On the basics of mathematics

Felix M Lev

Independent Researcher: felixlev314@gmail.com, San Diego, California, 92010 USA

Abstract

As shown in the works of Gödel and others, standard mathematics has foundational problems because it invokes the infinite ring of integers Z. At the same time, finite mathematics proceeds from a finite ring $R_p = (0, 1, 2, ...p-1)$ of residues modulo p. We give a new and simple proof that Z is the limit of R_p when $p \to \infty$. The proof involves only potential infinity and not actual infinity. We explain why the proof plays an important role in the foundations of mathematics and physics.

1 Problem statement

In the *technique* of Standard Mathematics (SM), infinity is understood only as potential infinity. In mathematical literature, the difference between actual infinity and potential infinity is well known. For example, a possible definition of the difference is: "Actual infinity is the concept of a completed, definite set with an infinite number of members, while potential infinity is a never-ending process that never reaches a final, complete state." In other words, potential infinity is understood only as a limit, and in the description of potential infinity, only finite sets are used. For example, the set of all natural numbers represents actual infinity, and as shown in Gödel's incompletes theorems, this leads to problems in the foundation of standard mathematics.

At the same time, the *basis* of SM does involve actual infinity: SM starts from the infinite ring of integers Z and, at least in standard textbooks on SM, it is not even posed a problem whether Z can be treated as a limit of finite rings. Moreover, in SM, the ring Z is generalized to the field of rational numbers, the field of reals etc., and actual infinities with different cardinalities are considered. On the other hand, Finite Mathematics (FM) proceeds only from finite sets. In FM there are no foundational problems since the assertion of the

truth or falsehood of any statement can be directly verified (at least theoretically) in a finite number of steps.

Usually, FM proceeds from the ring $R_p = (0, 1, 2, ...p-1)$ where addition, subtraction and multiplication are performed as usual but modulo p. In the literature, for the ring R_p, the notation Z/p is often used. In our opinion, this notation is not quite relevant for the following reasons. Firstly, it is unnatural to use the concept of infinite set in FM. Secondly, the notation Z/p for R_p may create the impression that R_p is a special case of Z. However, although Z contains more elements than R_p, Z is not a more general concept than R_p since Z does not contain operations modulo a number.

In what follows, M is a natural number. A natural question arises whether Z is the limit of R_p at $p \to \infty$ and how such a limit should be defined. In SM, the following standard definition is used: a sequence of natural numbers (a_n) goes to $+\infty$ at $n \to \infty$ if $\forall M\ \exists n_0$ such that $a_n \geq M\ \forall n \geq n_0$. This definition uses only potential infinity. Therefore, by analogy, a problem arises whether the proof that $R_p \to Z$ at $p \to \infty$ can be given by using only potential infinity.

We did not succeed in finding such a proof in the literature. However, the fact that $R_p \to Z$ at $p \to \infty$ follows from a sophisticated construction called ultraproducts. As shown e.g., in [1, 2], infinite fields of zero characteristic (and Z) can be embedded in ultraproducts of finite fields. This fact can also be proved by using only rings (see e.g., Theorem 3.1 in [3]).

The theory of ultraproducts (described in a wide literature — see e.g., monographs [4, 5] and references therein) is essentially based on classical results on infinite sets involving actual infinity. In particular, the theory is based on Łoŝ' theorem involving the axiom of choice.

Probably, the fact that $R_p \to Z$ at $p \to \infty$, can also be proved in approaches not involving ultraproducts. For example, Theorem 1.1 in [6] states:

Let S be a finite subset of a characteristic zero integral domain D, and let L be a finite set of non-zero elements in the subring $Z[S]$ of D. There exists an infinite sequence of primes with positive relative density such that for each prime p in the sequence, there is a ring homomorphism $\varphi_p : Z[S] \to Z/pZ$ such that 0 is not in $\varphi_p(L)$.

The theorem involves only primes, and the existence of homomorphism does not guarantee that operations modulo p are not manifested for a sufficient number of operations. However, even if those problems

can be resolved, the proof of the theorem is based on the results of SM for characteristic zero integral domains, and the proof involves real and complex numbers, i.e., the results involve actual infinity.

The approaches of [3, 4, 5, 6] are in the spirit of the way of thinking of many mathematicians that sets with characteristics 0 are general, sets of positive characteristics are their special cases and for investigating sets with characteristic 0 it is sometimes convenient to use properties of simpler sets of positive characteristics.

We conclude that the available proofs that $R_p \to Z$ at $p \to \infty$ are based on the results of SM involving all natural numbers and therefore, as shown in the works of Gödel and other mathematicians, the problem of foundation of these results remains open. Therefore, as noted above, it is desirable to investigate whether the fact that $R_p \to Z$ at $p \to \infty$ can be proved by using only potential infinity.

2 Proof that $R_p \to Z$ at $p \to \infty$ using only potential infinity

Note. To study the limit $p \to \infty$, it suffices to investigate the case $p > p_0$ where the explicit value of p_0 does not play a role. Below we define two functions: $k = k(p)$ and $n = n(p)$ such that the values of these functions be positive natural numbers. For this purpose it suffices to choose for p_0 any value greater than, for example, 100, but again, the explicit value of p_0 doesn't play a role.

Since all operations in R_p are carried out modulo p, one can represent R_p as a set $\{0, \pm 1, \pm 2, ..., \pm(p-1)/2)\}$ if p is odd and as a set $\{0, \pm 1, \pm 2, ..., \pm(p/2-1), p/2\}$ if p is even.

We define the meaning of the statement that $R_p \to Z$ at $p \to \infty$ as follows: Let $k = k(p)$ be a natural number depending on p such that $k < (p-1)/2$ if p is odd and $k < p/2 - 1$ if p is even. Let $S = S(k)$ be a set of numbers $(0, \pm 1, \pm 2, ... \pm k)$. Then $S \subset R_p$ and $S \subset Z$. Let $n = n(p)$ be a natural number depending on p such that:

- For any operation of summation, subtraction and multiplication involving m elements of S where m is any number such that $m \leq n$, the result is the same in R_p and Z.
- $k(p) \to \infty$ and $n(p) \to \infty$ when $p \to \infty$.

This means that for the set S and the number n there is no manifestation of operations modulo p, i.e., the results of any operations of addition, subtraction and multiplication of $m \leq n$ elements from S are the same in R_p and Z. This means that if experiments involve only such sets S and numbers n then it is not possible to conclude whether the experiments are described by a theory involving R_p with a large p or by a theory involving Z.

Theorem: $R_p \to Z$ *at* $p \to \infty$.

Proof. Define the function $h(p)$ such that $h(p) = (p-1)/2$ if p is odd and $h(p) = p/2 - 1$ if p is even. $\forall p > p_0$ there exists a unique natural $n = n(p)$ such that $2^{n^2} \leq h(p) < 2^{(n+1)^2}$. Then if $k = k(p) = 2^n$ *Theorem* is satisfied. □

This *Theorem* has already been proven in Chapter 6.3 of the book [7]. However, the above proof is significantly simpler than the one given in [7].

In [7] we have proposed the following

Definition. Let theory A contain a finite nonzero parameter and theory B be different from theory A. Suppose that theory A can reproduce any result of theory B by choosing a value of the parameter. Suppose that one can define a limit of A when the parameter goes to zero or infinity, and in this limit, A becomes B. On the contrary, when the limit is already taken, one cannot return back from B to A and B cannot reproduce all results of A. Then theory A is more general than theory B and theory B is a special degenerate case of theory A.

The proved *Theorem* shows that:

- a) Any result in Z can be obtained in R_p if p is chosen to be sufficiently large.
- b) In Z it is impossible to reproduce those results in R_p where operations modulo p are nontrivial.

Therefore, as follows from *Definition*:

Statement 1: R_p *is more general than* Z*, and* Z *is a special degenerate case of* R_p *at* $p \to \infty$.

In [7] we discuss why the properties a) and b) are important for physics.

3 Standard Quantum Theory vs. Finite Quantum Theory

The question arises as to what role a theorem just proven can play in the foundation of mathematics. Let's first discuss the question of how we should treat mathematics: i) as a purely abstract science or ii) as a science that should describe nature? My observation is that for physicists, only approach ii) is acceptable. However, when I discussed this issue with mathematicians and philosophers, I discovered that many of them treat mathematics only from the point of view of i) and arguments related to the description of nature are not significant for them. Approach i) can be called the approach of Hilbert, who was its most famous proponent. There was a great discussion between him and Gödel about whether Gödel's incompleteness theorems indicate that the approach has foundational problems.

The fact that Hilbert's approach does not raise the question of describing nature does not mean that this approach should be rejected out of hand. For example, Dirac's philosophy is: *"I learned to distrust all physical concepts as a basis for a theory. Instead, one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics."* Dirac also said that for him the most important thing in any physical theory is the beauty of formulas in this theory. That is, he meant that sooner or later, a physical meaning of any beautiful mathematical theory will be found. An example of Dirac being right: Hilbert spaces were studied since the first decade of the 20th century by Hilbert, Schmidt, and Riesz, but these spaces began to be used in quantum physics only starting in the 30s of the 20th century. Within the Hilbert approach, many beautiful and powerful results have been obtained in modern mathematics. However, since, following Cantor, Hilbert's approach involves actual infinity, the problem of foundation of this approach remains open.

Let us now consider the problem of foundation of mathematics from the point of view if ii). As shown in the extensive physics literature (see, e.g., [7]),

Statement 2: Classical (i.e., non-quantum) theory is a special degenerate case of quantum one in the formal limit $\hbar \to 0$ *where* $\hbar$ *is the Planck constant.*

Let standard quantum theory (SQT) be a quantum theory based

on SM, and finite quantum theory (FQT) be a quantum theory based on FM. Then, from the point of view of ii) and *Statement 2*, the question of which mathematics is more general, SM or FM, depends on which theory is more general, SQT or FQT.

Let's first discuss some properties of SQT. Here, physical states are elements of a separable Hilbert space $\mathcal{H}$. In quantum theory (both, SQT and FQT), any system is considered to consist of elementary particles described by irreducible representations (IRs) of a symmetry algebra. In nonrelativistic theory, the symmetry algebra is the Galilei algebra, in relativistic theory — the Poincare algebra, and in de Sitter (dS) and anti-de Sitter (AdS) theories — the dS and AdS algebras, respectively. In SQT, IRs of these algebras describing elementary particles are infinite-dimensional. The state vector of the entire system is the tensor product of the state vectors for the elementary particles in the system (regardless of whether these particles interact or not). Therefore, the Hilbert space $\mathcal{H}$ for the entire system is infinite-dimensional, even if the system consists of a single elementary particle.

A known result of the theory of Hilbert spaces is that [8]:

Statement 3: A Hilbert space is separable if and only if it admits a countable orthonormal basis $(e_1, e_2, ...e_n, ...)$ *and it is always possible to choose a basis such that the norm of each* e_j $(j = 1, 2, ...\infty)$ *is an integer.*

Let the complex numbers $(c_1, c_2...)$ be the coefficients of the decomposition of a vector $x \in \mathcal{H}$ over the basis $(e_1, e_2...)$. The only condition that the coefficients must satisfy is: $\sum_{j=1}^{\infty} |c_j|^2 < \infty$. The known result of the theory of Hilbert spaces is that [8]:

Statement 4: The set of all points $(c_1, c_2, ...)$ *with only finitely many nonzero coordinates, each a rational number, is dense in the separable Hilbert space.*

This implies that, *with any desired accuracy*, each element of $\mathcal{H}$ can be approximated by a finite linear combination

$$x = \sum_{j=1}^{n} c_j e_j \tag{1}$$

where $c_j = a_j + ib_j$ and all the numbers (a_j, b_j) $(j = 1, 2, ...n)$ are rational.

The next observation is that spaces in quantum theory are projective: for any complex number $c \neq 0$, the elements x and cx describe the

same state. The meaning of this statement is that not the probability itself but ratios of different probabilities have a physical meaning. As a consequence, both parts of Eq. (1) can be multiplied by a common denominator of all the nonzero numbers a_j and b_j. As a consequence,

Statement 5: Each element of a separable projective Hilbert space can be approximated with any desired accuracy by a finite linear combination (1) where all the numbers a_j and b_j are integers, i.e., belong to Z.

The important consequence for understanding SQT is that here there is a large excess of states: although formally the theory involves Hilbert spaces of states $(c_1, c_2, ...c_n, ...)$ where all the c_j are arbitrary complex numbers and the only limitation is the condition $\sum_{j=1}^{\infty} |c_j|^2 < \infty$, for describing experiments with any desired accuracy it suffices to involve only states where only a finite number of the coefficients $c_j = a_j + ib_j$ are non-zero and all the numbers (a_j, b_j) belong to Z.

Before discussing FQT, let us note that in SQT, as shown by Dyson [9], it follows even from purely mathematical considerations that:

1) nonrelativistic theory (NT) is a special degenerate case of relativistic one (RT) in the formal limit $c \to \infty$. The quantity c is usually associated with the speed of light but in fact this is only a constant of the theory;

2) RT is a special degenerate case of dS and AdS invariant theories in the formal limit $R \to \infty$ where R is the parameter of contraction from the dS or AdS Lie algebras to the Poincare Lie algebra;

3) In turn, since dS and AdS algebras are semisimple, dS and AdS theories cannot be obtained from any more general theories by contraction.

In a theory based on FM, there cannot be dimensional parameters (kg, m, s) which are taken from macroscopic theory. Therefore, in FQT there cannot be Galilei and Poincare symmetry algebras. In FQT, state spaces are not Hilbert spaces, but spaces over the ring $R_p + iR_p$ which is the complex extension of the ring R_p. As follows from the Zassenhaus theorem [10], all IRs of the algebras over the rings of nonzero characteristics (modular IRs) are finite-dimentional. An explicit construction of modular IRs of dS and AdS algebras is given in [11, 12]. Therefore, if a system consists of a finite (even large)

number of elementary particles, then the state space for this system is finite-dimensional.

Another fundamental difference between SQT and FQT is the following. In SQT, IRs describing elementary particles have the property that the energy of particles in such IRs can be either only positive or only negative, but there are no IRs in which there are energies with different signs. In physics literature, the first case is referred to as particles, and the second, as antiparticles. Particles and antiparticles are assigned quantum numbers of opposite signs, and the conservation laws of these quantum numbers prohibit particle$\leftrightarrow$antiparticle transitions. Experimental data currently show that some quantum numbers (for example, baryon number and electric charge) are conserved with very high precision, and to date, no cases have been discovered where these numbers were not conserved.

On the other hand, in FQT, one IR necessarily contains states with both positive and negative energies and, as shown in [7, 13], when $p \to \infty$, one IR splits into two IRs in SQT with positive and negative energies. It is clear that the case when there is one IR uniting positive and negative energies has greater symmetry than the case when there are two different IRs with positive and negative energies. One might think that, since in contrast to SQT, in FQT there are no conserved quantum numbers, SQT is a more fundamental physical theory than FQT because SQT supposedly agrees with experiment, while FQT does not. However, the impression that in SQT these conservation laws are valid comes from the fact that, at the present stage of the universe, p is a huge number and superpositions of states with positive and negative energies practically do not play a role at this stage.

As shown in [7, 14], FQT is a more general theory than SQT because SQT is a special degenerate case of FQT at $p \to \infty$. As follows from *Definition*, to prove this statement, it is necessary to prove that:

Statement 6A: For any result of SQT it is possible to find $p = p_1$ such that FQT reproduces this result for all $p \geq p_1$;

Statement 6B: There are phenomena for the description of which it is necessary to use operations modulo a number.

As shown in [7, 14], *Statement 6A* follows from *Theorem* in Sec. 2 and from *Statement 5*. For proving *Statement 6B* we consider two phenomena: gravity and baryon asymmetry of the universe.

At present, the theory of quantum gravity is non-renormalizable and contains irremovable divergences. But, as shown in [7], at least the Newtonian gravitational law can be derived from FQT in semiclassical approximation. In this approach, the gravitational constant G is not taken from the outside but depends on p as $1/ln(p)$. By comparing this result with the experimental value, one gets that $ln(p)$ is of the order of 10^{80} or more, and therefore p is a huge number of the order of $exp(10^{80})$ or more. One might think that since p is so huge then in practice p can be treated as an infinite number. However, since $ln(p)$ is "only" of the order of 10^{80}, gravity is observable. In the formal limit $p \to \infty$, G becomes zero and gravity disappears. Therefore, in our approach, gravity is a consequence of finiteness of nature.

Before considering baryon asymmetry of the universe, let us discuss the following question. In many publications (see e.g., [15] and references therein), arguments are given that our universe works like a computer. Then the number p that determines the laws of physics in our universe is not a fundamental number given by a theory, but is a number that is determined by the state of the universe at its present stage. And, since the state of the universe is changing, it is natural to expect that the number p describing physics at different stages of the evolution of the universe will be different at different stages. As noted above, in the situation where p is very large, it may seem that the electric charge and baryon number are conserved quantum numbers. The above result about gravity shows that, at the present stage of the universe, the number p is huge, and this might be a justification of the postulate of modern particle theory that the electric charge and baryon number are strictly conserved quantum numbers.

The paradox with the baryon asymmetry of the universe is formulated as follows. According to modern cosmological theories, at early stages of the universe, the numbers of baryons and antibaryons were the same. Then, as follows from the law of baryon number conservation, those numbers should be the same at the present stage of the universe. However, at this stage, the number of baryons is much greater than the number of antibaryons.

The above paradox arises if we assume that the number p was huge even in the early stages of the universe and therefore the laws of conservation of electric charge and baryon number held true even in these stages. However, there is no basis for this assumption, and therefore the baryon asymmetry paradox does not arise.

In [7, 14] we gave other examples when FQT can solve problems that SQT cannot solve. Therefore, the above arguments show that

Statement 7: FQT is a more general theory than SQT.

In turn, as explained above, if we accept ii), then it follows from *Statement 7* that

Statement 8: Finite Mathematics is a more general theory than Standard Mathematics.

In conclusion of this section, let us discuss the following question. As noted above, in SM, the ring Z is generalized to the case of various fields in which four operations are possible: addition, subtraction, multiplication, and division. However, when generalizing SM to FM, we considered only the ring R_p and its extensions. In FM, division can seem unnatural. For example, in the Galois field F_p, where p is a prime, $1/2$ is a large number $(p+1)/2$ if p is large. However, the main question is whether it is necessary to have division in FQT.

SQT is essentially based on the concept of infinitesimals introduced by Newton and Leibniz more than 300 years ago. This concept was in the spirit of the experience that any macroscopic object can be divided into arbitrarily large number of arbitrarily small parts. However, now we know about the existence of elementary particles. Even the name "elementary particle" itself suggests that such a particle cannot be divided into parts. For example, the energies of electrons in modern accelerators are millions of times greater than the electron rest energy, and such electrons experience many collisions with other particles. If the electron could be divided into parts, this would have been discovered long ago. So, in physics, division has limited applicability, since when we reach the level of elementary particles, further division is no longer possible. Standard macroscopic theory and standard geometry (the concepts of continuous lines and surfaces) can work well only in the approximation when sizes of atoms are neglected. It seems rather strange that, although most physicists understand this, they nevertheless consider the concept of infinitesimals not as only approximate but as fundamental.

4 Conclusion

A new result of this paper is a simple proof of *Theorem* in Sec. 2. This proof uses only potential (not actual) infinity, and it makes natural

the conclusion of Sec. 3 that Finite Mathematics is more general than Standard Mathematics. As a consequence:

Mathematics describing nature at the most fundamental level involves only a finite number of numbers, while the concepts of limit, infinitesimals and continuity are needed only in calculations describing nature approximately.

It is clear that the ultimate quantum theory can only be based on mathematics free of foundational problems. As noted above, finite mathematics indeed does not have such problems. At the same time, in the approach of Cantor, Hilbert and other mathematicians the following questions arise:

A) If we accept that actual infinity is necessary, then how does this correspond to the work of Gödel and other mathematicians, that if we start from the entire infinite series of natural numbers, then problems arise in the foundation of mathematics.

B) In this approach, there is the concept of infinitesimals but, as noted in the preceding section, it is not clear whether this concept is compatible with the existence of elementary particles.

So, although in the approach of Cantor, Hilbert and other mathematicians, many strong and beautiful results have been obtained, the question arises how to reconcile this approach with A) and B). The answer to this question is currently unknown. However, it is clear that the problem exists, and this could be a good incentive for further research in mathematics and physics.

The question now arises of how to use the above results to construct the ultimate quantum theory. As noted in [7, 13], the main difficulty in such a construction is the following. Our physical intuition comes from the existing theory, which contains particles and antiparticles and laws of conservation of electric charge and baryon number. As explained in [7, 13], such results arise because the symmetry algebras used in quantum physics (Galilei, Poincare, and anti-de Sitter) have the property that in irreducible representations of such algebras, the energies of particles can only be either positive or negative, and there are no irreducible representations in which particles have states with different energy signs. At the same time, in finite mathematics, irreducible representations necessarily contain both positive and negative energies. It is clear that the case when there is one irreducible representation uniting positive and negative energies has a higher symmetry than the case when this irreducible representation splits into

two independent irreducible representations with positive and negative energies. In the former case, there are no strict concepts such as particle-antiparticle, proton-antiproton, electron-positron, and so on. These concepts can only be approximate when the characteristics of the ring in finite mathematics is very large. Therefore, the challenge of constructing a fundamental quantum theory is to construct a theory without the assumption that p is anomalously large, and such a theory will be based on new physical concepts.

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