On the Existence of a Prime Number in the Interval $[n^2,\, n^2 + n/2]$

Hashem Sazegar
email:h.sazegar@gmail.com

Abstract

Oppermann's conjecture states that for every positive integer n, there exists at least one prime number between n^2 and $n^2 + n$. Prior to this, Legendre had conjectured that there is always at least one prime number between n^2 and $(n + 1)^2$. In this paper, we not only claim to prove Oppermann's conjecture but also propose a smaller interval, asserting that there exists at least one prime between n^2 and $n^2 + n/2$.

Keywords: Bertrand–Chebyshev theorem, Landau's problems, Goldbach's conjecture, twin primes.

1 Introduction

Bertrand stated that for every positive integer n, there is always at least one prime between n and $2n$. The first proof was given by Chebyshev in 1850, now known as the Bertrand–Chebyshev theorem. Ramanujan later provided a simpler proof in 1919 using properties of the Gamma function. In 1932, Erdős gave an elementary proof using the Chebyshev function and binomial properties.

Legendre's conjecture states that for every positive integer n, there is at least one prime between n^2 and $(n + 1)^2$. Landau highlighted this as one of his famous problems. He also presented three other conjectures, forming the so-called "Landau's four problems":

1. The Twin Prime Conjecture: there exist infinitely many primes p such that $p + 2$ is also prime.

2. Goldbach's Conjecture: every even integer greater than 2 can be written as the sum of two primes.

3. There exist infinitely many primes of the form $n^2 + 1$.

4. There is always a prime between two consecutive squares.

2 Main Theorem

Theorem. There exists at least one prime number between n^2 and $n^2 + n/2$.

2.1 Proof

We attempt to prove this statement by induction. For $n = 12$, between 144 and 150 we find the prime 149. Assume the statement holds for $n = k$, and prove it for $n = k + 1$. For convenience, denote $k + 1$ by n. Assume that all integers in the interval $[n^2, n^2 + n/2]$ are composite. Then the total number of composite numbers in this interval is approximately $[n/2]$, among which $[n/4]$ are even (hence discarded) and the remaining numbers are odd. Among these odd numbers, some are divisible by small primes $3, 5, 7, \ldots, q$, where q denotes the largest prime less than or equal to $[n/2]$.

Let $q = p_s$, and set

$$a = \left[\frac{n}{4}\right].$$

- Multiples of 3, $\dfrac{a}{3}$ numbers, are eliminated, leaving $a\left(1 - \frac{1}{3}\right)$.
- Multiples of 5, $\dfrac{a\left(1 - \frac{1}{3}\right)}{5}$ numbers, are eliminated.
- Multiples of 7,

$$\frac{a\left(\left(1 - \frac{1}{3}\right) - \dfrac{1 - \frac{1}{3}}{5}\right)}{7} = \frac{a\left(0.66 - \frac{0.66}{5}\right)}{7},$$

  are eliminated.

- Multiples of 11,
$$\frac{a\left(0.53-\frac{0.53}{7}\right)}{11},$$
are eliminated.

Continuing this process, at stage p_{s+1}, we would have
$$\frac{a\ell - a\ell/p_s}{p_{s+1}},$$
which must vanish if all numbers were composite.

However, we obtain
$$0=\frac{a\ell\left(1-\frac{1}{p_s}\right)}{p_{s+1}} > \frac{a\ell\left(1-\frac{1}{p_s}\right)}{2p_s}.$$

This expression never tends to zero, even if we let $p_s \to \infty$. In fact,
$$\lim_{p_s\to\infty}\frac{a\ell\left(1-\frac{1}{p_s}\right)}{2p_s} = \frac{a\ell}{2e}.$$

Therefore, the assumption that all numbers in the interval are composite is false, and hence there must exist at least one prime in the interval $[n^2, n^2+n/2]$.

2.2 Second proof

Let $a=[\frac{n}{2}]$ and let $p_{s+1}\leq n$ be the greatest prime number in $[n^2, n^2+\frac{n}{2}]$. We prove similar to the previous proof. We begin by p_{s+1} until to prime number 3. Therefore

p_{s+1}: $\frac{a}{p_{s+1}}$ numbers are removed.

p_s: $\frac{a-\frac{a}{p_{s+1}}}{p_s}$ numbers are removed.

p_{s-1}: $\frac{(a-\frac{a}{p_{s+1}})-\frac{(a-\frac{a}{p_{s+1}})}{p_s}}{p_{s-1}}$ numbers are removed.

$\vdots$

3: $\frac{al_1 - \frac{1}{5}}{3}$ numbers are removed.

2: $\frac{al_1' - \frac{1}{3}}{2}$ numbers are removed.

Note that $\frac{al_1 - \frac{1}{5}}{3}$ is not equal to zero, since $a, l_1 > 0$. Therefore there exists at least a prime number between n^2 and $n^2 + n/2$.

Based on this method, it can be shown that there exists at least a prime number in intervals $[n^2,\ n^2 + n/2]$, $[n^2,\ n^2 + n/3]$, ..., $[n^2,\ n^2 + \epsilon n]$ for $\epsilon > 0$

3 Example

Consider $n = 30$. Between 30^2 and $30^2 + 30$, we find several primes:

$$
\begin{aligned}
&900 = 2 \times 450,\\
&901 = 17 \times 53,\\
&903 = 3 \times 301,\\
&905 = 5 \times 181,\\
&907 \text{ is prime},\\
&909 = 3 \times 301 + 3 \times 2,\\
&911 \text{ is prime},\\
&913 = 11 \times 83,\\
&915 = 3 \times 301 + 3 \times 4,\\
&917 = 7 \times 131,\\
&919 \text{ is prime},\\
&921 = 3 \times 301 + 3 \times 6,\\
&923 = 13 \times 71,\\
&925 = 5 \times 181 + 5 \times 4,\\
&927 = 3 \times 301 + 3 \times 8.\\
&929 \text{ is prime},
\end{aligned}
$$

In order for a number in the interval to be composite, it must have a prime divisor not exceeding

$$q = \sqrt{30^2 + 30} \approx \sqrt{930}.$$

Hence, it suffices to check primes up to $q \leq 30$, where q is a prime factor.

For $n = 30$, the number of odd integers in the interval is 15. By eliminating multiples of small primes:

$$\begin{aligned}
&\text{Multiples of 3: } [15/3] = 5,\\
&\text{Multiples of 5: } [(15-5)/5] = 2,\\
&\text{Multiples of 7: } [(15-7)/7] = 1,\\
&\text{Multiples of 11: } 1,\\
&\text{Multiples of 13: } 1.
\end{aligned}$$

Thus, out of 15 odd numbers, four numbers are primes.

Our Methodology

We compute the successive elimination of multiples of primes as follows:

$$\begin{aligned}
\text{Multiples of 3}: &\ \frac{15}{3} &= 5,\\
\text{Multiples of 5}: &\ \frac{15-5}{5} &= 2,\\
\text{Multiples of 7}: &\ \frac{15-7}{7} &= 1.14,\\
\text{Multiples of 11}: &\ \frac{15-8.14}{11} &= 0.623,\\
\text{Multiples of 13}: &\ \frac{15-8.76}{13} &= 0.479,\\
\text{Multiples of 17}: &\ \frac{15-9.242}{17} &= 0.338,\\
\text{Multiples of 19}: &\ \frac{15-9.58}{19} &= 0.282,\\
\text{Multiples of 23}: &\ \frac{15-9.862}{23} &= 0.223,\\
\text{Multiples of 29}: &\ \frac{15-10.085}{29} &= 0.169.
\end{aligned}$$

From these calculations, we observe that the successive elimination of prime multiples never reaches zero. Therefore, the assumption that all numbers in the interval are composite leads to a contradiction. It follows that there must exist at least one prime in the interval.

This reasoning can be extended similarly to the intervals $[n^2, n^2 + n/2]$, and beyond.

Our Second Methodology In order for a number in the interval to be composite, it must have a prime divisor not exceeding

$$q = \sqrt{30^2 + 30} \approx \sqrt{930}.$$

Hence, it suffices to check primes up to $q \leq 30$, where q is a prime factor.

$$\begin{aligned}
&\text{Multiples of 29: } [30/29] = 1,\\
&\text{Multiples of 23: } [29/23] = 1,\\
&\text{Multiples of 19: } [28/19] = 1,\\
&\text{Multiples of 17: } [27/17] = 1,\\
&\text{Multiples of 13: } [26/13] = 2,\\
&\text{Multiples of 11: } [24/11] = 2,\\
&\text{Multiples of 7: } [22/7] = 3,\\
&\text{Multiples of 5: } [19/5] = 3,\\
&\text{Multiples of 3: } [16/3] = 5,\\
&\text{Multiples of 2: } [11/2] = 5.
\end{aligned}$$

Second method

$$\begin{aligned}
\text{Multiples of } 29: \ \frac{30}{29} &= 1.034,\\
\text{Multiples of } 23: \ \frac{28.996}{23} &= 1.26,\\
\text{Multiples of } 19: \ \frac{27.676}{19} &= 1.46,\\
\text{Multiples of } 17: \ \frac{26.216}{17} &= 1.54,\\
\text{Multiples of } 13: \ \frac{24.676}{13} &= 1.898,\\
\text{Multiples of } 11: \ \frac{22.778}{11} &= 2.0707,\\
\text{Multiples of } 7: \ \frac{20.707}{7} &= 2.958,\\
\text{Multiples of } 5: \ \frac{17.749}{5} &= 3.55,\\
\text{Multiples of } 3: \ \frac{14.299}{3} &= 4.733,\\
\text{Multiples of } 2: \ \frac{9.466}{2} &= 4.733.
\end{aligned}$$

Also, this method does not tends to zero.