-1} = S(S(\ \ldots(N-3)_{times}))}$.*

3 *These mathematical-models of kneading for the 1—dimensional ideals of dough are functions.*

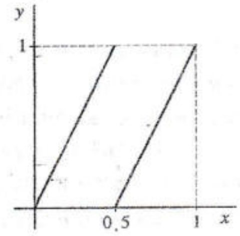
1 

Graph of the piecewise-linear tent-transformation T according to the equation above. The graph looks like a simplification of the parabola.

Stretch-and-fold-operation T is represented by following transformation $\mathrm{T(x) = \{[(2x)\ if\ (x}$

$\leq ½)],[(-2x+2) \text{ if } (x \geq ½)]\}$, *figure above shows the corresponding graph:*

1 *A justification of this model is almost self-evident. The dough is modelled by the unit-interval* $[0,1]$. *Stretching-operation is taken care by the factor* 2 *in front of* x. *The* 1^{st} *half of interval* $[0,1]$ *is only stretched and not folded. Thus,* 1^{st} *part from definition of* T *is in order of* $T(x) = 2x$ *if* $x \leq ½$. *The* 2^{cd} *half becomes* $[1,2]$ *after stretching and must be folded over its left end-point. This is equal to folding at* $x = 0$ *(multiplying with* -1 *and shifting to the right by* 2 *units).*

2 

Graph of saw-tooth-transformation S *according to the equation above. The graph justifies the name.*

Model of 2^{cd} *procedure, the stretch-cut-and-paste operation is another mathematical transformation. It's defined for* $x \in [0,1]$: $S(x) = \{[(2x) \text{ if } (x < ½)],[(2x-1) \text{ if } (x \geq ½)]\}$, *above figure shows the corresponding graph:*

1 *There is a notation for the function* S, *which differs from the above one. It uses a function which computes the fractional part* $\mathrm{Frac}(x)$ *of a number* x: $\mathrm{Frac}(x) = x-k$ *if* $k \leq x < k$, $k =$ *integer. With this notation the* S*—transformation can be written as* $S(x) = \mathrm{Frac}(2x)$ *for* $0 \leq x < 1$. *Only for the point* $x = 1$ *this formula does not work. But this is not significant, because* $x = 1$ *is fixed-point of operator* S *and, moreover, there are no other points in the unit-interval transformed to this fixed-point. Thus there is no loss to neglect the fixed-point* $x = 1$.

1 *Starting with* $0 \leq x_0 < 1$ *one computes* $x_1 = \mathrm{Frac}(2x_0) \rightarrow x_2 = \mathrm{Frac}(2x_1)$ *and generalizes* $x_{k+1} = \mathrm{Frac}(2x_k)$, $k = 0,1,2,\ldots$ *If one wants to know what is* x_k *for some very large value of* k *in terms of* x_0, *one only has to write* $x_k = \mathrm{Frac}(2^k x_0)$.

2 *If* x *is defined in binary-representation by* $0.a_1a_2a_3\ldots$ *with* a_k *as binary digits, the transformation* $S(x)$ *will be:* $S(0.a_1a_2a_3\ldots) = \mathrm{Frac}(2 \cdot 0.a_1a_2a_3\ldots) = 0.a_2a_3\ldots$, *because multiplication by* 2 *yields shifting all binary digits* 1 *place to left followed by erasing the digit moved in front of the point. Due to the type of this almost mechanical procedure the* S*—transformation is also called the shift-operator when interpreted in context of binary-representations.*

4 *On Chaos—Properties of S— and T—Transformation.*

One may start with the chaos-properties for the saw-thooth-(S)*-transformation using the closed-form-description to derive central properties of chaos: sensitivity, mixing and dense periodic-points. The substitution-property allows to carry-over these features to the iteration of* T*—transformation. In order to study the chaos-properties for* T*—transformation a formula is derived that allows for direct computation of any iterate of the* T*—transformation without to carry-out the iteration-process over and over again. In other words one tries to obtain an explicit expression for result* $T^k(x_0)$ *for initial-points* x_0 *and iteration-stages* k.

1 *The* 1^{st} *piece of solution for this is the substitution property of* T*— and* S*—transformation explained in* $\lll C_{2.1.1.1} \ggg$. *The* 2^{cd} *piece is the explicit formula for the iteration of* S*—function needed for evaluation of* $S^{k-1}(x_0)$ *as described under* $\lll C_{3.2.1} \ggg$.

2 *If assumed* x_0 *is given and one wants to know result of* x_k *after* k *(stretch-and-fold)* T*—operations. With aid of substitution-property, one will* 1^{st} *compute* S *based-on short-cut* $y = S^{k-1}(x_0) = \mathrm{Frac}(2^{k-1}x_0)$ *and applying the* T*—operation once,* $x_k = T(y) = \{[(2y) \text{ if } (y \leq 0.5)], [(-2y+2) \text{ if } (y > 0.5)]\}$.

1 *If* $0 \leq x \leq 1$ *passed to binary representation* $x = a_1x^{-1} + a_2x^{-2} + a_3x^{-3}\ldots$, *n, it may be written* $x = 0.a_1a_2a_3\ldots$ *and* $S(x) = \mathrm{Frac}(2x) = 0.a_2a_3a_4\ldots$ *in shift-opera-*

tor-form $\lll C_{3.2.2} \ggg$.

5 *Sensitivity on initial Conditions on base of S–Transformation*.

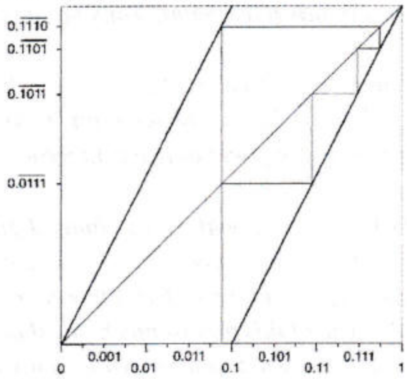
One may pick an initial number $x_0 = 0.a_1a_2a_3...$ but specify it up to $N = 100$ digits only, then the true number will differ from the specified one by at most 2^{-100}. Since digits $a_{101}, a_{102}, ...$ are undefined, it can be assumed, in each step of the calculation someone flips a coin and thereby determines those digits. One might say, the initial number is known only to some degree of uncertainty (sometimes called noise in the data). Following results will be obtained when running iteration $x = 0.a_1a_2a_3... \rightarrow S(x) = \mathrm{Frac}(2x) = 0.a_2a_3a_4...$:

1 *At the beginning everything is tame. But as one continues iterating the noise creeps closer and closer to the decimal-point and after precisely 100 iterations, the result will become perfectly random. This phenomenon is known as sensitive dependence on initial conditions.*

2 *Moreover, one can provide now an argument for the uniform distribution of the spices in dough after kneading. If the spice originally comes in a clump, the coordinates of the particles should be given as $0.a_1a_2...a_ka_{k+1}...$ where the first k digits are the same for all particles, because they are clustered. The remaining digits are uniformly distributed, modelling the random mixing of the spice in the cluster. After k applications of the shift the common coordinates are gone, and only the random digits are left, which yields the uniform distribution of spice throughout the entire dough.*

6 *Periodic Points on base of S–Transformation*.

Now to proceed in understanding another phenomenon going with chaos. There are points of a different nature: periodic-points. If iteration starts with a periodic-point, this leads to a certain number of intervals visited again and again. Theoretically one can find infinitely many points of this type in each sub-interval.



A cycle of period 4.
The point $0.\{0111\}...$ is a periodic point. The binary-expansion allows read off immediately the iterative behaviour (here visualized as graphical-iteration).

1 *What happens if one specifies: $z_0 = 0.\{a_1a_2a_3...a_k\}...$, i.e., an infinite string of binary di-gits repeated after k digits. Running the iteration means, after k steps one will see z_0 again. In other words, one gets a cycle of k iteration-steps and thus z_0 will be called point of period k with respect to binary-shift.*

1 *For any given number y_0 one can find a number w_0 arbitrarily close to y_0 which is periodic. This can be seen from $y_0 = 0.a_1a_2a_3...a_k...$ and $w_0 = 0.\{a_1a_2a_3 ...a_k\}...$ for some k. Thus y_0 and w_0 differ by at most 2^{-k} and w_0 is periodic. This means, periodic-points are dense.*

2 *For some subdivision of the unit-interval $[0,1] \supset I_{j=(1 \vee 2 \vee ... \vee n)}$ may be applied $[(x_0) \wedge w_0)] \in I_j \subset [0,1]$ and x_0 and w_0 are periodic with the same period.*

1 *The orbits starting from x_0 and w_0 will have a period of p. Orbit from x_0 will see x_0 again and again in a same way as orbit from w_0 does with respect to w_0. Both orbits, however, mostly may have different lengths because they most often will pass through different subintervals of the unit-interval.*

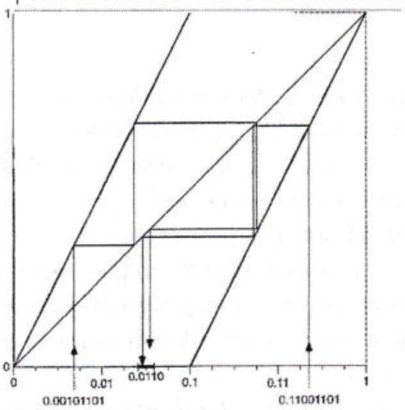
2 *As 2 orbits with the same period start from 2 points nearby and have different lengths, during every period they will initially move away from each other to get finally closer again.*

7 *Mixing–Behaviour on base of S–Transformation*.

An intuitive way to interpret mixing is to subdivide the unit-interval $[0,1]$ into subintervals and require that by iteration one can get from any starting-subinterval to any other target-subinterval. If this requirement is fulfilled for all finite subdivisions, one says, the system exhibits mixing. The point of a small interval of initial values finally may become spread over the whole unit-interval $[0,1]$.

1 *For any 2 open intervals I and J (which can be arbitrarily small, but must have a non-zero length) one can find initial values in I which when iterated will eventually lead to points in J.*

2 *For mixing, one requires that one can find a starting-point x_0 in I, whose orbit will enter the other interval J at some iteration:*



Mixing requires that the interval J can be reached from any other interval. Here 2 examples are shown how one can reach the small interval at 0.0110.

8 ## *Chaos–Properties on base of T–Transformation.*

It has to be recalled, that by means of the substitution property the iteration–T can be reduced to iteration given by the S–transformation. The k^{th} iterate x_k is obtained by $k-1$ binary-shifts followed by a single T–operation. Since the 1^{st} part is just a shift by $k-1$ binary-digits, one easily carries-over all the complicated dynamic-behaviour (sensitive dependence, denseness of periodic-points and mixing) of the shift-transformation to the T–transformation.

1 *One may ask-for: What are periodic-points for the iteration of the T–transformation? Or more precisely, x_0 is to be found so that $x_n = x_0$ applies for a given integer n, where $x_j = T(x_{j-1})$ for $j = 1,\dots,n$. All one has to do is to take a point w_0 which is periodic for the shift-transformation with period n, and apply a T–operation to obtain a periodic-point $x_0 = T(w_0)$ of T.*

2 *Indeed, let $w_0 = S^n(w_0)$ be a periodic-point of S. Then one checks whether $x_n = x_0$ using the definition of x_0 and the substitution-property of the 2 kneading-transformations: $x_n = T^n(x_0) = T^n(T(w_0)) = T^{n+1}(w_0)$ and $x_n = T(S^n(w_0)) = T(w_0) = x_0$. Hence, it is true: if w_0 is a periodic-point for the binary-shift, then $x_0 = T(w_0)$ is a periodic-point for the T–transformation with the same period. Using this result it is not difficult to reason that periodic-points of T are dense in the unit-interval.*

3 *Likewise its bit technical but not difficult too, deriving sensitivity and mixing for the T–transformation.*

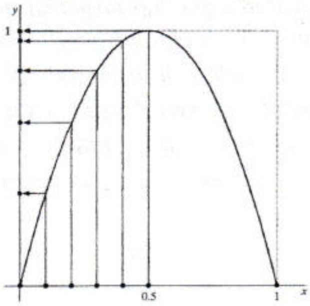
D ## *Non–uniform Kneading of Dough in Time– and Space–Dimension.*

The discourse of section ≪C≫ was devoted to the iteration of the uniform kneading-operators given by the S- and T–transformations. Now an argument shall be introduced which makes a connection between feedback-system $x \leftrightarrow 4x(1-x)$ and those for previous kneadings of dough.

1 *When the transformation $y = 4x(1-x)$ is graphed in a (x,y)–coordinate-system, one gets the generic parabola:*

1 *One only is interested here in x–values ranging $0 \rightarrow 1$. The corresponding y–values also range from $0 \rightarrow 1$, y–values monotonically increase for $x < ½$ and decrease monotonically in the range $x > ½$. One may notice that the interval $[0,½]$ on x–axis is stretched-out to interval $[0,1]$ on y–axis, and the same is true for the interval $[½,1]$. In other words, the transformation $4x(1-x)$ stretches both intervals to twice of their lengths. The stretching, however, is non-uniform. In fact, small intervals (close to 0 and 1) are stretched a great deal, while intervals close to*

the midpoint ½ are compressed.



Generic-parabola are characterized by the fact that their graphs precisely fits into a square which has 1 of its diagonals on the bi-sector of the (x,y)—coordinate-system.

2 ***A point is now be reached where connection is made with kneading. What happens if one applies the transformation $4x(1-x)$ to the interval $[0,1]$? It is already known that each half of the interval is stretched to twice its length. Moreover, checking the end-points of the interval, one finds $0 \to 0$, $½ \to 1$, $1 \to 0$. This means, the result of 1 application of the transformation $4x(1-x)$ to the interval $[0,1]$ can be interpreted as a combination of a stretch-and fold-kneading-operation.***

1 ***In other words, the iteration $f(x) \leftrightarrow 4x(1-x)$ is a relative of the uniform stretch-and-fold-kneading-operation. All complex, dynamic behaviour of chaotic-properties shown first for the shift-operator and then for the T-transformation (uniform kneading-operators) can now be found in non-uniform kneading due to quadratic-iterator $f(x) \leftrightarrow 4x(1-x)$.***

3 ***What follows now is describes the equivalence between iterations of uniform kneading-operator given by the tent-transformation $T(x)$ and the quadratic-iterator $f \leftrightarrow 4x(1-x)$, the non-uniform kneading.***

1 ***This equivalence is established by the non-linear change of coordinates given by the transformation $x' = h(x) = \sin^2(\pi x/2)$.***

1 ***Some logic around the trigonometric terms $\cos^2(\alpha) = 1-\sin^2(\alpha)$ and $\sin(2\cdot\alpha) = 2\cdot\sin(\alpha)\cdot\cos(\alpha)$ are essentual to evaluate coordinate-transformation $h(x)$:***

1 ***Iterating an start-point x_0 under T—function and iterating the transformed-point $x'_0 = \sin^2(x_0\pi/2)$ under parabola $f(x) = 4x\,(1-x)$ will produce functions that correspond to each other by means of equation $x' = h(x) = \sin^2(x\pi/2)$. To establish this algebraically, one starts with x_0 for the parabola. Thus, $x_0x_1\ldots$ is the iteration under T—function and $y_0y_1\ldots$ is the corresponding iteration under the parabola. One can show by induction that, in fact, $y_k = x'_k = h(x)$ for numbers $k = 0,1,\ldots$, proving the equivalence.***

2 ***It's started with the transformation $y_0 = \sin^2(x_0\pi/2)$, where $0 \leq x_0 \leq 1$. One substitutes y_0 in the formula for the quadratic-iterator $y_1 = 4y_0(1-y_0) = \sin^2(x_0\pi/2)(1-\sin^2(x_0\pi/2))$. By using the trigonometric-identity $\cos^2\alpha = 1-\sin^2\alpha$ leads to $y_1 = 4\cdot\sin^2(x_0\pi/2)\cdot\cos^2(x_0\pi/2)$. Using the double-angle-identity $\sin(2\cdot\alpha) = \sin(\alpha)\cdot 2\cdot\cos(\alpha)$, one gets $y_1 = \sin^2(x_0\pi)$.***

3 ***The 1^{st} iterate of x_0 under T—function is $x_1 = T(x_0)$. It's easily to be shown that y_1 in fact is identical to x_1 after change of coordinates, i.e., $x'_1 = h(x_1) = y_1$.***

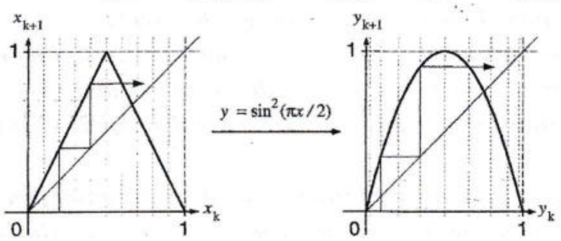
a. ***Begining with case $0 \leq x_0 \leq ½$, $x_1 = T(x_0) = 2x_0$ and $x'_1 = \sin^2(x_1\pi/2) = \sin^2(x_0\pi) = y_1$.***

b. ***Alternatively, for $½ < x_0 \leq 1$. One starts substituting $x_1 = T(x_0) = 2-2x_0$. Then $x'_1 = \sin^2(x_1\pi/2) = \sin^2(\pi - x_0\pi)$. By using initially $\sin^2(\pi+\alpha) = \sin^2(\alpha)$ and finally $\sin^2(-\alpha) = \sin^2(\alpha)$ one gets $x'_1 = \sin^2(-x_0\pi) = \sin^2(x_0\pi) = y_1$.***

4 ***The result shows $x'_1 = y_1$ and the conclusion $x'_k = y_k$ for $k = 0,1,2,\ldots$ (followed by induction). Thus, since $x'_k = h(T^k(x_0))$ and $y_k = f^k(h(x_0))$, one has shown the functional-equation $f^k(h(x)) = h(T^k(x))$, $k = 1,2\ldots$.***

2 ***When looking at an initial-point x_0 and its orbit $x_1\ x_2\ x_3\ \ldots$ under the T—transformation the***

orbit becomes $x_1 = T(x_0)$ $x_2 = T^2(x_0)$ $x_3 = T^3(x_0)$ ... $x_k = T^k(x_0)$.... *The* h*−transformed point* x_0 *is now* $y_0 = x'_0 = h(x_0)$ *the initial-point in new coordinates, belongs to the iteration of the parabola. Computing now the iteration of* $y_0 = x'_0$ *using* $f(y) = 4y\,(1-y)$ *one will obtain* $y_1 = f(y_0)$ $y_2 = f^2(y_0)$ ... $y_k = f^k(y_0)$....



Changing coordinates according to the function $h(x)$ *transforms graph of* T*−transformation to that of* $f(y) = 4y(1-y)$*. Iteration for* T *(left) is transformed into iteration for* f *(right) using* h*.*

1 *The claim is, this is the same orbit as the* 2^{cd} *one in* $\lll D_{3.2} \ggg$*. In other words, not only* $y_0 = h(x_0)$*, but also* $y_1 = h(x_1)$, $y_2 = h(x_2)$,..., $y_k = h(x_k)$, ... *holds. Thus, iterating* x_0 *under* T *produces an orbit which is (after changing of coordinates) the same as that of* $y_0 = x'_0 = h(x_0)$ *under the quadratic−*f*. In terms of the functions* f *and* T *this equivalence can be put into the form of the functionnal-equivalence* $f^k(h(x)) = h(T^k(x))$, $k = 1,2,3,..,$ *for all* $x \in [0,1]$.

4 *The iteration of the* T*−transformation and the parabola are totally equivalent. All the signs of chaos are found when iterating the quadratic-iterator* $f(x) = 4x(1-x)$*. However, these conclusions are not self-evident.*

1 *For dense periodic points and mixing property proofs are straightforward and require only to use properly the appropriate definitions together with the fundamental equation* $f^k h(x) = hT^k(x)$*. Dense periodic points for the* T*−transformation and the equivalence of* T *and* f *yield that also f has the dense periodic points. Mixing for* T *and the equivalence yield that also* f *is mixing.*

2 *Now, this approach does not work for the* 3^{rd} *property of chaos, sensitivity. Sensitivity on initial conditions is not generally inherited from one dynamical system to another which has iterations that are equivalent by change of coordinates. In contrast, properties of mixing and dense periodic-points are passed-over to such equivalence-systems.*

1 *Therefore, the derivation of sensitivity for the quadratic-transformation requires more than just the sensitivity of* T *and the equivalence of* f *and* T*. But there is a theorem verifying that properties of mixing and dense periodic points are already suffice to show the* 3^{rd} *property of chaos, sensitivity. In other words, if* f *is chaotic and* f *and* g *are equivalent via a change of coordinates* h*, i.e.* $f(h(x)) = h(g(x))$ *then also* g *is chaotic.*

5 *About Dependences of Chaos–Characteristics and how they are inherited.*

It's quite natural to ask whether the 3 *chaos-properties are independent or not. That is, whether* 1 *or* 2 *of these conditions could imply the other(s) or not. Another natural question is that of inheritance. Given that a mapping* f *is chaotic and that* g *is related to* f*, can one conclude that* g *is chaotic as well?*

1 X *and* Y *might be* 2 *subsets of the real-line and* p *and* q *be transformations* $p : X \rightarrow X$ *and* $q: Y \rightarrow Y$*. Then* p *and* q *are said to be topologically-conjugate provided* p *and* q *are continuous and there exists a homeomorphism* $h: X \leftrightarrow Y$ *such that the functional-equation* $h(p(x)) = q(h(x))$ *holds for all* $x \in X$*.*

1 *The consequences of having topological –conjugacy between* p *and* q *are strong. If* p *is mixing then* q *is mixing and the opposite is also true, when* p *has dense perio-dic points then so has* q*, and likewise vice-versa. The reason for this equivalence is that mixing and dense periodic points are topological-properties. But it must be noted, that the property of sensitive-dependence on initial conditions is in general not inherited under topological-conjugacy.*

2 *In discussions of* S*− and* T*−transformations the crucial relation* $TT = TS$ *had been established. If one defines* $h = T$ *one gets a functional-relation in the form* $hS = Th$*, but then arise problems in using this for a topological-conjugacy bet-ween* T *and* S*. Transformation* T *is continuous and* S *is not. If* h *would be a homeomorphism then also* S *would*

have to be continuous because of the functionnal-equation $hS = Th$ leads to $S = h^{-1}Th$, but this is not. The reason is that h is not a homeomorphism. It is continuous on–to, but not 1–to–1 (each $y \neq 1$ in [0,1] would have 2 preimages $x_1 \neq x_2$ such that $h(x_1) = y = h(x_2)$, and therefore there is no inverse transformation for h. This situation leads to a very useful modification of the notion of topological-conjugacy.

3 *X and Y might be 2 subsets of the real-line and f and g be transformations $f : X \to X$ and $g: Y \to Y$. Then g is said to be topologically- semi-conjugate to f provided there is a continuous on–to–transformation $h: X \to Y$ such that the functional-equation $h(f(x)) = g(h(x))$ holds for all $x \in X$.*

1 *One may note that this is not a equivalence-relation, because when g is semi-conjugate to f one may still have that f is not semi-conjugate to g. Moreover, it is not required that f or g has to be continuous.*

This is exactly the situation which was found for T and S in section ≪C≫. In other words, T is semi-conjugate to S. Thus, one may state the following fact:

2 *If one assumes that g is semi-conjugate to f via a continuous and on-to transformation h and f has dense periodic-points and is mixing, then g has dense periodic-points and is mixing too.*

4 *2 important consequences can now be drawn:*

1 *The functional-equation $hf = gh$ implies $hf^n = g^n h$ for any natural number n. Indeed, $hf = gh$ implies $ghf = g^2h$ and using $gh = hf$ implies $hf^2 = g^2h$. Likewise, the general case can be obtained by induction.*

2 *It shows that the work where chaos-properties are established for T from that of S has to be seen in a very general background:*

1 *One may assume that g is semi-conjugate to f via a continuous and on-to transformation h, and f may have dense periodic-points and is mixing, then also g has dense periodic-points and is mixing.*

5 *Finally one will come to the discussion whether the 3 properties of chaos are independent from each other:*

1 *If X is an arbitrary subset of the real-line and $f: X \to X$ is an continuous transformation which has the properties of mixing and dense periodic-points, then f has sensitive dependence on initial conditions too. In other words, if f is chaotic and g is topologically semi-conjugate to f, then g is chaotic too.*

E Conclusion.

In the sections of current discourse the cooperative oscillation of 2 test-masses floating in gravity caused by a 2–dimensional curved space-time, ought to be contrasted to lines occurring in a 2–dimensional, dynamic system that behaves chaotically.

1 *The gravity-field created by a 2–dimensional curved space-time insight the shaft caused the 2 earth-ships to perform completely identical movements on their courses from Atlanta to Taralga and vice versa. However, their different starting-times eventually led to the characteristic pattern of their world-lines.*

1 *Both ships always meet directly below each platform, when one ship is starting downwards already for a new turn into the shaft, while the other just comes-up from the shaft to complete its still incomplete turn shortly afterwards.*

2 *When both ships are in same direction towards an end-point of the shaft, the ship in front initially hurrys away from the one behind, up to a maximum-distance, to fall-back again towards the successsor afterwards on its way to the next meeting-point.*

1 *In the interval between 2 consecutive meeting-points, the world-lines for the 2 ships can be describes as geodesics running on a sphere. During this period the geodesics (after both ships having left starting-position) initially diverged up to a maximum-distance to approach each other again afterwards and reunite at the rear meeting-point finally.*

3 *Measuring curvature of space-time within this context means, to assign each elementary area of space-time its responsibility for bending of geodesics. Each of the relevant quantities in this context can be measured in an appropriate manner.*

2 *Now focus should be on sections ⋘C,D⋙ in order to get a firm idea about periodic-points in conjunction with the quadratic-iterator.*

1 *To begin with S—transformation:* $S(x) = Frac(2x)$ *for* $0 \leq x < 1$ *and reveal the interpretation by passing to binary representation of the real-number* x *between* 0 *and* 1*. One may recall that any real-number* x *from the unit-interval can be written as* $a = 0.a_1a_2a_3...$ *where the* a_k *are binary-digits, i.e., each* $a_k = \{0 \vee 1\}$ *and:* $x = a_1 2^{-1} + a_2 2^{-2} + a_3 2^{-3} + ...$

1 *What does the iteration of the S—transformation mean in terms of binary-expan-sion? Multiplication by* 2 *means passing from* $0.a_1a_2a_3...$ *to* $a_1.a_2a_3....$ *Thus* 1 *application of the transformation is accomplished by* 1^{st} *shifting all binary-digits* 1 *place to the left and then erasing the digit that is moved in front of the point:* $x = 0.a_1a_2a_3... \rightarrow S(x) = 0.a_2a_3a_4....$

2 *Now for the chaos-phenomenon of periodic-points.*

1 *What happens if one specifies* $x_0 = 0.\{a_1a_2a_3...a_k\}...$ *and runs the iteration? In other words, if one has an infinite string of binary-digits (repeated after* k *digits) and exposes it to a binary shift? Running the iteration means, after* k *steps* x_0 *will be seen again and again. A cycle of length* k *will be obtained. Therefore,* x_0 *is called periodic with respect to a binary-shift.*

2 *Important here is, for any given number* x_0*, one can find a number* w_0 *arbitrarily close to* x_0*, which is periodic. If* $x_0 = 0.a_1a_2a_3...$ *then one may choose* $w_0 = 0.\{a_1a_2a_3...a_k\}...$ *for some* k*. Then* x_0 *and* w_0 *differ at most by* 2^{-k}*, and* w_0 *is periodic, this means that periodic-points are dense.*

2 *By means of the substitution property the iteration of* T *can be reduced to the iteration given by the S—transformation.*

1 *The* k^{th} *iterate* x_k *is obtained by* $k-1$ *binary-shifts followed by a single T—operation. Since the* 1^{st} *part is just a shift by* $k-1$ *binary-digits, one can carry all the complicated dynamic-behaviour of denseness of periodic-points for the shift-transformation over to the T—transformation.*

2 *May be it's asked for, what are periodic-points for the T—transformation? Or, more precisely, one should find* x_0 *so that* $x_n = x_0$ *for a given integer* n*, where* $x_j = T(x_{j-1})$ *for* $j = 1,...,n$*. All one has to do is to take a point* w_0 *which is periodic for the shift-transformation with period* n *and to apply the T—transformation to obtain a periodic-point* $x_0 = T(w_0)$ *of* T*. Indeed, as* $w_0 = S^n(w_0)$ *is a periodic-point of* S*, then:* $T^n(x_0) = T^n(T(w_0)) = T^{n+1}(w_0) = x_n = T(S^n(w_0)) = T(w_0) = x_0$*.*

3 *Hence it is true, if* w_0 *is a periodic point for the binary-shift, then* $x_0 = T(w_0)$ *is a periodic-point for the T—transformation with the same period. Using this result it is not hard to reason that periodic-points of T—transformation are dense in the unit-interval too.*

3 *It's known that periodic-points of the T—transformation are dense in the unit-interval* $[0,1]$*, so it's claimed that periodic-points of the quadratic-iterator* $f(x) = 4x(1-x)$ *are dense too.*

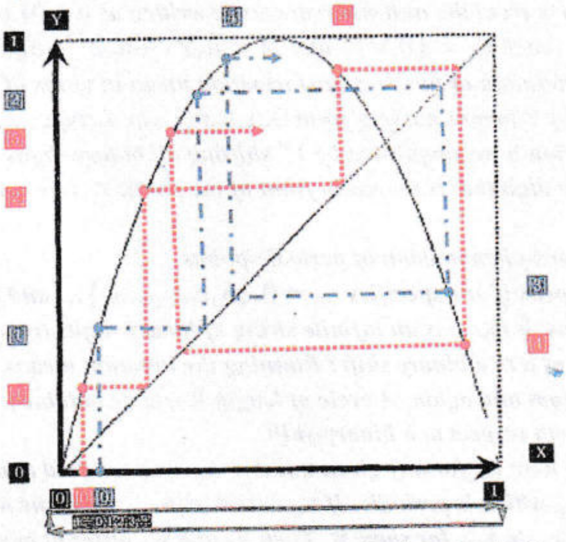
1 *If* $y \in [0,1]$*, then it has to be shown that there is a sequence of periodic-points from* f *with limit* y*. One chooses* x *as preimage of* y *under* h*, i.e.,* $h(x) = y$*,* h *is onto. Since periodic-points of* T *are dense in* $[0,1]$ *one will find a sequence of points* $x_1, x_2, x_3,...$ *with a limit* x *and such that each point* x_e *is periodic-point from* T *of some period* p_e*. Thus* $T^{p_e}(x_e) = x_e$ *for* $e = 1,2,3,....$ *It's claimed that sequence* $y_1, y_2, y_3,...$ *with limit* $y_e = h(x_e)$ *has limit* y *and is sequence of periodic-points from* f*.*

2 *The* 1^{st} *claim is true because* h *is continuous. One way of defining what continuity for a function* $f: X \rightarrow X$ *(where* X *for example is a subset of the real-line) means* x *is the following: The function* f *is said to be continuous provided that for any* $x \in X$ *and any sequence* $x_1, x_2, ...$ *with limit* x *one has that the sequence* $f(x_1), f(x_2),...$ *has its limit* $f(x)$*.*

3 *The* 2^{cd} *claim follows from* $f^k h = hT^k$*:* $f^{p_e}(y_e) = f^{p_e}(h(x_e)) = h(T^{p_e}(x_e)) = h(x_e) = y_e$.

4 ***Thus, it has been verified yet that periodic-points are dense for the quadratic-iterator too. Moreover, this fact regarding periodic-points must also be fulfilled for the sub-intervals of the unit-interval, no matter how the subdivision may be, as long as the width of the sub-intervals is greater than zero. This insight will be crucial for the further discussions.***

4 ***Graphical representation of the following generic parabola will support further discussions.***



1 ***The figure shows the generic parabola*** $4X(1-X)$ ***within coordinate-system*** X, Y ***above unit-interval*** $[0,1]$.

2 ***2 spice-grains are inserted into dough close to each other at periodic-points [0] and [0] (this is quite likely due to denseness of periodic points in any sub-interval of the unit-interval) and the quadratic-iterator*** $Y = X_{n+1} \leftarrow 4X_n(1-X_n)$ ***starts running for kneading-steps*** $n = 0,1,2,3,\dots$ ***from each of periodic-points [0] and [0]. Both spice-grains during these operations will move like: [1] → [2] → [3] → [4] → … and [0] → [1] → [2] → [3] → [4] →…..***

1 ***Periods of a complete cycle for [1] may be*** p ***and*** q ***for [1] with*** $[p = q] \vee [p \neq q]$.

3 ***All*** y***—values obtained during execution of the quadratic-iterator*** $y \leftarrow x_{j+1} = 4x_j(1-x_j)$ ***for*** $j = 1,2,3,\dots$ ***will be lined-up on the generic parabola*** Y.

1 ***It is supposed that from each of the 2 point [0] and [0] a series of iteration-steps is started. Then corresponding*** y_k***—values from the 2 series will have different gradients with the generic-parabola, because of the*** X***-distance [0] ↔ [0] between the grains. This means, both sequences*** (x_k, y_k) ***are shifted against each other, and will describe 2 parabolic geodesics on a parabolic surface.***

4 ***Both spice-grain thus provisionally are describe by positively-curved geodesics on a 2—dimensional, positively-curved*** (x,y)***—surface. But when iteration-steps of the quadratic-iterator are interpreted as events in time, the geodesics can be thought as being related in space-time as well.***

1 ***Between a space-unit*** x ***and a time-variable*** t ***may exist, e.g,*** $c = 3 \cdot 10^{10}\,[\mathrm{m\,sec}]^{-1} = x/t$ ***(where*** c ***represents the constant vacuum-ligth-speed) and thus*** $t \leftrightarrow x/c$.

5 ***This new perspective now makes the just mentioned curved-surface appear as structure in space-time, and this way it becomes comparable with the sphere in section ≪B≫ (where geodesics of the 2 boomerangers were embedded in). Although both surfaces are different in scaling and one is parabolic while the other is spheric, dy-namics on both are qualitatively similar. This is also unerderlined by the following:***

1 ***Because both spice-grain-cycles have different periods*** p ***and*** q***, after*** m ***periods***

(where m is equal to the smallest common multiple of p and q) the geodesics will always meet again at [1] and [1] just above [0] and [0] where they time-shifted started from at the very beginning, spatially very close to each other. On their courses the grain-geodesics first will move apart up to a maximum, then they approach each other again in order to have their shortest distance finally. Similar to what hab been observed already for the geodesics in section ≪B≫.

6 *In the view of surface's curvature and both geodesics running on it, the following can be noted finally:*

1 *The bend of a geodesic towards or away from a neighbouring geodesic (if they are passed in a similar way) proves the local existence of space-time-curvature. Curvature of geometry can be understood by distances in space and intervals in space-time.*

2 *The decrease in retreat-speed per time-unit is directly proportional to the average in this time-unit. Thus, in case of thought-experiment, this ratio remained constant permanently, because of the uniform curvature of the sphere. With respect to the quadratic-iterator this ratio has to be adapted permanently due to the non-uniform curvature of the paraboloid where the geodesics are running on.*

R *References.*

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