ON THE GENERALIZED SCHÖNHAGE-TYPE BOUND

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ABSTRACT. We prove an extension of the lower bound due to Schönhage on addition chains.

1. Introduction

An addition chain, introduced in [1], of length h leading to n is a sequence of numbers $s_o = 1, s_1 = 2, \ldots, s_h = n$ where $s_i = s_k + s_s$ for $i > k \geq s \geq 0$. The number of terms (excluding the first term) in an addition chain leading to n is the length of the chain. We call an addition chain leading to n with a minimal length an *optimal* or the *shortest* addition chain leading to n. In standard practice, we denote by $\ell(n)$ the length of an optimal addition chain that leads to n. Lower bounds for the length of the shortest addition chain leading to a fixed n have also been extensively studied. The best result along these lines is due to Schönhage [3], who proved in 1973 the following nontrivial lower bound:

Theorem 1.1. *[Schönhage]*

$$\ell(z) \geq \frac{\log z}{\log 2} + \frac{\log(\nu(z))}{\log 2} - 2.13$$

where $\nu(z)$ denotes the sum of all digits in the binary expansion of z.

In [2], it was asked if extended versions of the known lower bound for the optimal length of a classical addition chain due to Schönhage could be established in the framework that allows the construction of terms in a chain using a fixed number of previous terms. In this paper, we answer this question in an affirmative way by showing

Theorem 1.2. *[Generalized Schönhage-type bound]*
Let $d \geq 2$ be fixed and let $\nu_d(\cdot)$ denotes the number of nonzero digits in

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the base-d expansion of $\cdot$. Furthermore, let

$$s_0 = 1 < s_1 = 2 < \cdots < s_{\ell^d(n)} = n$$

be any optimal addition chain with fixed degree $d \geq 2$ and set

$$\alpha(n) := \max_{i \in K} \frac{s_i}{s_{i-1}}$$

where K is the set of all non d-dilate steps in the chain. Then $0 < \alpha := \alpha(n) < d$ and

$$\ell^d(n) \geq \frac{\log n}{\log d} + \left(1 - \frac{\log \alpha}{\log d}\right) \frac{\log(\nu_d(n))}{\log d}.$$

2. Addition chains with single fixed degrees

Definition 2.1. Let $n \geq 3$ and fix $d \geq 2$. We say that the sequence of positive integers

$$s_0 = 1 < s_1 = 2 < \cdots < s_h = n$$

is an addition chain with fixed degree $d \geq 2$ leading to n of length h if for each $1 \leq i \leq h$ the partition

$$s_i = \sum_{j \in [0, d-1]} s_j, \quad (s_j < s_i)$$

is possible. In particular, each term term in an addition chain with fixed degree $d \geq 2$ is the sum of at most d previous terms in the chain, with repetition allowed. We call the shortest addition chain with fixed degree $d \geq 2$ leading to a target an *optimal* addition chain with fixed degree d leading to the specified target. We denote the length of an optimal addition chain with fixed degree $d \geq 2$ leading to a target n with $\ell^d(n)$. The special case where the fixed degree $d = 2$ recovers the well-known concept of an addition chain.

Remark 2.2. As with addition chains, an optimal addition chain with fixed degree $d \geq 2$ does not have to be unique. In the family of all addition chains with fixed degree $d \geq 2$ leading to a target, it may be possible that at least two chains in the family have the shortest length.

Example 2.3. Choose the target $n = 21$ and fix the degree $d = 3$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 21$$

is an addition chain with fixed degree $d = 3$, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = s_4 + s_2 + s_0$.

Example 2.4. Choose the target $n = 63$ and fix the degree $d = 4$. The sequence

$$s_0 = 1, s_1 = 2, s_2 = 4, s_3 = 8, s_4 = 16, s_5 = 32, s_6 = 56, s_7 = 63$$

is an addition chain with fixed degree $d = 4$, because $s_2 = 2s_1, s_3 = 2s_2, s_4 = 2s_3, s_5 = 2s_4, s_6 = s_5 + s_4 + s_3, s_7 = s_6 + s_2 + s_1 + s_0$.

Proposition 2.5. *[The fixed degree monotonicity principle]*
For any target $n \geq 2$ and any fixed degrees d_1, d_2 with $2 \leq d_1 \leq d_2$, we have

$$\ell^{d_2}(n) \leq \ell^{d_1}(n).$$

In particular, allowing more summands at each step cannot increase the minimal chain length.

Proof. Let $d_1 \leq d_2$ and take $\mathcal{C}_n$ to be any addition chain with fixed degree d_1 that leads to n of length $h := \ell^{d_1}(n)$; that is, an optimal addition chain with fixed degree d_1. Because $d_1 \leq d_2$, it implies that $\mathcal{C}_n$ is also a valid addition chain with fixed degree d_2 that leads to n. Therefore, the minimal length of an addition chain with fixed degree d_2 cannot exceed the minimal length of an addition chain with fixed degree d_1 for any chosen target. □

The inequality in Proposition 2.5 is fundamental, but it could allow us to transfer the lower bound for the length of an optimal addition chain with fixed degree to a lower bound for the length of an optimal classical addition chain that leads to the same target. The converse argument also holds. The following are some immediate remarks on the implications of *monotonicity principle*:

- The inequality in Proposition 2.5 can be strict. For example, choose the target $n = 7$ and fix the degrees $d_1 = 2, d_2 = 3$. Under the $d_1 = 2$ constraint, we have an optimal addition chain of length $\ell^2(7) = \ell(7) = 4$ (eg. $1, 2, 4, 6, 7$), while with $d_2 = 3$ the chain $1, 2, 6, 7$ is optimal and of length $\ell^3(7) = 3$.
- Furthermore, we deduce from Proposition 2.5 that for all $n \geq 1$ and every $d \geq 2$, we have

  $$\ell^d(n) \leq \ell^2(n) := \ell(n).$$

  Hence, fixed degree minimal lengths cannot exceed the minimal length of addition chains (classical).

The following observation is terse, but suggests that in the regime where we allow some degree of freedom, the Scholz-type bound for the addition chain that leads to numbers of the form $2^n - 1$ is plausible.

Theorem 2.6. *The inequality*

$$\ell^d(2^n - 1) \leq n - 1 + \left\lfloor \frac{\log n}{\log 2} \right\rfloor$$

holds for all $n \geq 2$ with fixed $d := (n-1) - \lfloor \frac{n-1}{2} \rfloor + 1$.

Proof. We first construct a sequence leading to 2^{n-1} by repeated doubling

$$s_0 = 1, s_1 = 2, s_2 = 2^2, \ldots, s_{n-1} = 2^{n-1}$$

obtained from the sequence of additions

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.$$

We extend the sequence by introducing new terms that leads to $2^n - 1$. We extend this sequence by adding previous terms in the following way

- $s_n := s_{n-1} + 2^{n-2} \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-1}{2} \rfloor - 1}$
- $s_{n+1} = s_n + 2^{\lfloor \frac{n-1}{2} \rfloor - 2} + \cdots + 2^{t_2} + \cdots + 2^{\lfloor \frac{n-1}{2^2} \rfloor - 1}$
- $s_{n+1-i} = s_{n+i-2} + 2^{\lfloor \frac{n-1}{2^{i-1}} \rfloor - 2} \cdots + 2^{t_i} + \cdots + 2^{\lfloor \frac{n-1}{2^i} \rfloor - 1}$ for $i \geq 1$

where $\lfloor \frac{n-1}{2} \rfloor \leq t_1 \leq n-2$, $\lfloor \frac{n-1}{2^2} \rfloor \leq t_2 \leq \lfloor \frac{n-1}{2} \rfloor - 2, \ldots, \lfloor \frac{n-1}{2^k} \rfloor \leq t_k \leq \lfloor \frac{n-1}{2^{k-1}} \rfloor - 2$. The terms adjoined to the sequence of repeated doubling constructed can be recast into the forms

- $s_n = 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor - 1}$
- $s_{n+1} := 2^n - 2^{\lfloor \frac{n-1}{2^2} \rfloor - 1}$
- $s_{n+1-i} := 2^n - 2^{\lfloor \frac{n-1}{2^i} \rfloor - 1}$ for $i \geq 1$.

By induction, we can write

$$s_{l^d(2^n-1)} = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2^1 + 1 = 2^n - 1$$

where $l^d(2^n - 1)$ denotes the index of the last term $2^n - 1$ in the constructed sequence. We observe that this construction yields an addition chain with fixed degree

$$d = (n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

since

$$s_{n+1-i} = \overbrace{s_{n+i-2} + 2^{\lfloor \frac{n-1}{2^{i-1}} \rfloor - 2} \cdots + 2^{t_1} + \cdots + 2^{\lfloor \frac{n-1}{2^i} \rfloor - 1}}^{d_i \text{ terms}}$$

with the number of summands in the partition of s_{n+1-i} into previous terms being $d_i = \lfloor \frac{n-1}{2^{i-1}} \rfloor - \lfloor \frac{n-1}{2^i} \rfloor + 1 \leq (n-1) - \lfloor \frac{n-1}{2} \rfloor + 1$ for $i \geq 1$. The constructed addition chain with fixed degree

$$d =: (n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor + 1$$

is the following

$$s_0 = 1, s_1 = 2, s_2 = 2^2, \ldots, s_{n-1} = 2^{n-1}, s_n = 2^n - 2^{\lfloor \frac{n-1}{2} \rfloor - 1}, s_{n+1} = 2^n - 2^{\lfloor \frac{n-1}{2^2} \rfloor - 1}, \ldots, 2^n - 1.$$

We note that the terms adjoined to the sequence contributes exactly

$$\left\lfloor \frac{\log(n-1)}{\log 2} \right\rfloor \leq \left\lfloor \frac{\log n}{\log 2} \right\rfloor$$

terms while the number of doubling steps contributes $n-1$ terms to the sequence. Assembling both contributions yields the upper bound for the length of the optimal addition chain with fixed degree

$$d := (n-1) - \left\lfloor \frac{n-1}{2} \right\rfloor + 1.$$

□

Problem 2.7. *[Fixed degree Scholz problem-a generalization]*
Let $d \geq 2$ be fixed. Find all integers $n \geq 1$ satisfying

$$\ell^d(2^n - 1) \leq n - 1 + \ell^d(n).$$

The special case $d = 2$ is the Scholz conjecture on addition chains.

The fixed degree Scholz problem (Problem 2.7) suggests a natural way to investigate the upper and lower bounds for $\ell^d(n)$ for all $n \geq 1$ for fixed $d \geq 2$. The following bound, although crude, serves as a benchmark.

Theorem 2.8. *[The d-arity method]*
Let d be fixed and let $\nu_d(\cdot)$ denote the number of nonzero digits in the base-d expansion of $\cdot$. For all $n \geq 2$, we have

$$\ell^d(n) \leq \left\lfloor \frac{\log n}{\log d} \right\rfloor + \nu_d(n) + 1.$$

Proof. Expand n is based d ($d \geq 2$) in the following way

$$n = \zeta(a_1)d^{a_1} + \cdots + \zeta(a_{\nu_d(n)})d^{a_{\nu_d(n)}}$$

with $a_1 > \cdots > a_{\nu_d(n)-1} > a_{\nu_d(n)}$ and $1 \leq \zeta(a_1), \ldots, \zeta(a_{\nu_d(n)}) \leq d-1$. Construct a sequence in the following way

$$1, 2, d, d^2, \ldots, d^{a_1}.$$

We observe that

- $d = \overbrace{1 + \cdots + 1}^{\text{d times}}$
- $d^2 = \overbrace{d + \cdots + d}^{\text{d times}}$.
- $d^{a_i} = \overbrace{d^{a_i-1} + \cdots + d^{a_i-1}}^{\text{d times}}$ for $2 \leq a_i \leq a_1$.

We further extend the sequence constructed by adding previous terms in the sequence as follows

$$1, 2, \ldots, d, d^2, \ldots, d^{a_1}, \zeta(a_1)d^{a_1}, \zeta(a_1)d^{a_1} + \zeta(a_2)d^{a_2}, \ldots, \sum_{i=1}^{\nu_d(n)} \zeta(a_i)d^{a_i} = n.$$

We observe that

- $\zeta(a_1)d^{a_1} = \overbrace{d^{a_1} + \cdots + d^{a_1}}^{\zeta(a_1) \text{ times}}$ with $\zeta(a_1) \leq d-1$
- $\zeta(a_1)d^{a_1} + \zeta(a_2)d^{a_2} = \zeta(a_1)d^{a_1} + \overbrace{d^{a_2} + \cdots + d^{a_2}}^{\zeta(a_2) \text{ times}}$ with $\zeta(a_2) \leq d-1$
- $\left(\sum_{i=1}^{k} \zeta(a_i)d^{a_i}\right) + \zeta(a_{k+1})d^{a_{k+1}} = \left(\sum_{i=1}^{k} \zeta(a_i)d^{a_i}\right) + \overbrace{d^{a_{k+1}} + \cdots + d^{a_{k+1}}}^{\zeta(a_{k+1}) \text{ times}}$ with $\zeta(a_{k+1}) \leq d-1$ for $k \geq 2$.

The sequence

$$1, 2, d, d^2, \ldots, d^{a_1}, \zeta(a_1)d^{a_1}, \zeta(a_1)d^{a_1} + \zeta(a_2)d^{a_2}, \ldots, \sum_{i=1}^{\nu_d(n)} \zeta(a_i)d^{a_i} = n$$

is therefore a valid addition chain with fixed degree d leading to n. We now analyze the number of terms in the chain. We observe that the number of terms in this fixed degree d ($d \geq 2$) addition chain is the contribution of the first two terms leading to d, the count of the number of steps where d similar terms from previous steps were added and the number of steps where we repeatedly added previous terms in building the base-d expansion of n. The number of steps where d similar terms from previous steps were added is at most

$$\left\lfloor \frac{\log n}{\log d} \right\rfloor$$

and the number of steps where we repeatedly added previous terms in building the base-d expansion of n is at most $\nu_d(n)-1$, where $\nu_d(n)$ is the number of nonzero digits in the base-d expansion of n. Hence,

$$\ell^d(n) \leq 2 + \left\lfloor \frac{\log n}{\log d} \right\rfloor + \nu_d(n) - 1.$$

□

The upper bound offered by Theorem 2.8 can be viewed as a generalization of the special case $d=2$ given by the binary method. However, this d-arity method could be wasteful in the worst-case scenario where there are a few nonzero digits in the base-d expansion of n. It may be possible to improve the bound using analogues or slightly adapted versions of the Brauer method for addition chains. We now turn our attention to establishing a lower bound for $\ell^d(n)$ for a fixed d.

3. Main result

Definition 3.1. Let $d \geq 2$ be fixed and let

$$s_0 = 1 < s_1 = 2 < \cdots < s_h = n$$

be any addition chain with fixed degree d. Let

$$G := \{i \ : \ i_j = i-1, 1 \leq j \leq d\}$$

for

$$s_i := \sum_{j=1}^{d} s_{i_j}$$

and

$$K := \{i \ : \ i_j < i-1, \text{for some } j, \ 1 \leq j \leq d\}$$

for

$$s_i = \sum_{j \in [1,d]} s_{i_j}$$

where there are at most d terms in the sum. We call G the *d-dilate steps* and K the *non d-dilate steps*, respectively, in the chain. We denote by $|G|$ and $|K|$ the number of d-dilate and non d-dilate steps in the addition chain with fixed degree.

Example 3.2. Fix $d=3$ and choose the target $n=20$. We construct the addition chain with fixed degree $d=3$ as follows

$$s_0 = 1, s_1 = 2, s_2 = 6, s_3 = 18, s_4 = 20$$

with $s_1 = 1+1, s_2 = 2+2+2, s_3 = 6+6+6, s_4 = 18+2$. Hence, we have

$$G = \{2, 3\} \quad \text{and} \quad K = \{1, 4\}$$

as the 3-dilate and non 3-dilate steps, respectively.

Theorem 3.3. *[Generalized Schönhage-type bound]*
Let $d \geq 2$ be fixed and let $\nu_d(\cdot)$ denotes the number of nonzero digits in the base-d expansion of $\cdot$. Furthermore, let

$$s_0 = 1 < s_1 = 2 < \cdots < s_{\ell^d(n)} = n$$

be any optimal addition chain with fixed degree $d \geq 2$ and set

$$\alpha(n) := \max_{i \in K} \frac{s_i}{s_{i-1}}$$

where K is the set of all non d-dilate steps in the chain. Then $0 < \alpha := \alpha(n) < d$ and

$$\ell^d(n) \geq \frac{\log n}{\log d} + \left(1 - \frac{\log \alpha}{\log d}\right) \frac{\log(\nu_d(n))}{\log d}.$$

Proof. Fix d with $d \geq 2$ and expand n in base d as follows

$$n = \sum_{i \geq 1} \zeta(a_i) d^{a_i}$$

where $\zeta(a_i) \in \{0, 1, \ldots, d-1\}$. Consider an optimal addition chain with fixed degree d leading to n of the form

$$s_0 = 1 < s_1 = 2 < \ldots < s_{\ell^d(n)} = n.$$

Classify the consecutive steps in the optimal chain using the following criteria:

$$G := \{i \ : \ i_j = i - 1, 1 \leq j \leq d\}$$

for

$$s_i := \sum_{j=1}^{d} s_{i_j}$$

and

$$K := \{i \ : \ i_j < i - 1, \text{for some, } 1 \leq j \leq d\}$$

for

$$s_i = \sum_{j \in [1,d]} s_{i_j}$$

where there are at most d terms in the sum. We call G the *d-dilate steps* and K the *non d-dilate steps*, respectively. Denote by $|G|$ and

$|K|$ the number of d-dilate and non d-dilate steps, respectively in the optimal addition chain with fixed degree d. Hence

$$\ell^d(n) = |G| + |K|.$$

Now, we write

$$n = \sum_{j\in[1,d]} s_{i_j}.$$

If all the s_{i_j} are equal and the sum contains exactly d terms, then

$$n = ds_{i_j} \iff \nu_d(n) = \nu_d(ds_{i_j}) = \nu_d(s_{i_j})$$

since multiplying the base-d expansion of any positive integer by d does not change the number of nonzero digits of the number. On the other hand, if at least one of the s_{i_j} is distinct, then

$$\nu_d(n) \leq \sum_{j\in[1,d]} \nu_d(s_{i_j}) \leq d\nu_d(s_{i_k})$$

for some $k \in [1, d]$. Similarly, if $s_{i_k} = ds_{i_r}$ then

$$\nu_d(s_{i_k}) = \nu_d(ds_{i_r}) = \nu_d(s_{i_r}).$$

On the other hand, if

$$s_{i_k} := \sum_{v\in[1,d]} s_{i_v}$$

where at least one of the s_{i_v} is distinct, then

$$\nu_d(s_{i_k}) \leq \sum_{v\in[1,d]} \nu_d(s_{i_v}) \leq d\nu_d(s_{i_u})$$

for some $u \in [1, d]$. By induction, we deduce

$$\nu_d(n) \leq d^{|K|}$$

and

$$|K| \geq \frac{\log(\nu_d(n))}{\log d}.$$

Similarly (by induction), we write

$$n \leq d^{|G|}\alpha^{|K|}$$

for some $0 < \alpha := \alpha(n) < d$. We deduce

$$|G| \geq \frac{\log n}{\log d} - |K|\frac{\log \alpha}{\log d}.$$

The lower bound follows by using the partition

$$\ell^d(n) = |G| + |K|.$$

□

It is worth announcing that Theorem 3.3 answers in the affirmative the question posed in the paper [2], in which it was asked if extended versions of the known lower bound for the optimal length of a classical addition chain due to Schönhage could be established in the framework that allows building terms in a chain using a fixed number of previous terms.

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