Problem of foundation of mathematics

Felix M Lev

Independent Researcher, Email: felixlev314@gmail.com, Carlsbad, California, USA

Abstract

A common situation in physics involves two theories, $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ contains a nonzero parameter, and $\mathcal{B}$ arises as a limit of $\mathcal{A}$ as this parameter approaches zero or infinity. In such cases, $\mathcal{A}$ is more general and $\mathcal{B}$ is a degenerate case of $\mathcal{A}$. Well-known examples include relativistic theory being more general than non-relativistic theory and quantum theory being more general than classical theory. In this short review we argue that an analogous situation holds in mathematics. Classical mathematics (CM) is based on the infinite ring of integers Z, whereas finite mathematics (FM) is based on the finite ring $R_p = (0, 1, 2, ...p-1)$ of residues modulo p. CM has foundational difficulties (as highlighted by Gödel's incompleteness theorems) while FM does not. All attempts to construct a quantum theory of gravity within CM encounter unavoidable divergencies. The existence of elementary particles also suggests that infinitesimals do not exist in nature. Despite this, CM is usually regarded as fundamental theory, while FM merely as a tool useful only in some models. We argue instead that FM is the more general theory, with CM appearing as its degenerate limit as $p \to \infty$. The key points are: $R_p \to Z$ as $p \to \infty$, and this can be proved using only potential (not actual) infinity; quantum theory based on FM is more general than quantum theory based on CM.

1 Problem statement

In the *technique* of Classical Mathematics (CM), infinity is treated only as potential infinity. The distinction between potential and actual infinity is well known:

- **Actual infinity** refers to a completed set containing infinitely many elements.

- **Potential infinity** refers to an endless process that never reaches a final completed state.

Thus, potential infinity appears only as a limit, and only finite sets are used in its description. For example, the set of all natural numbers represents actual infinity, and—as implied by Gödel's incompleteness theorems—its use leads to foundational difficulties within CM.

Nevertheless, CM is fundamentally based on actual infinity: it begins with the infinite ring of integers Z, and standard textbooks do not even pose the question of whether Z can be understood as a limit of finite rings. Moreover, in CM the ring Z is extended to fields (e.g., such as reals), involving actual infinities of various cardinalities.

In contrast, Finite Mathematics (FM) is built solely on finite sets and has no foundational problems. The truth or falsity of any statement can, at least in principle, be checked in a finite number of steps.

Typically, FM begins with the ring $R_p = (0, 1, 2, \ldots, p-1)$, where addition, subtraction, and multiplication are defined modulo p. In the literature, R_p is often denoted Z/p, but this notation is somewhat misleading. First, it introduces an infinite set into a finite framework. Second, it may suggest that R_p is merely a special case of Z, although Z does not contain modular operations and is therefore not more general than R_p.

Let M be a natural number. A natural question is whether Z can be viewed as the limit of R_p as $p \to \infty$ and, if so, how this limit should be defined. In CM, the standard definition of a sequence going to $+\infty$ uses only potential infinity: a sequence (a_n) tends to $+\infty$ as $n \to \infty$ if for every M there exists n_0 such that $a_n \geq M$ for all $n \geq n_0$. By analogy, one may ask whether a proof that $R_p \to Z$ as $p \to \infty$ can be given using only potential infinity. Such a proof appears in our monograph [1], and a simpler version is given in Sec. 2 of this paper. Despite the fundamental nature of this fact, we have not found it elsewhere in mathematical literature.

Another way to prove that $R_p \to Z$ as $p \to \infty$ uses ultraproducts. As shown in [2, 3], infinite fields of characteristic zero (including Z) can be embedded into ultraproducts of finite fields. This also follows using only rings (see, e.g., Theorem 3.1 in [4]). However, ultraproducts rely on classical results involving actual infinity, including the Łoś's theorem using the axiom of choice. Therefore, ultraproduct-based arguments cannot be used to establish that FM is more general than CM.

The approaches [4, 5, 6] reflect a common viewpoint: structures of characteristic zero are regarded as general, while those of positive characteristic are seen as special cases that sometimes provide useful tools. However, all such arguments depend on actual infinity and thus inherit its foundational issues, as shown by Gödel, Tarski, Church, Turing, and others. Consequently, it becomes important to determine whether the fact that $R_p \to Z$ as $p \to \infty$ can be proved *purely within the framework of potential infinity.*

2 Proof that $R_p \to Z$ at $p \to \infty$ using only potential infinity

Note. To study the limit $p \to \infty$, it suffices to consider values of $p > p_0$ where the specific value of p_0 is irrelevant. In what follows, we define two functions $k = k(p)$ and $n = n(p)$, each taking positive integer values. Any fixed value of p_0 greater than, say, 100 would work, although its explicit value is unimportant.

Because all operations in R_p are carried out modulo p, one may represent R_p as

- $\{0, \pm 1, \pm 2, ..., \pm (p-1)/2)\}$ if p is odd,
- $\{0, \pm 1, \pm 2, ..., \pm (p/2-1), p/2\}$ if p is even.

We define the meaning of the statement that $R_p \to Z$ at $p \to \infty$ as follows. Let $k = k(p)$ satisfy $k < (p-1)/2$ if p is odd and $k < p/2-1$ if p is even. Let $S = S(k)$ be a set of numbers $(0, \pm 1, \pm 2, ... \pm k)$. Then $S \subset R_p$ and $S \subset Z$. Next, let $n = n(p)$ be a natural number such that:

- For any operation of summation, subtraction and multiplication involving m elements of S where m is any number such that $m \leq n$, the result is the same in both R_p and Z. That is, no modular "wraparound" occurs for any expression small enough compared to p.
- As $p \to \infty$, both $k(p) \to \infty$ and $n(p) \to \infty$.

This means that for the set S and number n, the effects of modular arithmetic never appear. Any experiment restricted to such values cannot distinguish whether the governing arithmetic is that of R_p or Z.

Theorem: $R_p \to Z$ as $p \to \infty$.

Proof. Define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. For every $p > p_0$ there exists a unique natural $n = n(p)$ satisfying $2^{n^2} \leq h(p) < 2^{(n+1)^2}$. Then if $k = k(p) = 2^n$, *Theorem* follows. □

This *Theorem* was originally proved in more elaborate form in Chapter 6.3 of [1] but the above proof is significantly simpler.

In [1] we have proposed the following

Definition. Suppose that:

- Theory $\mathcal{A}$ contains a finite nonzero parameter and theory $\mathcal{B}$ differs from $\mathcal{A}$.
- $\mathcal{A}$ can reproduce any result of $\mathcal{B}$ by choosing an appropriate value of the parameter.
- One can define a limit of $\mathcal{A}$ when the parameter approaches zero or infinity.
- After taking the limit and obtaining $\mathcal{B}$, one cannot recover $\mathcal{A}$ from $\mathcal{B}$; moreover, $\mathcal{B}$ cannot reproduce all results obtainable in $\mathcal{A}$.

Then $\mathcal{A}$ is more general than $\mathcal{B}$ and $\mathcal{B}$ is a degenerate special case of $\mathcal{A}$.

The proved *Theorem* shows that:

- a) Any result in Z can be obtained in R_p if p is chosen to be sufficiently large.
- b) In Z one cannot reproduce results in R_p that rely on genuinely modular operations.

Therefore, as follows from *Definition*:

Statement 1: R_p is more general than Z, and Z is a special degenerate case of R_p in the limit $p \to \infty$.

3 Standard Quantum Theory vs. Finite Quantum Theory

We now examine how *Theorem* proved in Sec. 2 contributes to the foundations of mathematics and physics. A key preliminary question is how mathematics itself should be understood:

- (1) As a purely abstract discipline, independent of nature;
- (2) As a discipline that must ultimately describe nature.

Most physicists accept only viewpoint (2), while many mathematicians and philosophers adopt viewpoint (1) which may be called Hilbert's approach. Hilbert did not require mathematics to reflect physical reality. Gödel's incompleteness theorems, however, raised deep questions about the foundations of this approach. Nevertheless, Hilbert's perspective should not be dismissed. Dirac, for example, emphasized trusting mathematical structure above physical intuition, arguing that mathematically beautiful theories eventually reveal physical meaning. A historical example is the use of Hilbert spaces: studied since the early twentieth century, they were not applied to quantum physics until the 1930s. Under Hilbert's approach, many powerful results have been achieved. However, since it relies on actual infinity, its foundational issues remain unresolved.

We will now consider the viewpoint (2). As shown in the extensive physics literature (see, e.g., Sec. 1.3 of [1]):

Statement 2: Classical (i.e., non-quantum) theory is a special degenerate case of quantum one in the limit $\hbar \to 0$ *where* $\hbar$ *is the Planck constant.*

Then the question of whether CM or FM is more general depends on which physical theory—Standard Quantum Theory (SQT) based om CM, or Finite Quantum Theory (FQT) based on FM—is more fundamental.

Let's first discuss some properties of SQT. Here, physical states live in a separable Hilbert space $\mathcal{H}$. In quantum theory (both, SQT and FQT), any system is considered to consist of elementary particles described by irreducible representations (IRs) of a symmetry algebra. In nonrelativistic theory, the symmetry algebra is the Galilei algebra, in relativistic theory — the Poincare algebra, and in de Sitter (dS) and anti-de Sitter (AdS) theories — the dS and AdS algebras, respectively.

In SQT, IRs of these algebras describing elementary particles are infinite-dimensional. The state vector of the entire system is the tensor product of the state vectors for the elementary particles in the system. Therefore, the Hilbert space $\mathcal{H}$ for the entire system is infinite-dimensional, even if the system consists of a single elementary particle.

A known result of the theory of Hilbert spaces is [7]:

Statement 3: A Hilbert space is separable if and only if it admits a countable orthonormal basis $(e_1, e_2, ...e_n, ...)$ *and it is always possible to choose a basis such that the norm of each* e_j $(j = 1, 2, ...\infty)$ *is an integer.*

Let the complex numbers $(c_1, c_2...)$ be the decomposition coefficients for a vector $x \in \mathcal{H}$ over the basis $(e_1, e_2...)$. The only requirement that the coefficients must satisfy is: $\sum_{j=1}^{\infty} |c_j|^2 < \infty$. The known result of the theory of Hilbert spaces is that [7]:

Statement 4: The set of all points $(c_1, c_2, ...)$ *with only finitely many nonzero coordinates, each a rational number, is dense in the separable Hilbert space.*

This implies that, *with any desired accuracy*, each element of $\mathcal{H}$ can be approximated by a finite linear combination

$$x = \sum_{j=1}^{n} c_j e_j \tag{1}$$

where $c_j = a_j + ib_j$ and all the numbers (a_j, b_j) $(j = 1, 2, ...n)$ are rational.

The next observation is that spaces in quantum theory are projective: for any complex number $c \neq 0$, the elements x and cx describe the same state. The meaning of this statement is that not the probability itself but ratios of different probabilities have a physical meaning. As a consequence, both parts of Eq. (1) can be multiplied by a common denominator of all the nonzero numbers a_j and b_j, and

Statement 5: Each element of a separable projective Hilbert space can be approximated with any desired accuracy by a finite linear combination (1) where all the numbers a_j *and* b_j *are integers, i.e., belong to* Z*.*

This shows that SQT contains an enormous redundancy: although it formally uses an uncountable set of complex-valued vectors, any experimental prediction can be approximated using only finitely many integer coefficients and all the numbers (a_j, b_j) belong to Z.

Before discussing FQT, let us note that in SQT, as shown by Dyson [8] (see also Sec. 1.3 in [1]), it follows even from purely mathematical considerations that:

- Nonrelativistic theory (NT) is a special degenerate case of relativistic one (RT) in the formal limit $c \to \infty$. The quantity c is usually associated with the speed of light but in fact this is only a constant of the theory.

- RT is a special degenerate case of dS and AdS invariant theories in the formal limit $R \to \infty$ where R is the parameter of contraction from the dS or AdS Lie algebras to the Poincare Lie algebra.

- In turn, since dS and AdS algebras are semisimple, dS and AdS theories cannot be obtained from any more general theories by contraction.

In FQT, no dimensional constants such as kilograms, meters, or seconds appear, since these derive from macroscopic physics. Thus, FQT cannot have Galilei or Poincare symmetry. Here quantum states live not in Hilbert spaces, but in vector spaces over the "complexified finite ring" $R_p + iR_p$. By the Zassenhaus theorem [9], all IRs of the algebras over the rings of nonzero characteristics (modular IRs) are finite-dimensional. Explicit modular IRs of the dS and AdS algebras were constructed in [10, 11]. Therefore, any system containing finitely many elementary particles has a finite-dimensional state space.

A fundamental difference between SQT and FQT concerns particle/antiparticle structure:

- In SQT, IRs contain either only positive-energy states or only negative-energy states. These correspond to particles and antiparticles, and conservation laws (electric charge, baryon number) prevent transitions between them.

- In FQT, each IR necessarily contains both positive and negative energy states [10, 11]. As $p \to \infty$, this single IR "splits" into two separate IRs corresponding to particles and antiparticles in SQT [1, 12].

Thus the unified FQT IR has greater symmetry than the corresponding pair of IRs in SQT. One might think this contradicts experiment because charge and baryon number appear conserved. But this impression is due to the fact that at the present stage of the universe p is extremely large, so transitions between positive- and negative-energy states are effectively suppressed.

As shown in [1, 13], FQT is a more general theory than SQT because SQT is a special degenerate case of FQT at $p \to \infty$. As follows from *Definition*, to prove this statement, it is necessary to prove that:

- *Statement 6A: For any prediction of SQT, one can choose p sufficiently large so that FQT reproduces the same result.*

- *Statement 6B: Some physical phenomena require modular arithmetic.*

As shown in [1, 13], *Statement 6A* follows from *Theorem* in Sec. 2 and from *Statement 5*. For proving *Statement 6B* we consider two phenomena: gravity and the baryon asymmetry of the universe.

Quantum gravity within SQT is non-renormalizable and plagued by divergences. In contrast, [1] shows that, at least Newtonian gravity emerges in the semiclassical limit of FQT. In this approach, the gravitational constant G is not taken from the outside but depends on p approximately as $1/ln(p)$. Matching this with experiment gives $ln(p) \approx 10^{80}$ ore more. and therefore p is a huge number of the order of $exp(10^{80})$ or more. One might think that since p is so huge then in practice p can be treated as an infinite number. However, since $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Before considering baryon asymmetry of the universe, let us discuss the following question. In many publications (see e.g., [14] and references therein), arguments are given that our universe works like a computer. Then the number p that determines the laws of physics in our universe is not a fundamental number given by a theory, but is a number that is determined by the state of the universe at its present stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the universe will be different at different stages. As noted above, in the situation where p is very large, it may seem that the electric charge and baryon number are conserved quantum numbers.

The above result about gravity shows that, at the present stage of the universe, the number p is huge, and this might be a justification of the postulate of modern particle theory that the electric charge and baryon number are strictly conserved quantum numbers.

The paradox with the baryon asymmetry of the universe is formulated as follows. According to modern cosmological theories, at early stages of the universe, the numbers of baryons and antibaryons were the same. Then, as follows from the law of baryon number conservation, those numbers should be the same at the present stage of the universe. However, at this stage, the number of baryons is much greater than the number of antibaryons. The above paradox arises if we assume that the number p was huge even in the early stages of the universe and therefore the laws of conservation of electric charge and baryon number held true even in these stages. However, there is no basis for this assumption, and therefore the baryon asymmetry paradox does not arise.

In [1, 13] we gave other examples when FQT can solve problems that SQT cannot solve. Therefore, the above arguments show that

Statement 7: FQT is a more general theory than SQT.

In turn, from the viewpoint (2), it follows from *Statement 7* that

Statement 8: Finite Mathematics is a more general theory than Standard Mathematics.

In conclusion of this section, let us discuss the following question. As noted above, in CM, the ring Z is generalized to the case of various fields in which four operations are possible: addition, subtraction, multiplication, and division. However, when generalizing CM to FM, we considered only the ring R_p and its complex extensions In FM, division may seem unnatural. For example, in the Galois field F_p, where p is a prime, $1/2$ is a large number $(p+1)/2$ if p is large. However, the main question is whether it is necessary to have division in FQT.

SQT is essentially based on the concept of infinitesimals introduced by Newton and Leibniz more than 300 years ago. This concept was in the spirit of the experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts. However, now we know about the existence of elementary particles. Even the name "elementary particle" itself suggests that such a particle cannot be subdivided into parts. For example, the energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with other particles. If the electron could be divided into parts, this would have been discovered long ago. So, in physics, division has limited applicability, since when we reach the level of elementary particles, further division is no longer possible. Standard macroscopic theory and standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected. It seems rather strange that, although most physicists understand this, they nevertheless consider the concept of infinitesimals not as only approximate but as fundamental.

4 Conclusion

The proof of *Theorem* in Sec. 2 using only potential (not actual) infinity is much simpler than in Sec. 6.3 of [1]. It makes natural the conclusion of Sec. 3 that Finite Mathematics is more general than Classical Mathematics. As a consequence:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

It is clear that the ultimate quantum theory can only be based on mathematics free of foundational problems. As noted above, finite mathematics indeed does not have such problems. At the same time, in the approach of Cantor, Hilbert and other mathematicians the following questions arise:

- A) If we accept that actual infinity is necessary, then how does this correspond to the work of Gödel and other mathematicians, that if we start from the entire infinite series of natural numbers, then problems arise in the foundation of mathematics.

- B) In this approach, there is the concept of infinitesimals but, as noted in the preceding section, it is not clear whether this concept is compatible with the existence of elementary particles.

So, although in the approach of Cantor, Hilbert and other mathematicians, many strong and beautiful results have been obtained, the question arises how to reconcile this approach with A) and B). The answer to this question is currently unknown. However, it is clear that the problem exists, and this could be a good incentive for further research in mathematics and physics.

The following historical analogy can be given here. In nonrelativistic theory, many strong results have been obtained, but here there is no limit on the magnitude of speed. There is such a limit in relativistic theory where the magnitude cannot exceed c. The relativistic theory does not refute the non-relativistic one, but indicates that the latter is applicable when speeds are much less than c, and the former should be applied when the problem involves speeds comparable to c. Note that in SQT, there is no limit on the magnitude of angular momentum but in FQT, where all the quantities are taken modulo p, the magnitude of any angular momentum cannot exceed p.

The question now arises of how to use the above results to construct the ultimate quantum theory. As noted in [1, 12], the main difficulty in such a construction is the following. Our physical intuition comes from the existing theory, which contains particles and antiparticles and laws of conservation of electric charge and baryon number. As explained in [1, 12], such results arise because the symmetry algebras used in quantum physics (Galilei, Poincare, and anti-de Sitter) have the property that in IRs of such algebras, the energies of particles can only be either positive or negative, and there are no IRs in which particles have states with different energy signs. At the same time, in finite mathematics, IRs necessarily contain both positive and negative energies (see e.g., [10, 11]). It is clear that the case when there is one IR uniting positive and negative energies has a higher symmetry than the case when this IR splits into two independent IRs with positive and negative energies. In the former case, there are no strict concepts such as particle-antiparticle, proton-antiproton, electron-positron, and so on. These concepts can only be approximate when the characteristics of the ring in finite mathematics is very large. Therefore, the challenge of constructing a fundamental quantum theory is to construct a theory without the assumption that p is anomalously large, and such a theory will be based on new physical concepts.

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