Analysis of the Van der Pol Oscillator

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Abstract

The Van der Pol oscillator is a nonlinear system known for its self-sustaining oscillation and behavior. This paper analyzes how the system evolves as the damping parameter μ changes, focusing on equilibrium points, phase plane trajectories, and limit cycles. Throughout the paper, we highlight how the equation also relate to physical systems, such as electrical circuits and biological rhythms, showing the significance and relevance of the Van der Pol oscillator in modeling real-world nonlinear behavior.

Contents

1 Introduction

In 1927, Dutch electrical engineer Balthasar Van der Pol discovered a special electric circuit which is modeled by what is now known as the Van der Pol Oscillator equation, see [9]. It was initially found to model certain types of electrical circuits, but is now used to model linear and nonlinear oscillations beyond electrical circuits. These oscillations are characterized by periodic behavior, often being connected to limit cycles. The Van der Pol Oscillator is also known for exhibiting relaxation oscillations, a type of oscillation characterized by a slow build-up followed by a sudden jump. Since most of the world is run by nonlinear oscillations that do not show linear responses or forces acting on them, we can establish multiple connections of this equation with the outside world. There will be multiple examples throughout this document connecting the equation to the world.

1.1 History

During the 1920s, electric circuits were analyzed using linear models, but Van der Pol figured out that circuits, in fact, did not exhibit linear or constant behaviors. After realizing that electrical circuits could not be explained using traditional linear differential equations, he formulated a new equation called the Van der Pol equation, which captured the nonlinearity of electrical circuits. In his equation, the circuits accounted for self-sustained oscillations, where energy was added and removed from the nonlinearity of the system periodically.

His work was influential in early radio technology, as vacuum tubes, which are devices that control the flow of electricity using a vacuum which often behaved in nonlinear ways. His experiments with the equation showed that certain nonlinear systems could change from regular and even harmonic oscillations to unpredictable, random behavior.

Beyond electrical circuits, the Van der Pol equation can be used to model different phenomenons. Today, the equation remains a fundamental tool for modeling and studying nonlinear dynamics.

1.2 Modeling

The Van der Pol oscillator is modeled by a second-order differential equation that displays a non-linear damping term that renders the model different from a simple harmonic oscillation. The equation is shown below:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$$

Here, the x represents the general state of the system, such as the voltage in a circuit. The first derivative, $\dot{x}$, corresponds to the rate of change of x, while the second derivative, $\ddot{x}$, represents the acceleration of x. The term $\mu(1-x^2)$ would be the nonlinear damping, where the damping depends on both $\dot{x}$ and x, and μ controls the strength of this effect. The "$+x$" at the end of the equation acts as a restoring force, like a spring that pulls an object back to its starting position.

To make this equation appear simpler, we will rewrite it as a system of first-order equations by introducing the auxiliary variable y as follows:

$$\dot{x} = y$$
$$\dot{y} = \mu(1 - x^2)y - x$$

In this form, we can plot the equation in a the $x - y$ plane, which is called the phase space, where we can visualize the behavior of the system. In a phase plane, we can see how the state of the system (x and y) evolves over time. Also, you can analyze the system's long-term behavior by showing all the orbits or trajectories of the system, allowing you to identify its equilibrium points and their stability.

2 Qualitative Analysis

A qualitative analysis of the Van der Pol oscillator will show the overall trend or the behavior of the system. This approach examines key features of the equation such as equilibrium points, its stability, and limit cycles, and the general structure of trajectories in the phase plane.

2.1 Equilibrium Points

An equilibrium point in a differential equation is where all variables remain constant over time unless an external force acts upon it. It represents the balance between opposing forces, meaning there are no net changes in these points. There are multiple types of equilibrium points, such as stable points where nearby trajectories approach over time. There are also unstable points, where nearby trajectories move away over time, and saddle points, where both stable and unstable qualities exist. This happens when trajectories in a certain direction move away from the saddle point and conversely, other trajectories with a different direction approach the point. Understanding equilibrium points is crucial since you can determine whether the system will settle, oscillate, or diverge over time.

Now we will calculate the equilibrium points for the Van der Pol oscillator:

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = \mu(1 - x^2)y - x$$

Since the equilibrium points (x, y) are those for which $dx/dt = dy/dt = 0$, we get that:

$$y = 0 \quad and \quad \mu(1 - x^2)y - x = 0$$
$$\downarrow$$
$$\text{Equilibrium point: } (x, y) = (0, 0)$$

The Van der Pol oscillator only has one equilibrium point, which lies at $(0, 0)$. We can also see that this point is unstable, since the trajectories at point $(0, 0)$ go outward and away from the origin in a phase plane. This outward trajectory from the origin drives the oscillator into a periodic motion known as the limit cycle.

2.2 Connections to Other Equations

The Van der Pol equation is connected to a lot of other equations, and we are going to look at an example. The Liénard's Equation is an equation which is very similar to the VdP equation. The Liénard's Equation is:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

This shows how the Van der Pol equation is a type of a Liénard's Equation where $f(x) = -\mu(1 - x^2)$ and $g(x) = -x$. We can also derive the VdP equation from Liénard's by using Liénard's transformation, which is a technique used to convert a second-order nonlinear differential equation into a system of first-order equations. From the equation, we will define a new variable y as

$$y = \dot{x} + F(x)$$

Where $F(x)$ is defined by

$$F(x) = \int_0^x f(t)\,dt$$

Rewriting the original equation and using the chain rule for y, we obtain this first-order system:

$$\dot{x} = y - F(x)$$

$$\dot{y} = \ddot{x} + f(x)\dot{x} = -g(x)$$

Since we know $f(x)$, we get an $F(x)$ of

$$F(x) = \int_0^x -\mu(1 - t^2)dt = -\mu\left(x - \frac{x^3}{3}\right)$$

Using Liénard's transformation:

$$y = \dot{x} - \mu\left(x - \frac{x^3}{3}\right)$$

And rewriting this into a first-order system:

$$\dot{x} = y + \mu\left(x - \frac{x^3}{3}\right)$$

$$\dot{y} = -x$$

This is the first-order representation of the VdP equation, which shows how you can derive the VdP equation from Liénard's Equation. However, the standard first-order system is used more often because of its simpler aspect of the representation of velocity($\dot{x}$). For more details, see [5].

2.3 Changes of x

As said before, the $-\mu(1 - x^2)y$ is the factor of damping in the VdP equation. As we can see, since the variable x is inside the notation, it has an important role in deciding what the damping will look like. There are 2 main situations:

When $|x|$ is small ($|x| < 1$), the term $1 - x^2$ is positive, so $-\mu(1 - x^2)y$ stays negative, which contributes negative damping to the system. This causes the oscillation to grow in amplitude which adds energy into the system.

Conversely, when $|x|$ is big ($|x| > 1$), the term $1 - x^2$ is negative, so $-\mu(1 - x^2)y$ turns positive, which contributes positive damping to the system. This causes the oscillation to decrease in amplitude, which removes energy from the system.

The VdP equation adds energy when x is small, which corresponds to a low-energy state of the system, and removes energy when x is large. These changes of x show how the Van der Pol oscillator is a self-sustainable oscillation and why the equation also prevents itself from diverging to infinity or decaying to zero, but has a limit cycle instead.

2.4 Effects of x on Energy Dynamics

Again, consider the Van der Pol oscillator:

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \tag{1}$$

where $\mu > 0$, and is the factor that controls nonlinearity and damping.

To show the energy behavior of this system, we start with the function

$$E(t) = \frac{1}{2}(\dot{x}^2 + x^2), \tag{2}$$

which is the total mechanical energy of a linear harmonic oscillator, where $\frac{1}{2}\dot{x}^2$ is the kinetic energy, and $\frac{1}{2}x^2$ is the potential energy. Taking the derivative of $E(t)$ gives:

$$\frac{dE}{dt} = \dot{x}\ddot{x} + x\dot{x} = \dot{x}(\ddot{x} + x). \tag{3}$$

since the original VdP equation gives

$$\ddot{x} + x = \mu(1 - x^2)\dot{x}, \tag{4}$$

therefore,

$$\frac{dE}{dt} = \dot{x} \cdot \mu(1 - x^2)\dot{x} = \mu(1 - x^2)\dot{x}^2. \tag{5}$$

This expression describes how the energy changes over time. Its sign depends on the value of x:

- When $|x| < 1$, we have $1 - x^2 > 0$, so $\frac{dE}{dt} > 0$. The energy increases, since $\frac{dE}{dt}$ will be positive.
- When $|x| > 1$, we have $1 - x^2 < 0$, so $\frac{dE}{dt} < 0$. The energy decreases, since $\frac{dE}{dt}$ will be negative.

This again shows that the Van der Pol oscillator exhibits self-regulating behavior, giving us the same results as section **2.3**, which shows again, how this mechanism leads the equation to converge towards a stable limit cycle.

2.5 Trajectories of the Limit Cycle

Here, we can ask ourselves another interesting question: Can the trajectories on the phase plane ever cross or intersect with the limit cycle?

The answer is no. As shown in sections **2.3** and **2.4**, the limit cycle acts as a stable closed orbit that attracts all nearby trajectories but is never crossed by them. Inside the limit cycle, where most of the trajectories are when $|x| < 1$, trajectories spiral outwards as energy increases, and vice versa. Over time, both types of trajectories converge towards the limit cycle, but once a trajectory reaches it, the system settles into a periodic orbit and remains there. The limit cycle represents the long term behavior of the energy's system and serves as a boundary that seperates growing oscillatios from decaying ones, explaining how it can be ever crossed by any trajectory or orbit.

3 Case of $\mu = 0$

In the case when $\mu = 0$, the equation gets turned into

$$\ddot{x} + x = 0$$

This represents the most simplest form of the Van der Pol equation, reducing it to a linear system that exhibits a classical harmonic motion. This value of μ serves as a foundation to understanding the concept of nonlinearity.

3.1 Harmonic Oscillator

In this case, when the value of μ is 0, the VdP equation becomes a simple harmonic oscillator. How do we know this? We can assume a general solution for a linear differential equation to be:

$$x(t) = e^{rt}$$

where r is the variable to be determined, we can assume this since exponentials have the property that their derivatives are proportional to themselves, making differentials easy. It is also natural for people to use this equation as a solution for linear differential equations in mathematics. Now plugging it back into the equation

$$\ddot{x} + x = 0 \quad \Longrightarrow \quad r^2 e^{rt} + e^{rt} = 0$$

Factor out the e^{rt}:

$$e^{rt}(r^2 + 1) = 0$$

Since $e^{rt} \neq 0$, we set the remaining factor to zero:

$$r^2 + 1 = 0 \quad \Longrightarrow \quad r = \pm i$$

Since the characteristic equation gave complex roots $r = \pm i$, the general solution is written as:

$$x(t) = C_1 e^{it} + C_2 e^{-it}$$

Using Euler's formula, we rewrite the solution in revisioned form:

$$e^{it} = \cos t + i \sin t, \quad e^{-it} = \cos t - i \sin t,$$

$$\downarrow$$

$$x(t) = C_1(\cos t + i \sin t) + C_2(\cos t - i \sin t).$$

Grouping real and imaginary terms:

$$x(t) = (C_1 + C_2)\cos t + i(C_1 - C_2)\sin t.$$

Letting $A = C_1 + C_2$ and $B = i(C_1 - C_2)$, we rewrite it as:

$$x(t) = A\cos t + B\sin t.$$

Which is the general form of a harmonic oscillator. For more details, see [2].

3.2 Phase Plane

As you can see in the graph, the trajectory becomes a perfect circle when put in a phase plane.

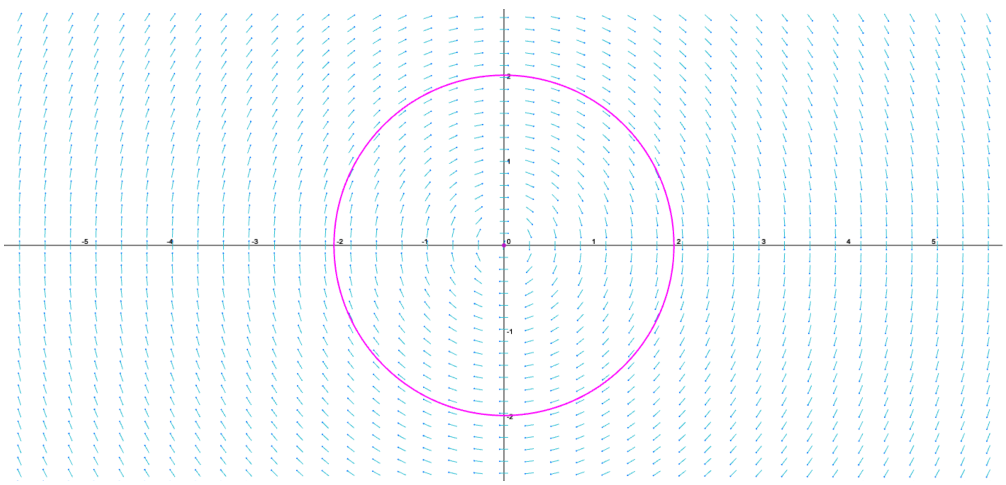

Figure 1: Phase plane when $\mu = 0$ with start point $(x, y) = (2, 0)$

When $\mu = 0$, the equilibrium point at $(0, 0)$ is stable, since trajectories neither converge nor diverge away from it, unlike a limit cycle. Instead, they trace out closed orbits in the phase plane, which indicates that the system exhibits undamped oscillations. This represents a system that oscillates forever without either growing or decaying. The trajectories on the phase plane trace a closed circle, which again indicates that this is in simple harmonic motion.

3.3 Connections

Harmonic oscillation is a fundamental concept that appears in many aspects of science. Harmonic motion occurs when an object experiences a restoring force proportional to the displacement from equilibrium, like a stretched spring bringing an object back to its starting position, or a pendulum swinging towards the lowest point due to gravity. The world we live in is almost entirely filled with nonlinear oscillations because of damping factors, external influences, and forces that do not exert perfect proportionality. Unlike an ideal harmonic oscillation, most oscillations require nonlinear models such as the VdP equation to model them properly. One such example of a perfect harmonic oscillator in our world would be the equation for an ideal LC circuit, which is derived from the damped harmonic oscillator equation for an RLC circuit:

$$L\ddot{Q} + R\dot{Q} + Q/C = E(t)$$

where Q is the charge on the capacitor, L is the inductance of the inductor, R is the resistance in the circuit, C is the capacitance of the capacitor, and $\dot{Q}$ represents the current (I), since current is the time derivative of charge. Since $R\dot{Q}$ is the damping term, when you make $R = 0$, the charge on the capacitor in this RLC circuit will stay in harmonic motion. For more details, see [1].

4 Case of Small μ

In the case of a small μ value, the Van der Pol oscillator shows behaviors that are nearly harmonic, with slight differences due to its weak nonlinearity. Small values of μ, which are typically within the range of $0 < \mu < 1$, show a system with weak damping, where the system seems almost sinusoidal with only small distortions. This behavior in damping is important for modeling systems in real-world situations, such as in electrical circuits and heartbeats, where the dynamics remain similar to harmonic motion. These values are also fundamental for the understanding of the progression of nonlinearity and relaxation oscillations, which we will see as the value of μ increases.

4.1 Limit Cycle

Here is another explanation on why the Van der Pol equation has a limit cycle:
The Poincaré–Bendixson theorem states that if there is a finite region of the plane lying between two closed curves A and B called R, and F is the velocity vector plane or phase plane for the system, and

1. if the system's movement, F, is always pointing inward at the edges of A and B, meaning nothing will escape outside
2. there are no fixed(critical/equilibrium) points

then the system has a closed trajectory lying inside R, or in other words, R contains a limit cycle.
For more details, see [6].
In the Van der Pol oscillator, there also is a finite region R where closed trajectories such as A and B circle a certain region. All of the trajectories typically point inward and towards the trajectories of A and B, helping limit the system's trajectory in a specific area. There is one equilibrium point in the phase plane of the VdP equation, which is at the origin. But since the oscillator's trajectories are not staying fixed at this certain point but rather getting attracted to it, as $t \to \infty$, this equilibrium point creates the limit cycle with the equilibrium point being in the middle. All of these factors in the VdP equation support the statement of the Poincaré–Bendixson theorem, showing again how the VdP equation creates a limit cycle in a phase plane.

4.2 Different Values for μ

In this section, we will explore how the patterns and behavior of the phase plane changes as the value of μ gradually increases. Here are the limit cycles for $\mu = 0.1$, $\mu = 0.5$, and $\mu = 1$ respectively:

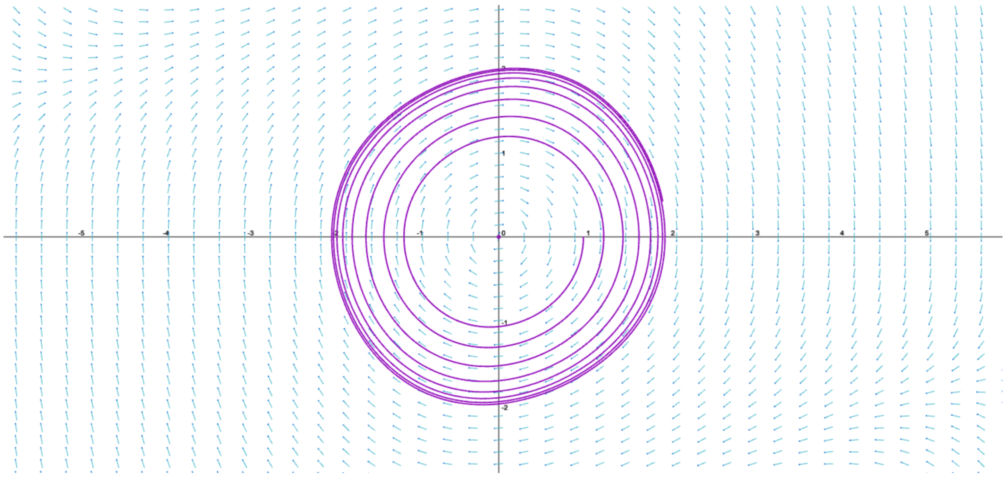

Figure 2: Phase plane when $\mu = 0.1$ with start point $(x, y) = (1, 0)$

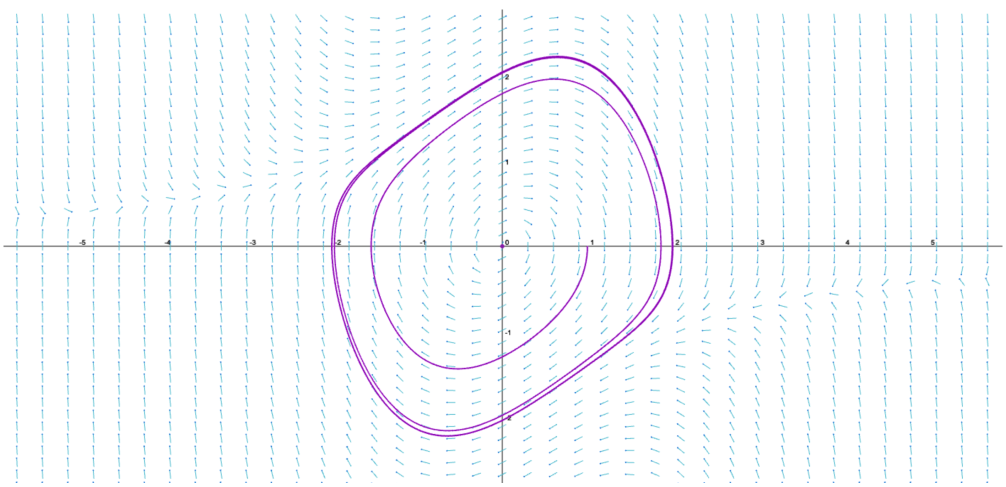

Figure 3: Phase plane when $\mu = 0.5$ with start point $(x, y) = (1, 0)$

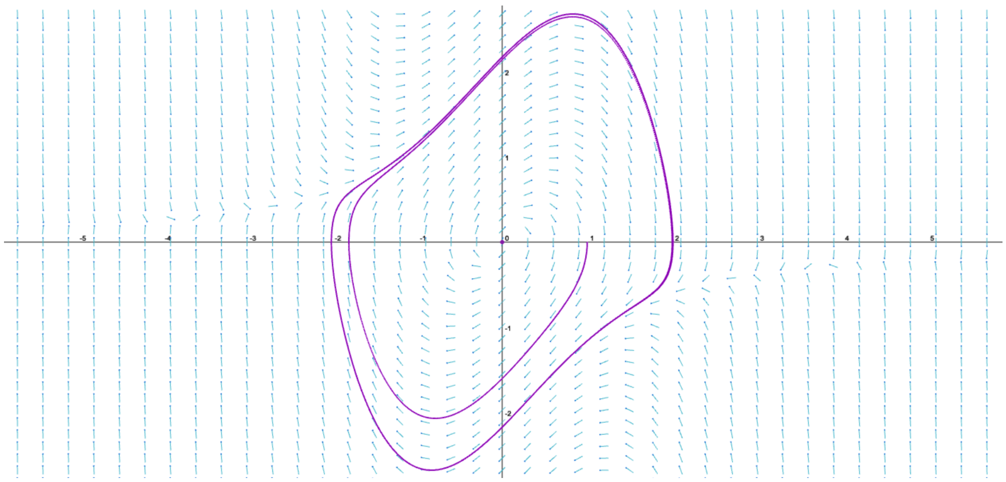

Figure 4: Phase plane when $\mu = 1$ with start point $(x, y) = (1, 0)$

As the value of μ increases, he behavior of the Van der Pol oscillator goes through significant changes, particularly in its limit cycle. When $\mu = 0.1$, the system exhibits a behavior that closely resembles a simple harmonic oscillator, much similar to when $\mu = 0$. As μ increases to 0.5 and 1, however, we can clearly see more visible changes on the limit cycle. The transition of increasing μ highlights how the oscillation goes from a near harmonic oscillator to a more nonlinear one, which is shown on the phase plane as the system looks more stretched horizontally as μ increases. This progression emphasizes the role of the damping term in the VdP equation, where smaller values of μ result in minimal damping, while larger values lead to more significant nonlinear damping effects, as expected from all the analysis we did before.

4.3 Connections

One of the examples of oscillations with small damping in real life would be the oscillations of breathing. For more details, see [7]. The breathing oscillation model includes states of inhaling and exhaling and the frequency of those states. A usual and common model of the breathing oscillation of someone at rest would mostly be depicted as a harmonic oscillator for simplicity and approximation, but as I said before, almost nothing in the real world is made up of simple harmonic oscillations due to external forces. Therefore, meanwhile almost harmonic, the breathing oscillation of someone at rest will have small damping in the oscillation, such as when the μ is small in the Van der Pol equation.
Another example of small damping in an oscillation would be similar to the example discussed in the "$\mu = 0$" part. Unlike the example above, when R is small, E(t) would look like a sinusoidal function with a little distortion due to the damping term R being small.

5 Case of Large μ

In this part of the analysis, we will see the changes in the behavior of the oscillator as the value of μ becomes larger. For large values of μ, the damping effect becomes more visible, causing more distortions and changes to the limit cycle. In this section, we will focus on how the limit cycle and the dynamics of the oscillator will evolve as μ increases to larger values.

5.1 Relaxation Oscillator

Van der Pol introduced the term "relaxation oscillator" to describe the nonlinear oscillations made by self-sustaining oscillating systems such as the VdP oscillator. For further insight, look at [8], [3], and [10] for reference. As the value of μ becomes large, the equation turns into a relaxation oscillator, which is characterized by its sharp drop and "fast to slow" behavior. The term "relaxation" actually comes from the behavior of these relaxation oscillators. In these types of oscillators, the system shows small changes for a period, but suddenly has a fast change; therefore, "relaxing."

5.2 Limit Cycle

The Poincaré–Bendixson theorem is a good way to prove that there is a limit cycle in the case of a small μ, but the existence of a limit cycle in large μs cannot be proven with the theorem. This is because in order for the Poincaré–Bendixson theorem to work, there must be a clear trapping region, which is a closed region where trajectories never enter or leave. In this region, the trajectories should eventually spiral inward or converge towards a limit cycle. However, when the μ is large, the phase plane gets more distorted, getting stretched horizontally more as μ gets bigger. This means the trajectories within the limit cycle do not exhibit a clear pattern of spiraling inward, but rather look horizontal and sometimes move outward. This can be shown in the "Different Values for μ" section and these examples:

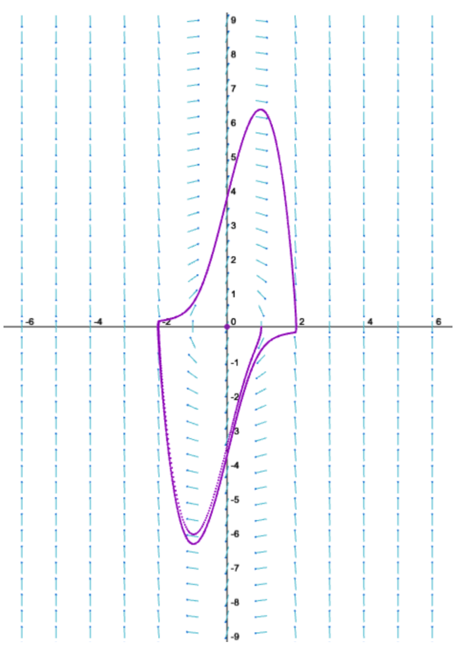

Figure 5: Phase plane when $\mu = 4$ with start point $(x, y) = (1, 0)$

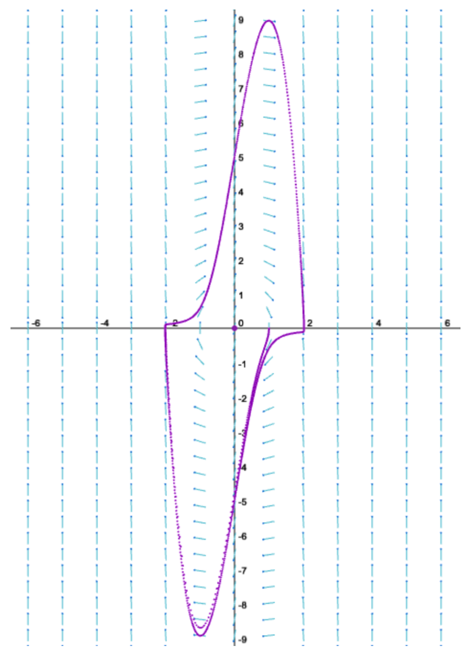

Figure 6: Phase plane when $\mu = 6$ with start point $(x, y) = (1, 0)$

In these examples, when the value of μ is large, we can see that the trajectories inside the limit cycle seem random and horizontal rather than an inward spiral, supporting the point that the Poincaré–Bendixson theorem does not work with large μs.

So, how else do we prove the existence of limit cycles in the case of a large μ? We can use the Levinson-Smith theorem which applies to second-order nonlinear differential equations. The Levinson-Smith Theorem states that when:

1. $f(x)$ is even and continuous
2. $g(x)$ is odd, $g(x) > 0$ if $x > 0$, and $g(x)$ is continuous for all x
3. $G(x) \to \infty$ as $x \to \infty$ where $G(x) = \int_0^x g(t)\,dt$
4. for some $k > 0$, we have

$$F(x) = \begin{cases} < 0, & \text{for } 0 < x < k, \\ > 0 \text{ and increasing}, & \text{for } x > k, \\ \to \infty, & \text{as } x \to \infty, \end{cases} \quad \text{where } F(x) = \int_0^x f(t)\,dt \tag{6}$$

Then, the nonlinear differential system has:

1. a unique critical point in the origin
2. a unique non-zero closed trajectory A, which is a stable limit cycle around the origin
3. all other non-zero trajectories spiralling towards A as $t \to \infty$

For more details, see [4]. The Van der Pol equation satisfies all of the conditions above since the standard form of the equation has a function $f(x) = -\mu(1 - x^2)$ which is even and continuous within the real plane since it is a quadratic function. Also, $G(x) = \int_0^x t\,dt = x^2/2$ which goes to $+\infty$ as x goes to $\pm\infty$. Lastly, $F(x) = \int_0^x -\mu(1 - t^2)\,dt = -\mu \int_0^x 1 - t^2\,dt = -\mu(x - x^3/3)$ which is negative when $x \in (0, \sqrt{3})$ and positive when $x \in (\sqrt{3}, +\infty)$ meaning that k would be $\sqrt{3}$. Also note that the value of μ cannot be negative since μ is the coefficient of nonlinear damping. Which makes $F(x) \to \infty$ as $x \to \infty$. This theorem proves that the VdP equation has a limit cycle in the case when the μ is large.

5.3 Connections

In the real world, the case of large values of μ is the most common because of external forces. For a system to have a non-large μ value, it has to be similar to sinusoidal, which is very unlikely to happen naturally. Some real-life connections would be ordinary things, such as the height of a wave at a beach, or a frequency of a heartbeat of a person. These patterns are unpredictable and inconsistent yet repeating, making them highly damped, but still an oscillation. These things would have a high nonlinear damping factor, causing them to exhibit more distorted and stretched limit cycles on a phase plane.

6 Conclusion

In this paper, we analyzed the Van der Pol oscillator and its dynamics, focusing on its behavior for different values of the parameter μ, which represents damping. We found that as μ increases, the limit cycle gets more distorted in the phase plane, but it will still have a limit cycle as long as it is bigger than 0. This behavior is significant in real-world systems where damping is nonlinear, such as the things we discussed throughout the paper. For further insights, readers may refer to the original Van der Pol paper, or [9].

Acknowledgements

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