below

ON THE NEED OF IMAGINARY NUMBERS (i) IN QUANTUM MECHANICS

(ANNIHILATION HIDDEN INTO MATHEMATICS)

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Abstract: *The need we have (with the Schrodinger's Equation) of running into the imaginary unit (i) is a simple way the nature uses to inform us that a particle, with its wave function Ψ, is half a reality and there exists a complex conjugate (-i) wave function (antiparticle) Ψ* that, together with Ψ, leads us, as to say, to a real "trigonometric"wave (with real values).*

We know from Euler that: $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ (or also: $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$) (1)

(see Appendix 1).

If now we consider the wave phenomena, which obviously have $(\vec{k}\cdot\vec{x} - \omega t)$ as argument, then the (1) becomes:

$$\cos(\vec{k}\cdot\vec{x} - \omega t) = \frac{e^{i(\vec{k}\cdot\vec{x}-\omega t)} + e^{-i(\vec{k}\cdot\vec{x}-\omega t)}}{2}. \tag{2}$$

Now, by considering the wave function of a free electron, we will have:

$$\Psi = A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)} = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E_k}{\hbar}t)} = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{p^2}{2m_0\hbar}t)} \tag{3}$$

and this one will obviously satisfy the Schrodinger's Equation:

(this is proved in the Appendix 2, Equations (A2.14)-------(A2.16))

$$\Delta\Psi + \frac{2m_0}{\hbar^2}(H - V)\Psi = 0 \qquad (E_k\text{=H-V}) \tag{4}$$

then the (2) tells us that the summation of two complex conjugate wave functions yields a trigonometric function (cos or sin).

Now, let's carry out some considerations:

1) The wave $\Psi = A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ is a solution of the Schrodinger's Equation, but if we combine it with its complex conjugate $\Psi^* = A\cdot e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$, then we get a $\Psi_{Trig} = \cos(\vec{k}\cdot\vec{x} - \omega t)$, BUT THIS IS **NOT** A SOLUTION FOR THE SCHRODINGER'S EQ.

2) With reference to the Appendix 3, the Electromagnetic Dirac Equation (A3.2):

$$(\hbar\frac{\partial}{\partial t} + \hbar c\vec{\alpha}\cdot\vec{\nabla} + i\beta m_0 c^2)\Psi = iq(c\vec{\alpha}\cdot\vec{A} - V)\Psi \tag{5}$$

leads us to conclude that the complex conjugate of Ψ (wave function of a **particle**) , that is Ψ* , is the wave function of the **antiparticle**, with its relevant conjugate Electromagnetic Dirac Equation (A3.5):

$$(\hbar\frac{\partial}{\partial t} + \hbar c\vec{\alpha}\cdot\vec{\nabla} + i\beta m_0 c^2)\Psi^* = -iq(c\vec{\alpha}\cdot\vec{A} - V)\Psi^* \tag{6}$$

where the electric charge q is changed into –q and the Ψ is turned into Ψ*.

3) But although $\Psi_{Trig} = \cos(\vec{k}\cdot\vec{x} - \omega t)$ is NOT a solution for the Schrodinger's Equation, actually, conversely it IS a solution for the d'Alembert Wave Equation:

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \qquad (7)$$

(this is proved in the Appendix 2, equations (A2.7)---------(A2.13))

4) And, by chance, a particle Ψ and its antiparticle Ψ^* , in case they meet, they annihilate, so disappearing and leaving two photons like Ψ_{Trig} ("true" waves, trigonometric ones, not imaginary with the i), which, as photons, satisfy the d'Alembert's Equation (7), indeed!

5) So the whole universe comes from a large creation of pairs particle-antiparticle that, due to the Heisenberg Uncertainty Principle, have to disappear by reannihilation, with an obvious mutual reattraction (in order not to violate the Energy Conservation Principle) and such a reattraction determines the existence of the (fundamental) Coulomb's Force (see Appendix 4) and among all these appearing/disappearing pairs, it also determines the global mutual reattraction of all the matter, in terms of Gravitation (see Appendix 5).

((The Ψ^* (with the (-i)) as well could be accepted by the Schrodinger's Equation, provided that it is renormalized to (+i), that is the (limited) direction under which the Schrodinger's Equation (which is limited, in its classical/non reletivistic energy approximation) has been conceived, indeed.))

APPENDIX 1: THE EULER'S FORMULAS

We have an f(x) and we suppose it equals, in a certain neighbourhood of x_0 , a generic series: $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$; now, if we also suppose that x_0 is the origin: (x_0=0) , $\rightarrow$ $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Now, as: $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$, it follows that:

$f(0) = a_0 + \sum_{n=1}^{\infty} a_n 0^n = a_0$ and then also: $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$ and:

$f'(0) = a_1 + \sum_{n=2}^{\infty} n a_n 0^{n-1} = a_1 + 0 = a_1$ and:

$$f''(x) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2(2-1) a_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2}$$

so: $f''(0) = 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n 0^{n-2} = 2a_2 + 0 = 2a_2 \quad \rightarrow \quad a_2 = \frac{1}{2} f''(0)$ and so on, so getting: $a_n = \frac{1}{n!} f^{(n)}(0)$ and finally: $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$, which is the Taylor Series Expansion (a MacLaurin's one, in this case). Now, by using it on the function $f(x) = e^x$, we have:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ... + \frac{1}{n!}x^n .$$

By considering the following three Taylor series expansions:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + ... + \frac{1}{n!}x^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 - \frac{1}{2} x + ... + \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = x - \frac{1}{6} x^3 + ... + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

if in the first one we put ix in place of x, we'll have:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot (ix)^n = 1 + ix + i^2 \frac{1}{2} x^2 + i^3 \frac{1}{6} x^3 + ... + \frac{1}{n!} i^n x^n = 1 + ix - \frac{1}{2} x^2 + ii^2 \frac{1}{6} x^3 + ... + \frac{1}{n!} i^n x^n =$$

$= 1 + ix - \frac{1}{2} x^2 - i \frac{1}{6} x^3 + ... + \frac{1}{n!} i^n x^n$, while from the third expansion we have:

$i \sin x = \sum_{n=0}^{\infty} \frac{i}{n!} f^n(0) \cdot x^n = ix - i \frac{1}{6} x^3 + ... + \frac{i(-1)^n}{(2n+1)!} x^{2n+1}$, so we understand that about e^{ix} :

$e^{ix} = 1 + ix - \frac{1}{2} x^2 - i \frac{1}{6} x^3 + ... + \frac{1}{n!} i^n x^n = \cos x + i \sin x$; similarly: $e^{-ix} = \cos x - i \sin x$, so, by using both of them, we get the following Euler's Formulas:

$\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. If in the $e^{ix} = \cos x + i \sin x$ we put x=π, we get: $e^{i\pi} + 1 = 0$.

Moreover:

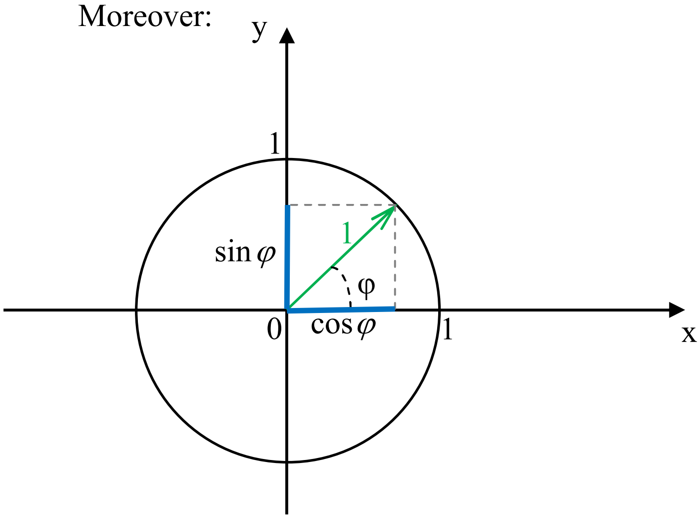


Fig. 1: The unit circle.

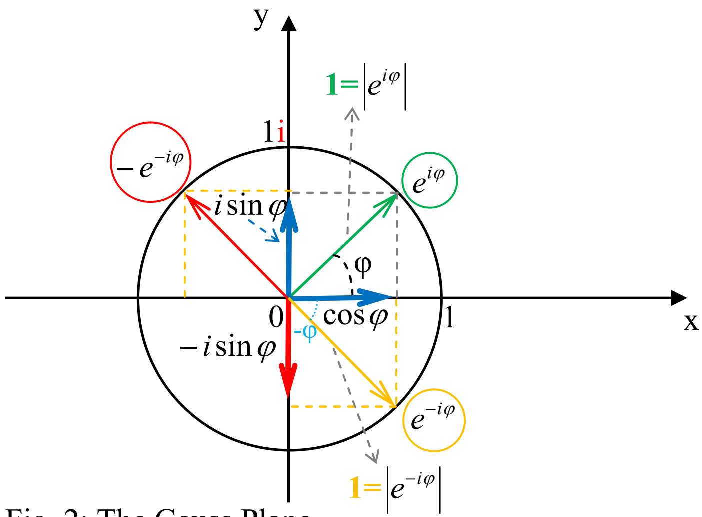


Fig. 2: The Gauss Plane.

by preliminarily reminding the Euler's Formulas just proved: $e^{i\varphi} = \cos\varphi + i \sin\varphi$ and $e^{-i\varphi} = \cos\varphi - i \sin\varphi$ and by considering such two equations as vector equations, then the former (for example) would look like that: $\vec{v} = \vec{v}_x + i\vec{v}_y$ (where $\vec{v}_x = (\cos\varphi)\hat{x}$ and $\vec{v}_y = (\sin\varphi)\hat{y}$), so:

$\boxed{\vec{v} = (\cos\varphi)\hat{x} + i(\sin\varphi)\hat{y} \,,}$ where $|\vec{v}| = 1 = |e^{i\varphi}|$, as its components are cosφ and sinφ .

Regarding the latter: $\vec{v} = \vec{v}_x - i\vec{v}_y$ (where $\vec{v}_x = (\cos\varphi)\hat{x}$ and $\vec{v}_y = (\sin\varphi)\hat{y}$), so:

$\boxed{\vec{v} = (\cos\varphi)\hat{x} - i(\sin\varphi)\hat{y} \,,}$ where $|\vec{v}| = 1 = |e^{-i\varphi}|$, as its components are cosφ and sinφ

[sin(-φ)=-sinφ].

Not only, but the two vectors, the green and the orange ones $e^{i\varphi}$ and $e^{-i\varphi}$ (in Fig. 2) give

2cosφ on the horizontal axis ($e^{i\varphi}+e^{-i\varphi}=2\cos\varphi$) , so we get the: $\boxed{\cos\varphi=\frac{e^{i\varphi}+e^{-i\varphi}}{2}}$ of Euler, just proved.

Finally, the two vectors, the green and the red ones $e^{i\varphi}$ and $-e^{-i\varphi}$ give i2sinφ on the vertical axis ($e^{i\varphi}+[-e^{-i\varphi}]=e^{i\varphi}-e^{-i\varphi}=i2\sin\varphi$) , so we get the: $\boxed{\sin\varphi=\frac{e^{i\varphi}-e^{-i\varphi}}{2i}}$ of Euler, just proved.

APPENDIX 2: THE WAVE FUNCTIONS

We know from the relativity that the total energy E is:

$$E^2=p^2c^2+m_0^2c^4 \tag{A2.1}$$

This is the most general formula we have for the energy and is suitable for a relativistic particle indeed.

Now, for a photon (a particle whose rest mass is equal to zero), we have: $E^2=p^2c^2$, and:

$$E=pc \tag{A2.2}$$

For a non relativistic particle, we know its kinetic energy is: $E_k=\frac{1}{2}m_0v^2$, but this is hidden in the (A2.1), which is more general, indeed. In fact, the (A2.1) can be rewritten in this way:

$$E=m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2} \tag{A2.3}$$

and for the developments of Taylor, we have:

$f(x)=\sqrt{1+x}=(1+x)^{1/2}\approx 1+\frac{1}{2}x$, from this, for the (A2.3):

$E=m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2}\approx m_0c^2(1+\frac{p^2}{2m_0^2c^2})=m_0c^2+\frac{p^2}{2m_0}$ and, for the kinetic energy, we have: $E_k=E-m_0c^2=\frac{p^2}{2m_0}=\frac{1}{2}m_0v^2$ qed.

Now, let's take the general expression for a wave:

$$\Psi=A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}=A\cdot e^{i(\frac{2\pi}{\lambda}\hat{k}\cdot\vec{x}-\frac{2\pi}{\lambda}vt)}, \tag{A2.4}$$

as: $\vec{k}=\frac{2\pi}{\lambda}\hat{k}$, $\omega=\frac{2\pi}{T}=2\pi f=2\pi\frac{v}{\lambda}$.

Such a wave simultaneously propagates in space (x) and oscillates in time t; in fact, if we fix t=0, we see we have an oscillation along x ($\Psi=A\cdot e^{i(\vec{k}\cdot\vec{x})}$) and if we fix x=0 we have an oscillation in time ($\Psi=A\cdot e^{-i(\omega t)}$).

We also know that:

$$E=hf=\frac{h}{2\pi}2\pi f=\hbar\omega \tag{A2.5}$$

and being the (A2.2) standing, we have:

$pc = \hbar\omega$, from which :

$$p = \hbar\frac{\omega}{c} = \hbar\frac{2\pi}{\lambda} = \hbar k = p \qquad \text{(A2.6)}$$

and the (A2.4) becomes:

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E}{\hbar}t)} \qquad \text{(A2.7)}$$

By simply putting such a Ψ in the following equations:

$$(i\hbar\frac{\partial}{\partial t})\Psi = E\Psi = (\hbar\omega)\Psi \qquad \text{(A2.8)}$$

$$(\frac{\hbar}{i}\vec{\nabla})\Psi = \vec{p}\Psi = (\hbar\vec{k})\Psi \;; \qquad \text{(A2.9)}$$

we have that they give identities, sot they are correct.

In one dimension: $(\frac{\hbar}{i}\frac{\partial}{\partial x})\Psi = p\Psi = (\hbar k)\Psi$; (∇ gradient)

So, we can deduce the following operatorial identities:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad \text{(A2.10)}$$

$$\vec{p} \rightarrow \frac{\hbar}{i}\vec{\nabla} \qquad \text{(A2.11)}$$

As the (A2.2) stands: $E^2 = p^2c^2$, we have:

$$(i\hbar\frac{\partial}{\partial t})^2\Psi = c^2(\frac{\hbar}{i}\vec{\nabla})^2\Psi \,, \qquad \text{(A2.12)}$$

that is:

$$\boxed{\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0} \qquad \text{(A2.13)}$$

or also ($\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, laplacian, divergence of a gradient): $\Delta\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$,

which is the **d'Alembert Wave Equation**.

Please notice such an equation, derived in a '' relativistic'' environment (photon, i.e. a particle propagating by speed c and with a zero rest mass) is invariant under a Lorentz Transformation.

If now we consider non relativistic particles (atoms are like that, ordinarily), we will get a non relativistic ''wave'' equation, which is the Schrodinger Equation. In fact, if in the (A2.7) we no longer consider $E = pc$, but $E_k = \frac{1}{2}m_0v^2$ (a non relativistic equation, indeed), we get:

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E_k}{\hbar}t)} = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{p^2}{2m_0\hbar}t)} \qquad \text{(A2.14)}$$

and as well as we got the (A2.12), by a direct use of the (A2.14) in the following equation:

$$(i\hbar\frac{\partial}{\partial t})\Psi = (-\frac{\hbar^2}{2m_0}\nabla^2)\Psi \qquad \text{(A2.15)}$$

($(i\hbar\frac{\partial}{\partial t})\Psi = (-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2})\Psi$, in one dimension)

we get an identity. Therefore, the (A2.15) is true. Please notice that in the (A2.14) we have no longer used a total E, but just an E_k, and we are going to take that into account.

The left side of the (A2.15) is $(i\hbar\frac{\partial}{\partial t})\Psi = E_k\Psi$, but we know that E_k=H-V, so, still in force of the (A2.15): $-\frac{\hbar^2}{2m_0}\Delta\Psi = (H-V)\Psi$, that is:

$$\boxed{\Delta\Psi + \frac{2m_0}{\hbar^2}(H-V)\Psi = 0} \tag{A2.16}$$

which is the **Schrodinger Equation**.

Let' get into a more general situation, where we have a relativistic particle with a rest mass not equal to zero.

As well as we did before, as for the (A2.1) we have: $E = \sqrt{p^2c^2 + m_0^2c^4}$, then, by using such an E still in the (A2.7) $\Psi = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E}{\hbar}t)}$, we will have:

$$\Psi = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{\sqrt{p^2c^2+m_0^2c^4}}{\hbar}t)} \tag{A2.17}$$

and, as usual, still by introduction of an equation into another, we see that such a Ψ is a solution for the following:

$$\boxed{(\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}) - \frac{m_0^2c^2}{\hbar^2}\Psi = 0} \tag{A2.18}$$

which is nothing but the **Klein-Gordon Equation** and it is similar to that of d'Alembert, but has an item more.

Let's really carry out the introduction of the (A2.17) in the (A2.18), to see that all this really stands.

We have: $\nabla^2\Psi = (i)^2\frac{p^2}{\hbar^2}\Psi = -\frac{p^2}{\hbar^2}\Psi$ and

$-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -\frac{1}{c^2}(-i)^2\frac{E^2}{\hbar^2}\Psi = \frac{1}{c^2\hbar^2}(p^2c^2 + m_0^2c^4)\Psi$ and so:

$-\frac{p^2}{\hbar^2}\Psi + \frac{1}{c^2\hbar^2}(p^2c^2 + m_0^2c^4)\Psi - \frac{m_0^2c^2}{\hbar^2}\Psi = 0$, that is 0=0.

Let's set $l = \frac{m_0c}{\hbar}$; such an l is dimensionally like the wave vector k. By such an l, we have that the (A2.17) and the (A2.18) can be rewritten as follows:

$$\boxed{\Psi = A\cdot e^{i(\vec{k}\cdot\vec{x} - \sqrt{(k^2+l^2)}ct)} = A\cdot e^{i(\vec{k}\cdot\vec{x} - \omega' t)}} \tag{A2.19}$$

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - l^2\Psi = 0 \qquad \text{(A2.20)}$$

where $\boxed{\omega' = \sqrt{(k^2+l^2)}c}$.

Let's rewrite the Klein-Gordon Equation (A2.20) in the following way:

$$-\frac{\partial^2\Psi}{\partial t^2} + c^2\nabla^2\Psi - l^2c^2\Psi = 0 \qquad \text{(A2.21)}$$

and after taking into account that $i^2 = -1$ and $(a-b)(a+b) = a^2 - b^2$, we have that such an equation can be rewritten like this:

$$\{[i\frac{\partial}{\partial t} - (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})][i\frac{\partial}{\partial t} + (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\}\Psi = 0\,, \qquad \text{(A2.22)}$$

or also:

$$[i\frac{\partial}{\partial t} - (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\Psi = 0$$

$$[i\frac{\partial}{\partial t} + (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\Psi = 0 \qquad \text{(A2.23)}$$

and the (A2.22) can be developed as:

$$[-\frac{\partial^2}{\partial t^2} + c^2(\vec{\alpha}\cdot\vec{\nabla})^2 + i\frac{m_0c^3}{\hbar}\beta(\vec{\alpha}\cdot\vec{\nabla}) + i\frac{m_0c^3}{\hbar}(\vec{\alpha}\cdot\vec{\nabla})\beta - \beta^2\frac{m_0^2c^4}{\hbar^2}]\Psi = 0 \qquad \text{(A2.24)}$$

This equation is equal to the (A2.21) if:

$\beta^2 = 1$, $\alpha\beta + \beta\alpha = 0$, $\alpha_i\alpha_j = 1^2$ if i=j and $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ if $i \neq j$

The last two conditions on alphas make us have only ∇^2 and not mixed terms in $\vec{\nabla}$. The (A2.23), here reported:

$$\boxed{(i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0} \qquad \text{(A2.25)}$$

can be considered as the **Dirac Equation**, which is usually provided in the following form, in natural units ($\hbar = c = 1, \beta = 1$):

$$\boxed{(i\gamma^\mu\partial_\mu - m_0)]\Psi = 0}\,, \qquad \text{(A2.26)}$$

where $i\gamma^\mu\partial_\mu = i\gamma^\mu \partial/\partial x^\mu$, which contains a summation under the Einstein convention, gives, under the values of µ, the derivative under the time $\frac{\partial}{\partial t}$ and under x, y and z of $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$:

$$i\gamma^\mu\partial_\mu \to i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla}\ .$$

At last, from the basic Dirac Equation (A2.25), that is:

$(i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0$, we can get to the **Electromagnetic Dirac Equation**:

$$\boxed{[(i\hbar\frac{\partial}{\partial t}-qV)-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}-q\vec{A})-\beta m_0c^2]\Psi=0} \qquad \text{(A2.27)}$$

by simply considering that the introduction of the electromagnetic phenomenon determines algebraic additions such as the energy component (qV) to the energy operator:

$$E\rightarrow i\hbar\frac{\partial}{\partial t} \qquad \rightarrow(i\hbar\frac{\partial}{\partial t}-qV)$$

and the momentum component ($q\vec{A}$) to the momentum operator $\vec{p}$, indeed:

$$\vec{p}\rightarrow\frac{\hbar}{i}\vec{\nabla} \qquad \rightarrow(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}).$$

All the details in the Appendix 6.

APPENDIX 3: THE ANTIPARTICLES

Let's consider the Electromagnetic Dirac Equation (A2.27):

$$[(i\hbar\frac{\partial}{\partial t}-qV)-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}-q\vec{A})-\beta m_0c^2]\Psi=0 \qquad \text{(A3.1)}$$

(which is treated deeply enough at the above mentioned link), together with the Dirac Hamiltonian: $H_D=qV+c\vec{\alpha}(\vec{p}-q\vec{A})+\beta m_0c^2$, as the (A3.1) can be rewritten like that:

$[(H_D-qV)-c\vec{\alpha}\cdot(\vec{p}-q\vec{A})-\beta m_0c^2]\Psi=0$.

Well, let's consider such an Electromagnetic Dirac Equation (A3.1) and let's write it again in this way:

$$(\hbar\frac{\partial}{\partial t}+\hbar c\vec{\alpha}\cdot\vec{\nabla}+i\beta m_0c^2)\Psi=iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi\ . \qquad \text{(A3.2)}$$

And, with reference to Appendix 2, we also remind of the conditions which must be respected by all the α and β: $\beta^2=1$, $\alpha_i\beta+\beta\alpha_i=0$, $\alpha_i\alpha_i=1^2$, $\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}$ (i,j=1,2,3) and in order to have just α coefficients, we name: $\beta=\alpha_4$, so, in a more concise style, the conditions become: $\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}$ (i,j=1,2,3,4) . Now, maybe numbers which satisfy such conditions do not exist, but there surely exist matrixes which can do that, such as:

$$\sigma_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \qquad \text{(A3.3)}$$

$$\rho_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},\quad \rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \qquad \text{(A3.4)}$$

where: $\boxed{\alpha_1=\rho_1\sigma_1\ ,\quad \alpha_2=\rho_3\ ,\quad \alpha_3=\rho_1\sigma_3\ ,\quad \alpha_4=\rho_1\sigma_2}$; in fact, we just need to carry out all the matrix products, such as:

$$\alpha_1=\rho_1\sigma_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$ and check: $\rightarrow$

$$\rightarrow\ \alpha_1\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\rightarrow|\alpha_1\alpha_1|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1$$, then:

$$\alpha_2=\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\ \alpha_2\alpha_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ and:

$$|\alpha_2\alpha_2|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1$$, then: $2=\alpha_1\alpha_1+\alpha_1\alpha_1=2\delta_{11}=2=\alpha_2\alpha_2+\alpha_2\alpha_2=2\delta_{22}=2$.

Then: $$\alpha_3=\rho_1\sigma_3=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$

$$\alpha_3\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ and: $$|\alpha_3\alpha_3|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

then: $2=\alpha_3\alpha_3+\alpha_3\alpha_3=2\delta_{33}=2$, then:

$$\alpha_4=\beta=\rho_1\sigma_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},$$

$$\alpha_4\alpha_4=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ and: $$|\alpha_4\alpha_4|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

then: $\alpha_4\alpha_4+\alpha_4\alpha_4=2\delta_{44}=2$.

Then: $\alpha_i\beta+\beta\alpha_i=(\alpha_i\alpha_4+\alpha_4\alpha_i)=0$, in fact:

$$\alpha_1\alpha_4+\alpha_4\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=$$

$$=\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}+\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ ;$$

$$\alpha_2\alpha_4+\alpha_4\alpha_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ .$$

$$\alpha_3\alpha_4+\alpha_4\alpha_3=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}+\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ .$$

Now we notice that α_4 is the only imaginary matrix, while α_1 , α_2 and α_3 are real; well, after that, if we take the complex conjugate of the (A3.2), we get the following: (turn all the i in –i and all the Ψ into Ψ*)

$$(\hbar\frac{\partial}{\partial t}+\hbar c\vec{\alpha}\cdot\vec{\nabla}+i\beta m_0c^2)\Psi^*=-iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi^* \qquad \text{(A3.5)}$$

and we see that in the last term on the left side of the equation the +i is unchanged, as that i has turned into –i, but simultaneously the β matrix ($\beta=\alpha_4$) contains an imaginary matrix (α_4) and all its i inside turn into –i, (and vice versa), so, in the end, we have no final change in that term.

Well, the (A3.5) has the same look of the (A3.2), but the sign of the charge has changed (q→-q); **from this the prediction of the existence of the positron (antimatter).**

APPENDIX 4: COULOMB FROM HEISENBERG

Here is the Heisenberg Uncertainty Principle:

$$\boxed{\begin{array}{l}\Delta p \cdot \Delta x \geq \hbar/2 \\ \Delta E \cdot \Delta t \geq \hbar/2\end{array}}$$

As it absolutely rules the appearing (pre annihilation) of a pair particle (+) / antiparticle (-), like positron/electron, it follows that when the two particles appear, then they part (in a time Δt) with velocity c of light and up to reaching the mutual distance Δx=r=cΔt and we also know from the physics that the force F is: F=Δp/Δt, so:

$$\Delta p \cdot \Delta x \approx \hbar/2 = h/4\pi \approx F\Delta t \cdot r = F\frac{r}{c} \cdot r = F\frac{r^2}{c} \rightarrow$$

$$\boxed{\rightarrow F = (\frac{hc}{4\pi})\frac{1}{r^2} \propto \frac{1}{r^2}}$$

which is exactly the relation between F and $1/r^2$ in the Coulomb's Law (*):

$$\boxed{F = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r^2} \propto \frac{1}{r^2}} \quad .$$

(*): those two constants are not identical, that is: $\frac{e^2}{4\pi\varepsilon_0} \neq \frac{hc}{4\pi}$, but rather:

$\frac{e^2}{4\pi\varepsilon_0} = (2\alpha)\frac{hc}{4\pi} \cong (\frac{2}{137})\frac{hc}{4\pi}$, where α is the Fine Structure Constant; all this (the additional coefficient 2α) stands because the Heisenberg Uncertainty Principle is shaped over an atom and not over a particle/antiparticle pair; in fact, as long as a particle and its own antiparticle, while showing up, also part at a distance r by travelling the distance of r/2 each from the centre of mass, then, conversely, in case of an H atom, the proton is so massive, with respect to the electron ($m_p=1836m_e$), that in practice it doesn't move at all and stays in the centre of mass of the system, so travelling for a length of 0r, while all the distance r is run by the light electron (1r), from which we get an average value of (0r+1r)/2=r/2 and the 2 is so justified.

At last, in order to justify the 137, we just notice that, still due to the difference in mass between the proton and the electron, the parting velocity is no longer c, but rather that (the only one, as the proton is not moving and stays in the centre of mass) had by the electron around the proton in an H atom, that is (obviously) c/137.

APPENDIX 5: NEWTON FROM HEISENBERG

In Appendix 4 we showed that Coulomb comes from Heisenberg. Now, here we show that Newton comes from Coulomb, so, finally, because of the transitive property, also Newton will come from Heisenberg.

Proof:

A PROPOSAL OF A UNIVERSE WHICH SHOWS THAT COULOMB IS INTO NEWTON, INDEED.

Here is such a universe:

$$\boxed{\begin{aligned} M_{Univ} &= 1{,}59486 \cdot 10^{55} kg \\ R_{Univ} &= 1{,}17908 \cdot 10^{28} m \\ T_{Univ} &= \frac{2\pi R_{Univ}}{c} = 2{,}47118 \cdot 10^{20} s \end{aligned}}$$

In such a universe we have that the mass is more than that visible (as well as stated nowadays) and the density is the observed one:

$$\rho = M_{Univ} / (\frac{4}{3}\pi \cdot R_{Univ}^3) = 2{,}32273 \cdot 10^{-30} kg / m^3$$

The composition of all the small electric interactions in the universe builds up the universe itself; in fact, as chance would have it:

$$\boxed{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_e} = G\frac{m_e M_{Univ}}{R_{Univ}}}$$ (UNIFICATION GRAVITATION - ELECTROMAGNETISM)

so Newton is into Coulomb.

Moreover, still by chance:

$$m_e c^2 = G\frac{m_e M_{Univ}}{R_{Univ}} \;,\; G\frac{M_{Univ}}{R_{Univ}^2} = a_{Univ} = G\frac{m_e}{r_e^2} = g_{EL} = 7{,}62 \cdot 10^{-12} m/s^2 \;,\quad h = \frac{2m_e c^2}{T_{Univ}} \text{ (just numerically)}$$

$$R_{Univ} = \sqrt{N} r_e \quad \text{(being } N = M_{Univ} / m_e = 1{,}74 \cdot 10^{85}\text{)}, \qquad T_{Univ}\frac{Gm_e^2}{hr_e} = \frac{1}{137} = \alpha \;,$$

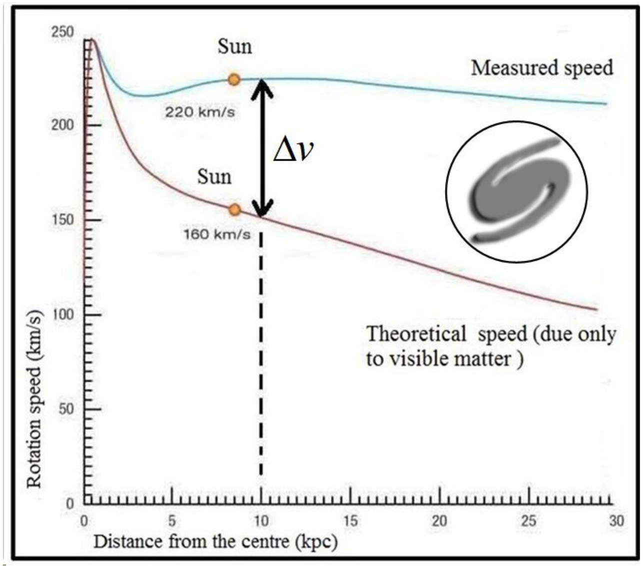
$$T_{CMBR} = (\frac{h}{2m_e\sigma}\frac{M_{Univ}}{4\pi R_{Univ}^2})^{1/4} \cong 2{,}7K$$

(where $\sigma = 5{,}67 \cdot 10^{-8} W / m^2 K^4$ is the Stefan-Boltzmann's Constant). Moreover:

$$T_{CMBR} = (\frac{3c^3}{80\pi\sigma}\frac{M_{Univ}}{R_{Univ}^3})^{1/4} = (\frac{72Gc^{11}h^3\varepsilon_0^4 m_e^6}{\pi^2 e^8 k^4})^{1/4} = 2{,}72846(02218319896)K \cong 2{,}72846K$$

-Then, being, according to Heisenberg: $\Delta p \cdot \Delta x = \hbar / 2$ (with the equality sign, out of simplicity, and $\hbar / 2 = h / 4\pi = 0{,}527 \cdot 10^{-34} J \cdot s$), we can say that:

$$\Delta p \cdot \Delta x = m_e c \cdot \Delta x = m_e c \cdot [\frac{1}{(2\pi)^2} G\frac{M_{Univ}}{R_{Univ}^2}] = 0{,}527 \cdot 10^{-34} \quad \text{(numerically)}.$$



-In our galaxy (the Milky Way) the Sun is at a distance of 8,5kpc from the centre and should have a rotation velocity of 160 km/s, if it was due just to the baryonic matter, that is that coming from all the stars and all the visible matter. But we know that the velocity of the Sun is 220 km/s. So there is a discrepancy Δv of 60 km/s: (Δv=220-160=60 km/s).
(1kpc=1000pc ; 1pc=1 Parsec=$3{,}26\_l.y.=3{,}08\cdot 10^{16}m$; 1 light year l.y.=$9{,}46\cdot 10^{15}m$)
($R_{Gal-Sun}=8{,}5kpc=27{,}71\cdot 10^{3}\_l.y.=2{,}62\cdot 10^{20}m$ is the distance from the Sun of the centre of the Milky Way). If the Sun was at a distance R_{GAL} of 30 kpc, it would have had the same velocity of 220 km/s, but the discrepancy Δv would have been larger. In general, we know from the rotation

curves that: $\Delta v = k\sqrt{R_{Gal-Sun}}$, where k =constant; now, by chance: → $\boxed{k=\sqrt{2G\frac{m_e}{r_e^2}}=\sqrt{\frac{2GM_{Univ}}{R_{Univ}^2}}}$

APPENDIX 6: ON THE ELECTROMAGNETIC DIRAC EQUATION

$$[(i\hbar\frac{\partial}{\partial t}-qV)-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}-q\vec{A})-\beta m_0c^2]\Psi=0$$

or:

$$[(E-qV)-c\vec{\alpha}\cdot(\vec{p}-q\vec{A})-\beta m_0c^2]\Psi=0$$

Electromagnetic Dirac Equation.

-<u>Preamble on the basic Dirac equation</u>:

$$[i\hbar\frac{\partial}{\partial t}-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla})-\beta m_0c^2]\Psi=0 \quad \textbf{or:} \quad [E-c\vec{\alpha}\cdot\vec{p}-\beta m_0c^2]\Psi=0$$

we start our reasoning from the basic Dirac Equation, where we started from the Klein-Gordon

Equation: $-\frac{\partial^2\Psi}{\partial t^2}+c^2\nabla^2\Psi-l^2c^2\Psi=0 \quad (l=\frac{m_0c}{\hbar})$ (A6.1) and by taking into account the well known quantum operators: $E \rightarrow i\hbar\frac{\partial}{\partial t}$ and $\vec{p} \rightarrow \frac{\hbar}{i}\vec{\nabla}$ and after taking into account that $i^2 = -1$ and $(a-b)(a+b) = a^2 - b^2$, we have that such an equation can be rewritten like this:

$$\{[i\frac{\partial}{\partial t} - (ic\vec{\alpha}\cdot\vec{\nabla} - \beta\frac{m_0 c^2}{\hbar})][i\frac{\partial}{\partial t} + (ic\vec{\alpha}\cdot\vec{\nabla} - \beta\frac{m_0 c^2}{\hbar})]\}\Psi = 0, \tag{A6.2}$$

or also like this:

$$[i\frac{\partial}{\partial t} - (ic\vec{\alpha}\cdot\nabla - \beta\frac{m_0 c^2}{\hbar})]\Psi = 0$$

$$[i\frac{\partial}{\partial t} + (ic\vec{\alpha}\cdot\nabla - \beta\frac{m_0 c^2}{\hbar})]\Psi = 0 \tag{A6.3}$$

and the (A6.2) can be developed as:

$$[-\frac{\partial^2}{\partial t^2} + c^2(\vec{\alpha}\cdot\vec{\nabla})^2 + i\frac{m_0 c^3}{\hbar}\beta(\vec{\alpha}\cdot\vec{\nabla}) + i\frac{m_0 c^3}{\hbar}(\vec{\alpha}\cdot\vec{\nabla})\beta - \beta^2\frac{m_0^2 c^4}{\hbar^2}]\Psi = 0 \tag{A6.4}$$

This equation is equal to the (A6.1) if:

$\beta^2 = 1$, $\alpha\beta + \beta\alpha = 0$, $\alpha_i\alpha_j = 1^2$ if i=j and $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ if $i \neq j$

The last two conditions on alphas make us have only ∇^2 and not mixed terms in $\vec{\nabla}$. The (A6.3), here reported:

$$\boxed{(i\frac{\partial}{\partial t} + ic\vec{\alpha}\cdot\vec{\nabla} - \beta\frac{m_0 c^2}{\hbar})\Psi = 0} \tag{A6.5}$$

can be considered as the basic **Dirac Equation**, which is usually provided in the following form, in natural units ($\hbar = c = 1, \beta = 1$):

$$\boxed{(i\gamma^\mu\partial_\mu - m_0)]\Psi = 0}, \tag{A6.6}$$

where $i\gamma^\mu\partial_\mu = i\gamma^\mu \partial/\partial x^\mu$, which contains a summation under the Einstein convention, gives, under the values of μ, the derivative under the time $\frac{\partial}{\partial t}$ and under x, y and z of $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$:

$$i\gamma^\mu\partial_\mu \rightarrow i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla}$$

-<u>Preamble on the electromagnetism</u>:

we know that the electric and magnetic fields can be expressed as a function of the electrodynamic potentials (see App. 8):

$$\vec{B} = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V$$

and we also remind the Lorentz force: $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$. Well, from those equations we get:

$\vec{F} = q[\vec{E} + (\vec{v}\times\vec{B})] = q[-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}V + (\vec{v}\times\vec{\nabla}\times\vec{A})]$; now, as: $\vec{\nabla}V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$, it follows that: $(\vec{\nabla}V)_x = \frac{\partial V}{\partial x}$; moreover, as:

$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$, then it follows that:

$$[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ (\vec{\nabla}\times\vec{A})_x & (\vec{\nabla}\times\vec{A})_y & (\vec{\nabla}\times\vec{A})_z \end{vmatrix}_x = v_y(\vec{\nabla}\times\vec{A})_z - v_z(\vec{\nabla}\times\vec{A})_y =$$

$$= v_y(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) - v_z(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) = v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z} \qquad \text{(A6.7)}$$

then, on a time derivative basis, we have: $\frac{dA_x}{dt} = \frac{\partial A_x}{\partial t} + (\frac{\partial x}{\partial t}\frac{\partial A_x}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial A_x}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial A_x}{\partial z}) =$

$= \frac{\partial A_x}{\partial t} + (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z})$, so: $-\frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} = -(v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z})$ and then, because of the (A6.7), we have:

$[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x = \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - (v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}) = \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$, that is, at last: $[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x = \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}$; finally, about the x component of the Lorentz force, we have:

$$(\vec{F})_x = F_x = q[\vec{E} + (\vec{v}\times\vec{B})]_x = q[(-\frac{\partial\vec{A}}{\partial t} - \vec{\nabla}V) + (\vec{v}\times\vec{\nabla}\times\vec{A})]_x = q[-(\frac{\partial\vec{A}}{\partial t})_x - (\vec{\nabla}V)_x + (\vec{v}\times\vec{\nabla}\times\vec{A})_x] =$$

$$= q\{-\cancel{\frac{\partial A_x}{\partial t}} - \frac{\partial V}{\partial x} + [\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt} + \cancel{\frac{\partial A_x}{\partial t}}]\} = q[-\frac{\partial V}{\partial x} + \frac{\partial}{\partial x}(\vec{v}\cdot\vec{A}) - \frac{dA_x}{dt}] = q[-\frac{\partial}{\partial x}(V - \vec{v}\cdot\vec{A}) - \frac{dA_x}{dt}] =$$

$= q\{-\frac{\partial}{\partial x}(V - \vec{v}\cdot\vec{A}) - \frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})]\}$; in fact:

$$\frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})] = \frac{d}{dt}[\frac{\partial}{\partial v_x}(v_xA_x + v_yA_y + v_zA_z)] = \frac{d}{dt}[\frac{\partial}{\partial v_x}(v_xA_x) + \frac{\partial}{\partial v_x}(v_yA_y) + \frac{\partial}{\partial v_x}(v_zA_z)] =$$

$= \frac{d}{dt}[A_x + 0 + 0] = \frac{dA_x}{dt}$. Therefore, we finally have:

$$F_x = q\{-\frac{\partial}{\partial x}(V - \vec{v}\cdot\vec{A}) - \frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})]\}. \qquad \text{(A6.8)}$$

The last one, that is the (A6.8), indeed, is the same as the following: $F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x}$, where U is obviously an energy: $U = qV - q(\vec{v}\cdot\vec{A})$; so, F_x obviously depends also on v ; moreover, as in the relativity we have the potential 4-vector $\underline{\Phi} = (A_x, A_y, A_z, \frac{V}{c})$ (see App. 9), then it follows that A refers to the space part (S) of the electromagnetic field, while V refers to the time one (t); then, also for the energy U we can say: $U = qV - q(\vec{v}\cdot\vec{A}) = U_t + U_S$.

Now, going back to the (A6.5) and considering the very well known quantum operators:

space: $\vec{p} \to \frac{\hbar}{i}\vec{\nabla}$ and time: $E \to i\hbar\frac{\partial}{\partial t}$

(in fact, also in the quantum mechanics the energy E correlates with time: $\Delta E \cdot \Delta t \geq \hbar/2$, while the impulse correlates with space: $\Delta p \cdot \Delta x \geq \hbar/2$)

and being: $\boxed{U_t = qV}$ and $U_S = -q(\vec{v}\cdot\vec{A})$ and, as a consequence:

$(U_S = -q(\vec{v}\cdot\vec{A}) \to p_{x\_S} = \int F_{x\_S}dt = -\int(\frac{d}{dt}\frac{\partial U}{\partial v_x})dt = -\int d(\frac{\partial U}{\partial v_x}) = -\frac{\partial U}{\partial v_x} \to$

$\to \vec{p}_S = -\frac{\partial U}{\partial \vec{v}} = -\frac{\partial(-q\vec{v}\cdot\vec{A})}{\partial \vec{v}} = q\vec{A}$, that is: $\boxed{\vec{p}_S = q\vec{A}}$), it follows that we have new quantum operators: space: $\vec{p} \to (\frac{\hbar}{i}\vec{\nabla} - q\vec{A})$ and time: $E \to (i\hbar\frac{\partial}{\partial t} - qV)$; so, we were saying that because of the (A6.5) (-i=1/i):

$$(i\frac{\partial}{\partial t} + ic\vec{\alpha}\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0 = [i\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{1}{i}\vec{\nabla}) - \beta\frac{m_0c^2}{\hbar}]\Psi = \hbar\cdot 0 = \hbar[i\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{1}{i}\vec{\nabla}) - \beta\frac{m_0c^2}{\hbar}]\Psi =$$

$= [i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}) - \beta m_0c^2]\Psi = 0$ and with the new operators:

$$\boxed{[(i\hbar\frac{\partial}{\partial t} - qV) - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}) - \beta m_0c^2]\Psi = 0\,.} \qquad \text{(A6.9)}$$

and besides we define the Dirac Hamiltonian: $H_D = qV + c\vec{\alpha}(\vec{p} - q\vec{A}) + \beta m_0c^2$, as the (A6.9) can be rewritten like that: $[(H_D - qV) - c\vec{\alpha}\cdot(\vec{p} - q\vec{A}) - \beta m_0c^2]\Psi = 0$.

Therefore, by putting the (A6.5) and the (A6.9) one over the other, we have:

$$\boxed{[i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}) - \beta m_0c^2]\Psi = [E - c\vec{\alpha}\cdot\vec{p} - \beta m_0c^2]\Psi = 0} \quad \text{basic Dirac Equation}$$

$$\boxed{[(i\hbar\frac{\partial}{\partial t} - qV) - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}) - \beta m_0c^2]\Psi = [(E - qV) - c\vec{\alpha}\cdot(\vec{p} - q\vec{A}) - \beta m_0c^2]\Psi = 0} \quad \text{EM Dirac Eq.}$$

-<u>Prediction of the spin</u>:

now we remind, from quantum mechanics, the eigenvalue equations of the angular momentums: $L_z\Psi = l\Psi$, $L^2\Psi = \lambda\Psi$, $l = m\hbar$, $-l \leq m \leq l$, $\lambda = \hbar^2 l(l+1)$. Well, if we call $\Psi_\alpha(\vec{r})$ the wave function of a particle, then the wave function of a particle with spin is: $\Psi = \Psi_\alpha(\vec{r})\cdot\chi_s$.

For particles with spin 1/2 (s=+1/2 and s=-1/2), we have two χ_s : $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ and $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$; so, as we have two wave functions $|+\rangle$ and $|-\rangle$, which can be written more concisely as: $|\pm\rangle$, then from the $L^2\Psi = \lambda\Psi = \hbar^2 l(l+1)\Psi$ readapted from the orbital angular momentums L to the spin ones S, we have:

$S^2\Psi = \lambda\Psi = \hbar^2 s(s+1)\Psi = \hbar^2 \frac{1}{2}(\frac{1}{2}+1)\Psi = \frac{3}{4}\hbar^2\Psi$, from which: $S^2|\pm\rangle = \frac{3}{4}\hbar^2|\pm\rangle$ and $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$; now, as $|\pm\rangle$ represents two wave functions, then we write again all in a matrix form, with 2x2 matrixes and to each of the two rows a wave function will correspond, indeed:
$S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, S^2 = \frac{3}{4}\hbar^2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and then we can evaluate also the matrixes S_+ and S_-: $S_\pm = S_x \pm iS_y$, so: $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ and $S_x = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$; in a more concise way we can write: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, where: $\sigma_1 = \sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_3 = \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$; those are the **Pauli Matrixes** and they follow the following properties, as we can directly prove:
$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k$,
$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, just like the matrixes α above treated, **so this is the first sign that those α matrixes predict the existence of the spin.**

Now, let's consider the following identity:
$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{N}) = [(\sigma_1\hat{1} + \sigma_2\hat{2} + \sigma_3\hat{3})(M_1\hat{1} + M_2\hat{2} + M_3\hat{3})][(\sigma_1\hat{1} + \sigma_2\hat{2} + \sigma_3\hat{3})(N_1\hat{1} + N_2\hat{2} + N_3\hat{3})] =$
$= (\sigma_1^2 M_1N_1 + \sigma_1\sigma_2 M_1N_2 + \sigma_2\sigma_1 M_2N_1) + (\sigma_2^2 M_2N_2 + \sigma_2\sigma_3 M_2N_3 + \sigma_3\sigma_2 M_3N_2) +$
$+ (\sigma_3^2 M_3N_3 + \sigma_1\sigma_3 M_1N_3 + \sigma_3\sigma_1 M_3N_1)$.
Now, we remember that: $\sigma_1\sigma_2 = i\sigma_3$; in fact: $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\cdot\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and also: $\sigma_2\sigma_1 = -i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_2 = -i\sigma_1$, $\sigma_1\sigma_3 = i\sigma_2$, $\sigma_3\sigma_1 = -i\sigma_2$, while for what the square values are concerned: $\sigma_1^2 = \sigma_1\sigma_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \equiv 1 = \sigma_2^2 = \sigma_3^2$, then: $(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{N}) =$
$= (\sigma_1^2 M_1N_1 + \sigma_1\sigma_2 M_1N_2 + \sigma_2\sigma_1 M_2N_1) + (\sigma_2^2 M_2N_2 + \sigma_2\sigma_3 M_2N_3 + \sigma_3\sigma_2 M_3N_2) +$
$+ (\sigma_3^2 M_3N_3 + \sigma_1\sigma_3 M_1N_3 + \sigma_3\sigma_1 M_3N_1) =$
$= (\sigma_1^2 M_1N_1 + i\sigma_3 M_1N_2 - i\sigma_3 M_2N_1) + (\sigma_2^2 M_2N_2 + i\sigma_1 M_2N_3 - i\sigma_1 M_3N_2) + (\sigma_3^2 M_3N_3 + i\sigma_2 M_1N_3 - i\sigma_2 M_3N_1) =$
$= (\sigma_1^2 M_1N_1 + \sigma_2^2 M_2N_2 + \sigma_3^2 M_3N_3) + i\sigma_3(M_1N_2 - M_2N_1) + i\sigma_1(M_2N_3 - M_3N_2) + i\sigma_2(M_1N_3 - M_3N_1) =$
$= (M_1N_1 + M_2N_2 + M_3N_3) + i\sigma_3(M_1N_2 - M_2N_1) + i\sigma_1(M_2N_3 - M_3N_2) + i\sigma_2(M_1N_3 - M_3N_1) =$
$= (M\cdot N) + i[\sigma\cdot(M\times N)] = (\sigma\cdot M)(\sigma\cdot N)$ and for $\vec{M} = \vec{N} \rightarrow$
$$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{M}) = (\vec{M}\cdot\vec{M}) + i[\vec{\sigma}\cdot(\vec{M}\times\vec{M})] \qquad \text{(A6.10)}$$
Now, we have already said that:
$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k},$$
and by considering a z oriented field, we have:
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = 0 + 0 + B_z\hat{k} = B_z\hat{k}, \qquad \text{(A6.11)}$$
that is: $B_x = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) = 0$, $B_y = (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) = 0$ and $B_z = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$ (A6.12)

so we can set:

$$A_y = \frac{1}{2}B_x\,,\ \ A_x = -\frac{1}{2}B_y \ \text{ and } \ A_z = 0 \qquad \text{(A6.13)}$$

in fact, the (A6.13), if inserted into the (A6.12), will confirm the (A6.11). So we can say that:

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} \qquad \text{(A6.14)}$$

in fact: $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ x & y & z\end{vmatrix} = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & B_z\\ x & y & z\end{vmatrix} = \frac{1}{2}[\hat{i}(0-yB_z)+\hat{j}(xB_z-0)+\hat{k}(0-0)] =$

$= \hat{i}\frac{(-yB_z)}{2}+\hat{j}\frac{(xB_z)}{2}+\hat{k}(0) = \hat{i}A_x+\hat{j}A_y+\hat{k}A_z$ and, as a cross check:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{(-yB_z)}{2} & \frac{(xB_z)}{2} & 0\end{vmatrix} = \hat{i}(0-0)+\hat{j}(0-0)+\hat{k}[\frac{\partial}{\partial x}(\frac{xB_z}{2})-\frac{\partial}{\partial y}(\frac{-yB_z}{2})] = B_z$$

Now, we put $\vec{M} = (\vec{p}-q\vec{A})$ into the (A6.10); as: $\vec{M}\times\vec{M} = (\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A}) =$

$= \vec{p}\times\vec{p}-q(\vec{p}\times\vec{A})-q(\vec{A}\times\vec{p})+q^2(\vec{A}\times\vec{A}) = 0-q(\vec{p}\times\vec{A})-q(\vec{A}\times\vec{p})+q^2(0) = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})]$

then $\vec{M}\times\vec{M} = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})]$ and after inserting into this the well known $\vec{p}\to -i\hbar\vec{\nabla}$, then we get: $\vec{M}\times\vec{M} = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})] = -q\{(-i\hbar\vec{\nabla}\times\vec{A})+[\vec{A}\times(-i\hbar\vec{\nabla})]\}$ and after inserting into this the (A6.14) and also the $\vec{B} = \vec{\nabla}\times\vec{A}$, we get: $\vec{M}\times\vec{M} = i\hbar q(\vec{B}+\frac{1}{2}\vec{B}\times\vec{r}\times\vec{\nabla}) = i\hbar q(\vec{B}+\frac{1}{2}\vec{\nabla}\times\vec{B}\times\vec{r})$, but:

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_x & B_y & B_z\end{vmatrix} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & B_z\end{vmatrix} = \hat{i}\frac{\partial B_z}{\partial y}-\hat{j}\frac{\partial B_z}{\partial x}+\hat{k}0 = 0+0+0 = 0$$, so:

$\vec{M}\times\vec{M} = i\hbar q(\vec{B}+\frac{1}{2}\vec{B}\times\vec{r}\times\vec{\nabla}) = i\hbar q(\vec{B}+0) = i\hbar q\vec{B}$; therefore, with reference to the (A6.10), we have: $(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{M}) = (\vec{\sigma}\cdot\vec{M})^2 = [\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 = (\vec{M}\cdot\vec{M})+i[\sigma\cdot(\vec{M}\times\vec{M})] = (\vec{p}-q\vec{A})^2+i(\vec{\sigma}\cdot i\hbar q\vec{B}) =$

$$= (\vec{p}-q\vec{A})^2-\hbar q(\vec{\sigma}\cdot\vec{B}) \qquad \text{(A6.15)}$$

We defined the Dirac Hamiltonian: $H_D = qV+c\vec{\alpha}(\vec{p}-q\vec{A})+\beta m_0c^2$, so, after reminding that, in relativity: V=cA$_0$, and also that the matrixes α_i: ($\alpha_1 = \rho_1\sigma_1$, $\alpha_2 = \rho_3$, $\alpha_3 = \rho_1\sigma_3$ $\alpha_4 = \rho_1\sigma_2$), and even that if we swap α_2 and α_4 , then all the properties of the matrixes α_i still hold (see Appendix 7), then we have: ($\alpha_1 = \rho_1\sigma_1$, $\alpha_2 = \rho_1\sigma_2$, $\alpha_3 = \rho_1\sigma_3$, $\alpha_4 = \rho_3 = \beta$) and so:

$H = qV+c\vec{\alpha}(\vec{p}-q\vec{A})+\beta m_0c^2 = +qcA_0+[c\alpha_1(\vec{p}-q\vec{A})_1+c\alpha_2(\vec{p}-q\vec{A})_2+c\alpha_3(\vec{p}-q\vec{A})_3]+\rho_3m_0c^2$,

as, due to the swapping, β becomes, for the time being, α_4(=ρ_3). Going on:

$H = +qcA_0+[c\alpha_1(\vec{p}-q\vec{A})_1+c\alpha_2(\vec{p}-q\vec{A})_2+c\alpha_3(\vec{p}-q\vec{A})_3]+\rho_3m_0c^2 =$

$= +qcA_0+[\rho_1\sigma_1c(\vec{p}-q\vec{A})_1+\rho_1\sigma_2c(\vec{p}-q\vec{A})_2+\rho_1\sigma_3c(\vec{p}-q\vec{A})_3]+\rho_3m_0c^2 = +qcA_0+\rho_1c[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]+\rho_3m_0c^2$

so: $(H-qcA_0)^2 = \{\rho_1c[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]+\rho_3m_0c^2\}^2 =$

$$= \rho_1^2 c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + \rho_3^2 m_0^2 c^4 + \rho_1\rho_3 m_0 c^3[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] + \rho_3\rho_1 m_0 c^3[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4$$

so: $(H - qcA_0)^2 = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4$, (A6.16)

as: $\rho_1^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv 1$,

$\rho_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv 1$ and:

$\rho_1\rho_3 = c\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to |\rho_1\rho_3| = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1$.

$\rho_3\rho_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -\rho_1\rho_3$.

and at last, by using the (A6.15)into the (A6.16), we get:

$(H - qcA_0)^2 = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$ (A6.17)

Now, for an electron which moves with a small impulse, we have: $H = H_1 + m_0 c^2$ ($H_1 << m_0 c^2$); so, with such an H into the (A6.17), we get:

$(H - qcA_0)^2 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B}) = (H_1 + m_0 c^2 - qcA_0)^2 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$

and by approximating the square term $(H_1 + m_0 c^2 - qcA_0)^2$ with $(H_1 + m_0 c^2 - qcA_0)$, as the H is small by supposition (and around the vertex, we approximate the parabola with a line), we get: ((($H_1 + m_0 c^2 - qcA_0 \to c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$))) , where, conversely, on the right side we have to carry out a dimensional counterbalance, due to the approximation just done, that is, regarding the term $c^2(\vec{p}-q\vec{A})^2$, it is a squared impulse x c^2, indeed, so, as in classical physics (that is non relativistic) we have: $E_k = \frac{1}{2}m_0 v^2 = \frac{1}{2m_0}m_0^2 v^2 = \frac{1}{2m_0}p^2$, then: $c^2(\vec{p}-q\vec{A})^2$ →

$\to \frac{1}{2m_0}(\vec{p}-q\vec{A})^2$ (that is [J^2] → [J]); then, regarding the term $-\hbar c^2 q(\vec{\sigma}\cdot\vec{B})$ we have to make the adaptation (-[J^2] → -[J]) so, here too, we will multiply by $\frac{1}{2m_0}$ and then we will divide by c^2: →

$\to -\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$; at last, regarding the left term: $m_0^2 c^4$, the same thing follows, that is:

([J^2] → $\sqrt{\ }$ → [J]) → $m_0 c^2$; finally: $H_1 + m_0 c^2 - qcA_0 \approx \frac{1}{2m_0}(\vec{p}-q\vec{A})^2 + m_0 c^2 - \frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B}) \to$

$\to H_1 - qcA_0 \approx \frac{1}{2m_0}(\vec{p}-q\vec{A})^2 - \frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$; (A6.18) the quantity $-\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$ is an additional potential energy:

$$\left([\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})]=[\frac{JsC}{kg}(1,T)]=[\frac{JsC}{kg}(\frac{Wb}{m^2})]=[\frac{JsC}{kg}(\frac{Vs}{m^2})]=[J(CV)\frac{1}{kg\frac{m^2}{s^2}}]=[J(J)\frac{1}{J}]=[J] \right)$$

and it tells us that the electron has an intrinsic magnetic momentum $\frac{\hbar q}{2m_0}\sigma \equiv \frac{\hbar q}{2m_0}$. Now, let's put an atom in a low magnetic field, whose induction vector is B, and such a B extends along z, just like that above examined; well, we have, from the electromagnetism: $H=\frac{1}{2m_0}(\vec{p}-q\vec{A})^2-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$, but if we consider the components of $\vec{A}$ we found before, on occasion of the proof of equation (A6.14), then we have: $H=\frac{1}{2m_0}\{(\hat{i}p_x+\hat{j}p_y+\hat{k}p_z)-q[\hat{i}(-\frac{yB_z}{2})+\hat{j}(\frac{xB_z}{2})+\hat{k}0)]\}^2-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}=$

$$=\frac{1}{2m_0}[(\hat{i}(p_x+\frac{qyB_z}{2})+\hat{j}(p_y-\frac{qxB_z}{2})+\hat{k}p_z]^2-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}=$$

$$=\frac{1}{2m_0}[(p_x+\frac{qy}{2}B_z)^2+(p_y-\frac{qx}{2}B_z)^2+p_z^2]-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}=\frac{(p_x^2+p_y^2+p_z^2)}{2m_0}+$$

$+\frac{q}{2m_0}B_z(yp_x-xp_y)+\frac{1}{2m_0}(\frac{qx}{2})^2B_z^2+\frac{1}{2m_0}(\frac{qy}{2})^2B_z^2-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$ and having, by supposition, a low B, then the terms in B_z^2 will be neglected, with respect to those in B_z , so:

$H=\frac{p^2}{2m_0}+\frac{q}{2m_0}B_z(yp_x-xp_y)-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$, but being, for an atom:

$\vec{L}=\vec{r}\times\vec{p}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x & y & z\\ p_x & p_y & p_z\end{vmatrix}=\hat{i}(yp_z-zp_y)+\hat{j}(zp_x-xp_z)+\hat{k}(xp_y-yp_x)\rightarrow L_z=-(yp_x-xp_y)$, then:

$H=\frac{p^2}{2m_0}-\frac{q}{2m_0}B_zL_z-\frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$; now, let's name: $H_P=-\frac{q}{2m_0}B_zL_z=-\frac{q}{2m_0}\vec{B}\cdot\vec{L}$, as the only B component is B_z ; now, if we have not only the L from the revolution around the nucleus, but also an intrinsic L'=S (spin), then we will get an H_p' such that: (and we remind we proved that $\vec{S}=\frac{\hbar}{2}\vec{\sigma}$)

$$H'_P=-\frac{q}{2m_0}\vec{B}\cdot 2\vec{S}=-\frac{q\hbar}{2m_0}\vec{B}\cdot\vec{\sigma} \; ; \qquad \text{(A6.19)}$$

the 2 which multiplies S is an experimental evidence and it can be expressed by saying that the gyromagnetic spin ratio g_s is twice the orbital one g_l :

$g_l=\frac{\vec{\mu}_l}{\vec{L}}=+\frac{q}{2m_0}=\frac{g_s}{2}$. The (A6.19) lead us to the same term ($-\frac{\hbar q}{2m_0}(\sigma\cdot\vec{B})$) which showed up in the (A6.18), but we derived the (A6.18) as a consequence of the Electromagnetic Dirac Equation, so we can finally say that such an equation, that is, **the Electromagnetic Dirac Equation, indeed, is a prediction of the existence of the spin.**

APPENDIX 7: ON THE VALIDITY OF THE PROPERTIES OF MATRIXES $\vec{\alpha}$ EVEN WITH α_2 AND α_4 SWAPPED.

We remind of the conditions which must be respected by all the α and β: $\beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i = 0$, $\alpha_i\alpha_i = 1^2$, $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ (i,j=1,2,3) and in order to have just α coefficients, we name: $\beta = \alpha_2$, so, in a more concise style, the conditions become: $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ (i,j=1,2,3,4) .

$$\boxed{\begin{array}{l} \sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ \rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \end{array}}$$

where: $\boxed{\alpha_1 = \rho_1\sigma_1 \ , \ \alpha_2 = \rho_1\sigma_2 \ , \ \alpha_3 = \rho_1\sigma_3 \ , \ \alpha_4 = \rho_3}$; in fact, we just need to carry out all the matrix products, such as:

$$\alpha_1 = \rho_1\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ and check: } \rightarrow$$

$$\rightarrow \alpha_1\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow |\alpha_1\alpha_1| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \text{, then:}$$

$$\alpha_2 = \beta = \rho_1\sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\alpha_2\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and: } |\alpha_2\alpha_2| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1,$$

then: $2 = \alpha_1\alpha_1 + \alpha_1\alpha_1 = 2\delta_{11} = 2 = \alpha_2\alpha_2 + \alpha_2\alpha_2 = 2\delta_{22} = 2$, then:

$$\alpha_3 = \rho_1\sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$\alpha_3\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\text{ and: }|\alpha_3\alpha_3|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

then: $2=\alpha_3\alpha_3+\alpha_3\alpha_3=2\delta_{33}=2$, then:

$$\alpha_4=\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\quad\rightarrow\alpha_4\alpha_4=\rho_3\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$

and: $|\alpha_4\alpha_4|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1$ and: $2=\alpha_4\alpha_4+\alpha_4\alpha_4=2\delta_{44}=2$, then:

$\alpha_i\beta+\beta\alpha_i=(\alpha_i\alpha_2+\alpha_2\alpha_i)=0$, in fact:

$$\alpha_1\alpha_2+\alpha_2\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}+\begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\,;$$

$$\alpha_3\alpha_2+\alpha_2\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}+\begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\,,$$

$$\alpha_4\alpha_2+\alpha_2\alpha_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=$$

$$=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\,.$$

APPENDIX 8 - THE ELECTRODYNAMIC POTENTIALS

We define a vector $\vec{A}$ so that:

$$\vec{\nabla}\times\vec{A}=\vec{B} \tag{A8.1}$$

Then, according to the Third Equation of Maxwell, we have:

$\vec{\nabla}\times\vec{E}=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{A})=-\vec{\nabla}\times(\frac{\partial\vec{A}}{\partial t})$, so: $\vec{\nabla}\times(\vec{E}+\frac{\partial\vec{A}}{\partial t})=0$, therefore $\vec{E}+\frac{\partial\vec{A}}{\partial t}$ is irrotational an can be written as a gradient of a function –V:

$$-\vec{\nabla}V=\vec{E}+\frac{\partial\vec{A}}{\partial t}$$, or:

$$\vec{E}=-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}V \tag{A8.2}$$

Now, making a system out of equations (A8.1), (A8.2), the 1st and 4th Eq. of Maxwell and the following vectorial equality:

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=-\nabla^2\vec{A}+\vec{\nabla}(\vec{\nabla}\cdot\vec{A})$, we get:

$$\nabla^2 V+\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A})=-\rho/\varepsilon \tag{A8.3}$$

$$\nabla^2\vec{A}-\varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2}-\vec{\nabla}(\vec{\nabla}\cdot\vec{A}+\varepsilon\mu\frac{\partial V}{\partial t})=-\mu\vec{J} \tag{A8.4}$$

These two equations are electrodynamic equations not de-coupled. Now we de-couple them by a gauge transformation:

$$\vec{A}'=\vec{A}+\vec{\nabla}\varphi \qquad V'=V-\frac{\partial\varphi}{\partial t}$$

In fact: $\vec{\nabla}\times\vec{A}'=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times(\vec{\nabla}\varphi)=\vec{\nabla}\times\vec{A}+0=\vec{B}$ (curl of gradient = 0).

$$-\vec{\nabla}V'-\frac{\partial\vec{A}'}{\partial t}=-\vec{\nabla}V+\vec{\nabla}(\frac{\partial\varphi}{\partial t})-\frac{\partial\vec{A}}{\partial t}-\frac{\partial}{\partial t}(\vec{\nabla}\varphi)=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}=\vec{E}$$

so, the equations for A and V are the same for A’ and V’, too.

The transformation of gauge is chosen in order to respect the Lorenz Condition:

$$\vec{\nabla}\cdot\vec{A}+\varepsilon\mu\frac{\partial V}{\partial t}=0\,. \tag{A8.5}$$

In fact, we see it is respected also with A’ and V’, that is: $\vec{\nabla}\cdot\vec{A}'+\varepsilon\mu\frac{\partial V'}{\partial t}=0$; in fact:

$0=\vec{\nabla}\cdot\vec{A}'+\varepsilon\mu\frac{\partial V'}{\partial t}=\vec{\nabla}\cdot\vec{A}+\varepsilon\mu\frac{\partial V}{\partial t}+(\nabla^2\varphi-\varepsilon\mu\frac{\partial^2\varphi}{\partial t^2})$, so φ must respect the equation:

$(\nabla^2\varphi-\varepsilon\mu\frac{\partial^2\varphi}{\partial t^2})=-(\vec{\nabla}\cdot\vec{A}+\varepsilon\mu\frac{\partial V}{\partial t})$, but, by assumption, the right side is zero, and so does the left one; therefore, the Lorenz Condition is always respected.

By putting (A8.5) in (A8.4) and then considering that, in (A8.3), for the Lorenz Condition $\vec{\nabla}\cdot\vec{A}=-\varepsilon\mu\frac{\partial V}{\partial t}$, we get the De-Coupled Electrodynamic Equations:

$$\nabla^2\vec{A}-\varepsilon\mu\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\vec{J}$$

$$\nabla^2 V-\varepsilon\mu\frac{\partial^2 V}{\partial t^2}=-\frac{\rho}{\varepsilon}$$

By introducing the d’Alembertian operator: $\Box=\nabla^2(*)-\varepsilon\mu\frac{\partial^2(*)}{\partial t^2}$, we can write:

$$\boxed{\begin{aligned}&\square\vec{A} = -\mu\vec{J} \\ &\square V = -\frac{\rho}{\varepsilon}\end{aligned}}$$ (A8.6) (A8.7)

We can prove, by direct insertion, that the solutions are the following ones:

$$\boxed{\begin{aligned}&\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int_\tau \frac{\vec{J}(\vec{r}\,',t-\Delta r/V)}{\Delta r}d\tau' \\ &V(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int_\tau \frac{\rho(\vec{r}\,',t-\Delta r/V)}{\Delta r}d\tau'\end{aligned}}$$ (A8.8) (A8.9)

In fact, $\nabla^2 V = \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\rho}{r})d\tau'] = \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \rho(\nabla\frac{1}{r})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \frac{1}{r}(\nabla\rho)d\tau'] =$

$$= \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \rho(-\frac{1}{r^2})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \frac{1}{r}(\nabla\rho)d\tau'] = -\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\rho}{r^2})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' =$$

$$= -\frac{1}{4\pi\varepsilon}\int_{S'} \frac{\rho}{r^2}dS' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{1}{4\pi\varepsilon}\int_{S'} \rho d\Omega' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{\rho}{\varepsilon} + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'$$

Then: $\nabla^2 V - \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{\rho}{\varepsilon}$; now, if we prove that:

$\boxed{\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'} = \boxed{\varepsilon\mu\frac{\partial^2 V}{\partial t^2}}$, then the (A8.9) is proved. Let's see:

$$\boxed{\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'} = \frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{\nabla c\rho}{r})d\tau' = \frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{\nabla J}{r})d\tau' = \boxed{-\frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau'}$$

$\boxed{\varepsilon\mu\frac{\partial^2 V}{\partial t^2}} = \varepsilon\mu\frac{\partial}{\partial t}(\frac{\partial V}{\partial t}) = \varepsilon\mu\frac{\partial}{\partial t}(-\frac{1}{\varepsilon\mu}\nabla A) =$ (where we have used the (A8.5)); going ahead:

$$= -\frac{\partial}{\partial t}(\nabla A) = -\frac{\partial}{\partial t}[\frac{\mu}{4\pi}\int_\tau \nabla(\frac{J}{r})d\tau'] = -\frac{\mu}{4\pi}\int_\tau \nabla(\frac{1}{r}\frac{\partial J}{\partial t})d\tau' = \tag{A8.10}$$

$$= -\frac{\mu c}{4\pi}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau' = \boxed{-\frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau'}$$ qed.

And as in the step (A8.10) the (A8.8) has been crucial, in this case we just hold that as a proof for the (A8.8) itself, that is, by the way, already symmetric, with respect to the (A8.9).

APPENDIX 9: ELECTROMAGNETISM AND RELATIVITY.

We know that with a charge q we have a force over a test charge q_t (in P) which is $\vec{F}_{orce} = \frac{1}{4\pi\varepsilon_0}\frac{qq_t}{(\Delta r)^2}\hat{\Delta r}$ and an electric field $\vec{E}(\vec{r}) = \frac{\vec{F}_{orce}}{q_t} = \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2}\hat{\Delta r}$; moreover, as, by definition, the potential is: $dV = -\vec{E}\cdot d\vec{l}$, then we have a potential: $V = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Delta r}$.

$(V = \int dV = -\int \vec{E}\cdot d\vec{l} = -\int \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2}\hat{\Delta r}\cdot d\vec{l} = -\int \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2}d(\Delta r) = -\frac{q}{4\pi\varepsilon_0}\int\frac{d(\Delta r)}{(\Delta r)^2} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Delta r}$).

Now, by considering the charge q as incorporated in its small volume, then we have a certain charge density ρ , so, for the potential V: $V = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Delta r} = \frac{\int \rho dV_{ol}}{4\pi\varepsilon_0\Delta r} = \frac{\int \frac{\rho}{\varepsilon_0}dV_{ol}}{4\pi\Delta r}$.

Then, we know from physics that the current density is $\vec{j} = \rho\vec{v}$.

So, as a consequence, we define the current density four-vector in this way:

$\underline{j} = (\vec{j}, \rho c) = (j_x, j_y, j_z, \rho c) = (\gamma\rho_0\vec{v}, \gamma\rho_0 c)$; please, notice the similarity with the energy-momentum 4-vector: $\underline{p} = (\vec{p}, mc) = (\gamma m_0\vec{v}, \gamma m_0 c)$. And then we have: $|\underline{j}|^2 = |\vec{j}|^2 - j_4^2 = -\rho_0^2 c^2$.

Then, we know from the electromagnetism that the electric and magnetic fields (induction vector B) can be espresse as a function of the electrodynamic potentials V and A:

$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \vec{\nabla}\times\vec{A}$ and we also know the Lorenz Condition:

$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \varepsilon\mu\frac{\partial V}{\partial t} = 0$, and also the Continuity Equation: $\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$.

Still from the electromagnetism, we also know that:

$\Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} = \Box V$ and $\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{j} = \Box\vec{A}$.

Now, on the basis of all that, we are inclined to define the Potential 4-vector or 4-potential $\underline{\Phi}$:

$\underline{\Phi} = (A_x, A_y, A_z, \frac{V}{c}) = (A_x, A_y, A_z, A_t) = (A_x, A_y, A_z, A_0)$; in fact, so doing the previous equations can be summarized like that: $\Box\Phi_k = -\mu j_k$ and by also defining the 4-divergence:

$div_4 = \vec{\nabla}_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x} + \frac{\partial}{\partial(ct)} = (\vec{\nabla} + \frac{\partial}{\partial(ct)})$ we also have, of course: $\vec{\nabla}_4\underline{j} = 0$, due to the Continuity Equation, and $\vec{\nabla}_4\underline{\Phi} = 0$, due to the Lorenz Condition.

SULLA NECESSITÀ DEI NUMERI IMMAGINARI (i) IN MECCANICA QUANTISTICA

(ANNICHILAZIONE PREVISTA DALLA MATEMATICA)

Leonardo ***Rubino***

Ottobre 2025

Abstract: *Il fatto che, con l'Equazione di Schrodinger, abbiamo la necessità di imbatterci nell'unità immaginaria (i) è un modo semplice che la natura utilizza per informarci del fatto che una particella, con la sua Ψ, è una realtà a metà ed esiste una funzione d'onda (antiparticella) Ψ* complessa coniugata (-i) che, unitamente a Ψ, ci riporta, per così dire, ad un'onda "trigonometrica"reale (a valori reali).*

Sappiamo da Eulero che: $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$ (od anche: $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$) (1)

(vedere Appendice 1).

Considerando ora i fenomeni ondulatori, ovviamente ad argomento $(\vec{k}\cdot\vec{x}-\omega t)$, la (1) diviene:

$$\cos(\vec{k}\cdot\vec{x}-\omega t) = \frac{e^{i(\vec{k}\cdot\vec{x}-\omega t)}+e^{-i(\vec{k}\cdot\vec{x}-\omega t)}}{2}. \quad (2)$$

Considerando ora la funzione d'onda di un elettrone libero, avremo:

$$\Psi = A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)} = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x}-\frac{E_k}{\hbar}t)} = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x}-\frac{p^2}{2m_0\hbar}t)} \quad (3)$$

e questa soddisfa ovviamente l'Equazione di Schrodinger:
(cosa verificata in Appendice 2, Eq.ni (A2.14)-------(A2.16))

$$\Delta\Psi + \frac{2m_0}{\hbar^2}(H-V)\Psi = 0 \qquad (E_k=H-V) \quad (4)$$

dacché la (2) ci dice che la somma di due funzioni d'onda complesse coniugate dà una funzione trigonometrica (coseno o seno che sia).

Facciamo ora alcune considerazioni:

1) L'onda $\Psi = A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ è soluzione della Equazione di Schrodinger, ma se la combiniamo con la complessa coniugata $\Psi^* = A\cdot e^{-i(\vec{k}\cdot\vec{x}-\omega t)}$, otteniamo una $\Psi_{Trig} = \cos(\vec{k}\cdot\vec{x}-\omega t)$, CHE PERÒ **NON** È SOLUZIONE DELLA EQUAZIONE DI SCHRODINGER.

2) Con riferimento all'Appendice 3, l'Equazione Elettromagnetica di Dirac (A3.2):

$$(\hbar\frac{\partial}{\partial t} + \hbar c\vec{\alpha}\cdot\vec{\nabla} + i\beta m_0 c^2)\Psi = iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi \quad (5)$$

ci porta a concludere che la complessa coniugata di Ψ (Ψ funzione d'onda di una **particella**) , ossia Ψ* , è la funzione d'onda dell'**antiparticella**, con la relativa Equazione Elettromagnetica Coniugata di Dirac (A3.5):

$$(\hbar\frac{\partial}{\partial t} + \hbar c\vec{\alpha}\cdot\vec{\nabla} + i\beta m_0 c^2)\Psi^* = -iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi^* \quad (6)$$

con la carica elettrica q cambiata in –q e la Ψ cambiata in Ψ*.

3) Succede però che $\Psi_{Trig} = \cos(\vec{k} \cdot \vec{x} - \omega t)$, pur NON essendo soluzione della Equazione di Schrodinger, è soluzione della Equazione delle Onde di d'Alembert:

$$\nabla^2 \Psi - \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} = 0 \tag{7}$$

(cosa verificata in Appendice 2, Eq.ni (A2.7)---------(A2.13))

4) E, guarda caso, una particella Ψ e la sua antiparticella Ψ* , se si incontrano, si annichiliscono, scomparendo e dando origine a due fotoni del tipo Ψ_{Trig} (onde vere e proprie, trigonometriche, non immaginarie con l'i), che, in quanto fotoni, soddisfano appunto l'Equazione di d'Alembert (7)!

5) È allora vero che tutto l'Universo deriva da una enorme creazione di coppie particella-antiparticella, che per il Principio di Indeterminazione di Heisenberg devono tendere alla scomparsa per riannichilazione, con ovvia riattrazione reciproca (per salvaguardare il Principio di Conservazione dell'Energia) e tale riattrazione determina l'esistenza della Forza (fondamentale) di Coulomb (vedere Appendice 4) e, nel complesso di tutte queste coppie comparse che si riattraggono, determina anche la riattrazione globale di tutta la materia, in termini di Gravitazione (vedere Appendice 5).

((Anche la Ψ* (col (-i)) potrebbe essere digerita dalla Equazione di Schrodinger, a patto di rinormalizzarla a (+i), ossia la direzione (limitata) sotto la quale l' Equazione di Schrodinger (che è limitata, nella sua approssimazione classica/non relativistica dell'energia) è stata appunto concepita.))

APPENDICE 1: Le Formule di Eulero

Ho una f(x) e suppongo che essa sia uguale, in un certo intorno del valore x_0 , ad una certa serie generica: $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$; ora, supponendo che x_0 sia l'origine: (x_0=0) → $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Essendo ora che: $f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 x^0 + \sum_{n=1}^{\infty} a_n x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n$, segue che:

$f(0) = a_0 + \sum_{n=1}^{\infty} a_n 0^n = a_0$ e poi anche: $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + \sum_{n=2}^{\infty} n a_n x^{n-1}$ e:

$f'(0) = a_1 + \sum_{n=2}^{\infty} n a_n 0^{n-1} = a_1 + 0 = a_1$ e:

$$f''(x) = \sum_{n=1}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = 2(2-1) a_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2} = 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n x^{n-2}$$

da cui: $f''(0) = 2a_2 + \sum_{n=3}^{\infty} n(n-1) a_n 0^{n-2} = 2a_2 + 0 = 2a_2 \quad \rightarrow \quad a_2 = \frac{1}{2} f''(0)$ e così via, ottenendo: $a_n = \frac{1}{n!} f^{(n)}(0)$ e quindi: $f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) x^n$, che è lo Sviluppo in Serie di Taylor (di MacLaurin, in questo caso). Applicandolo ora alla funzione $f(x) = e^x$, si ha:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + ... + \frac{1}{n!} x^n .$$

Consideriamo i tre seguenti sviluppi di Taylor:

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + ... + \frac{1}{n!} x^n$$

$$\cos x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = 1 - \frac{1}{2}x + ... + \frac{(-1)^n}{(2n)!}x^{2n}$$

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot x^n = x - \frac{1}{6}x^3 + ... + \frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

se nel primo consideriamo ix in luogo di x, avremo:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{1}{n!} f^n(0) \cdot (ix)^n = 1 + ix + i^2 \frac{1}{2}x^2 + i^3 \frac{1}{6}x^3 + ... + \frac{1}{n!} i^n x^n = 1 + ix - \frac{1}{2}x^2 + ii^2 \frac{1}{6}x^3 + ... + \frac{1}{n!} i^n x^n =$$

$= 1 + ix - \frac{1}{2}x^2 - i\frac{1}{6}x^3 + ... + \frac{1}{n!} i^n x^n$, mentre il terzo sviluppo dà:

$i \sin x = \sum_{n=0}^{\infty} \frac{i}{n!} f^n(0) \cdot x^n = ix - i\frac{1}{6}x^3 + ... + \frac{i(-1)^n}{(2n+1)!}x^{2n+1}$ e allora si intuisce che, per e^{ix} :

$e^{ix} = 1 + ix - \frac{1}{2}x^2 - i\frac{1}{6}x^3 + ... + \frac{1}{n!} i^n x^n = \cos x + i \sin x$; similmente: $e^{-ix} = \cos x - i \sin x$, da cui, facendo sistema tra le due, si ottengono le seguenti Formule di Eulero: $\cos x = \frac{e^{ix} + e^{-ix}}{2}$, $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$. Se nella $e^{ix} = \cos x + i \sin x$ poniamo x=π , otteniamo: $e^{i\pi} + 1 = 0$.

Inoltre,

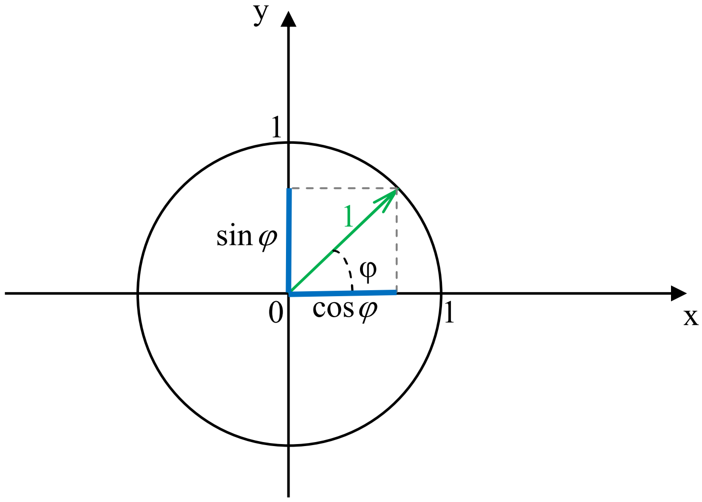


Fig. 1: Il cerchio trigonometrico.

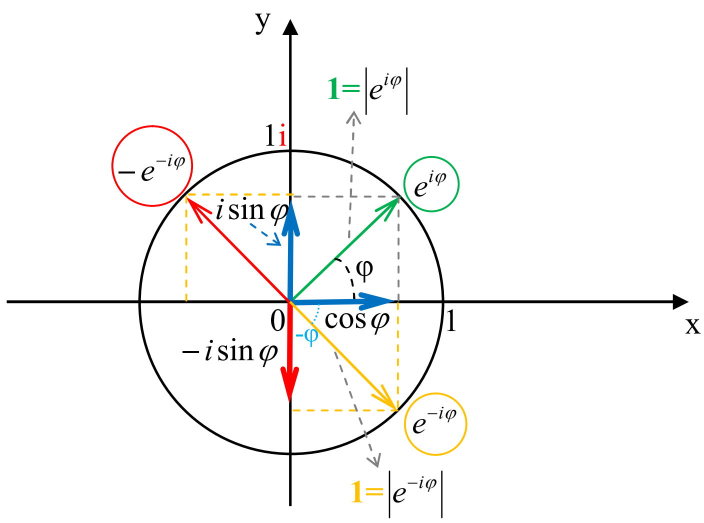


Fig. 2: Il Piano di Gauss.

ricordando le Formule di Eulero appena dimostrate:

$e^{i\varphi} = \cos\varphi + i \sin\varphi$ ed $e^{-i\varphi} = \cos\varphi - i \sin\varphi$ ed “immaginando” queste due equazioni come equazioni vettoriali, la prima, ad esempio, sarebbe del tipo: $\vec{v} = \vec{v}_x + i\vec{v}_y$ (con $\vec{v}_x = (\cos\varphi)\hat{x}$ e $\vec{v}_y = (\sin\varphi)\hat{y}$) e dunque:

$\boxed{\vec{v} = (\cos\varphi)\hat{x} + i(\sin\varphi)\hat{y} \,,}$ con $|\vec{v}| = 1 = |e^{i\varphi}|$, viste le sue componenti cosφ e sinφ .

Riguardo invece la seconda: $\vec{v} = \vec{v}_x - i\vec{v}_y$ (con $\vec{v}_x = (\cos\varphi)\hat{x}$ e $\vec{v}_y = (\sin\varphi)\hat{y}$) e dunque:

$\boxed{\vec{v} = (\cos\varphi)\hat{x} - i(\sin\varphi)\hat{y} \,,}$ con $|\vec{v}| = 1 = |e^{-i\varphi}|$, viste le sue componenti cosφ e sinφ [sin(-φ)=-sinφ].

Non solo, ma i due vettori verde ed arancione $e^{i\varphi}$ ed $e^{-i\varphi}$ (di Fig. 2) vanno a comporre

2cosφ sull'asse orizzontale ($e^{i\varphi}+e^{-i\varphi}=2\cos\varphi$), da cui la: $\boxed{\cos\varphi=\frac{e^{i\varphi}+e^{-i\varphi}}{2}}$ di Eulero, sempre appena dimostrata.

E, per finire, i due vettori verde e rosso $e^{i\varphi}$ e $-e^{-i\varphi}$ vanno a comporre i2sinφ sull'asse verticale ($e^{i\varphi}+[-e^{-i\varphi}]=e^{i\varphi}-e^{-i\varphi}=i2\sin\varphi$), da cui la: $\boxed{\sin\varphi=\frac{e^{i\varphi}-e^{-i\varphi}}{2i}}$ di Eulero, sempre appena dimostrata.

APPENDICE 2: LE FUNZIONI D'ONDA

Sappiamo dalla relatività che, per l'energia totale E, si ha:

$$E^2=p^2c^2+m_0^2c^4 \tag{A2.1}$$

Questa è l'espressione, per l'energia, più generale che abbiamo e vale appunto per una particella anche relativistica.

Ora, per un fotone, che è poi una « particella » con massa a riposo nulla, si ha $E^2=p^2c^2$, ossia:

$$E=pc \tag{A2.2}$$

Per una particella non relativistiva, sappiamo invece che vale, per la sua energia cinetica, la seguente espressione: $E_k=\frac{1}{2}m_0v^2$, ma quest'ultima è nascosta proprio nella (A2.1), che è di valore più generale, appunto. Infatti, la (A2.1) può essere così riscritta:

$$E=m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2} \tag{A2.3}$$

e ricordando che, per gli sviluppi di Taylor, si ha:

$f(x)=\sqrt{1+x}=(1+x)^{1/2}\approx 1+\frac{1}{2}x$, segue che, per la (A2.3):

$E=m_0c^2(1+\frac{p^2}{m_0^2c^2})^{1/2}\approx m_0c^2(1+\frac{p^2}{2m_0^2c^2})=m_0c^2+\frac{p^2}{2m_0}$ e, per l'energia cinetica, si ha dunque: $E_k=E-m_0c^2=\frac{p^2}{2m_0}=\frac{1}{2}m_0v^2$ cvd.

Consideriamo ora l'espressione generale di un'onda:

$$\Psi=A\cdot e^{i(\vec{k}\cdot\vec{x}-\omega t)}=A\cdot e^{i(\frac{2\pi}{\lambda}\hat{k}\cdot\vec{x}-\frac{2\pi}{\lambda}vt)}, \tag{A2.4}$$

in quanto: $\vec{k}=\frac{2\pi}{\lambda}\hat{k}$, $\omega=\frac{2\pi}{T}=2\pi f=2\pi\frac{v}{\lambda}$;

Tale onda, contemporaneamente, si propaga nello spazio (x) ed oscilla nel tempo t; infatti, se si pone t=0, si vede che si ha un'oscillazione lungo x ($\Psi=A\cdot e^{i(\vec{k}\cdot\vec{x})}$) e se si pone x=0 si ha una oscillazione nel tempo ($\Psi=A\cdot e^{-i(\omega t)}$).

Sappiamo inoltre che:

$$E = hf = \frac{h}{2\pi} 2\pi f = \hbar\omega \qquad \text{(A2.5)}$$

e, valendo anche la (A2.2), si ha:

$pc = \hbar\omega$, da cui :

$$p = \hbar\frac{\omega}{c} = \hbar\frac{2\pi}{\lambda} = \hbar k = p \qquad \text{(A2.6)}$$

e la (A2.4) diventa :

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E}{\hbar}t)} \qquad \text{(A2.7)}$$

Per semplice sostituzione diretta di tale Ψ nelle seguenti equazioni:

$$(i\hbar\frac{\partial}{\partial t})\Psi = E\Psi = (\hbar\omega)\Psi \qquad \text{(A2.8)}$$

$$(\frac{\hbar}{i}\vec{\nabla})\Psi = \vec{p}\Psi = (\hbar\vec{k})\Psi \; ; \qquad \text{(A2.9)}$$

si ha che esse danno delle identità, ossia sono giuste.

Nel caso monodimensionale : $(\frac{\hbar}{i}\frac{\partial}{\partial x})\Psi = p\Psi = (\hbar k)\Psi$; (∇ gradiente)

Dunque, possiamo rilevare le seguenti corrispondenze operatoriali:

$$E \rightarrow i\hbar\frac{\partial}{\partial t} \qquad \text{(A2.10)}$$

$$\vec{p} \rightarrow \frac{\hbar}{i}\vec{\nabla} \qquad \text{(A2.11)}$$

Valendo poi la (A2.2), ossia: $E^2 = p^2c^2$, si ha:

$$(i\hbar\frac{\partial}{\partial t})^2\Psi = c^2(\frac{\hbar}{i}\vec{\nabla})^2\Psi \, , \qquad \text{(A2.12)}$$

ossia:

$$\boxed{\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0} \qquad \text{(A2.13)}$$

o anche ($\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, laplaciano, divergenza del gradiente): $\Delta\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0$,

che è l'**Equazione delle Onde, o di d'Alembert**.

Si noti che tale equazione, di derivazione '' relativistica'' (fotone, ossia particella che si propaga a velocità c e con massa di riposo zero) è invariante per trasformazioni di Lorentz.

Passando ora al caso di particelle non relativistiche (gli atomi, ordinariamente, sono tali), otterremo un'equazione ''d'onda'' non relativistica, ossia l'Equazione di Schrodinger. Infatti, se nella (A2.7) consideriamo invece non più $E = pc$, ma $E_k = \frac{1}{2}m_0v^2$ (equazione appunto non relativistica), otteniamo:

$$\Psi = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E_k}{\hbar}t)} = A \cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{p^2}{2m_0\hbar}t)} \qquad \text{(A2.14)}$$

e, proprio come abbiamo fatto per ottenere la (A2.12), per sostituzione diretta della (A2.14) nella seguente equazione:

$$(i\hbar\frac{\partial}{\partial t})\Psi = (-\frac{\hbar^2}{2m_0}\nabla^2)\Psi \tag{A2.15}$$

($\quad (i\hbar\frac{\partial}{\partial t})\Psi = (-\frac{\hbar^2}{2m_0}\frac{\partial^2}{\partial x^2})\Psi$, nel caso monodimensionale $\quad$)

si ottiene un'identità. Dunque, la (A2.15) è vera. Attenzione, però, perché nella (A2.14) abbiamo usato non più una E totale, ma solo la E_k, fatto di cui teniamo conto.

Il primo membro della (A2.15) vale $(i\hbar\frac{\partial}{\partial t})\Psi = E_k\Psi$, ma sappiamo che E_k=H-V, da cui, sempre per la (A2.15): $-\frac{\hbar^2}{2m_0}\Delta\Psi = (H-V)\Psi$, ossia:

$$\boxed{\Delta\Psi + \frac{2m_0}{\hbar^2}(H-V)\Psi = 0} \tag{A2.16}$$

che altro non è che l'**Equazione di Schrodinger**.

Consideriamo ora il caso più generale, ossia particella relativistica e con massa a riposo non nulla.

Come abbiamo fatto in precedenza, visto che per la (A6.1) si ha: $E = \sqrt{p^2c^2 + m_0^2c^4}$, allora, sostituendo tale E sempre nella (A2.7) $\Psi = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{E}{\hbar}t)}$, si avrà:

$$\Psi = A\cdot e^{i(\frac{\vec{p}}{\hbar}\cdot\vec{x} - \frac{\sqrt{p^2c^2+m_0^2c^4}}{\hbar}t)} \tag{A2.17}$$

e, come al solito, sempre per sostituzione, si vede che tale Ψ è soluzione della seguente:

$$\boxed{(\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}) - \frac{m_0^2c^2}{\hbar^2}\Psi = 0} \tag{A2.18}$$

che altro non è che l'**Equazione di Klein-Gordon** e che è simile a quella di d'Alembert, ma ha un elemento in più.

Proviamo ad effettuare veramente tale sostituzione della (A2.17) nella (A2.18), per verificare che davvero vale tutto ciò. Si ha che $\nabla^2\Psi = (i)^2\frac{p^2}{\hbar^2}\Psi = -\frac{p^2}{\hbar^2}\Psi$ e

$-\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -\frac{1}{c^2}(-i)^2\frac{E^2}{\hbar^2}\Psi = \frac{1}{c^2\hbar^2}(p^2c^2 + m_0^2c^4)\Psi$ e dunque:

$-\frac{p^2}{\hbar^2}\Psi + \frac{1}{c^2\hbar^2}(p^2c^2 + m_0^2c^4)\Psi - \frac{m_0^2c^2}{\hbar^2}\Psi = 0$, ossia 0=0.

Poniamo ora $l = \frac{m_0c}{\hbar}$; tale l ha le dimensioni del vettore d'onda k. Con tale l, si ha che le (A2.17) ed (A2.18) si riscrivono così:

$$\boxed{\Psi = A \cdot e^{i(\vec{k}\cdot\vec{x} - \sqrt{(k^2+l^2)}ct)} = A \cdot e^{i(\vec{k}\cdot\vec{x} - \omega' t)}} \tag{A2.19}$$

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} - l^2\Psi = 0 \tag{A2.20}$$

con $\boxed{\omega' = \sqrt{(k^2+l^2)}c}$.

Riscriviamo ora l'Equazione di Klein-Gordon (A2.20) in questo modo:

$$-\frac{\partial^2\Psi}{\partial t^2} + c^2\nabla^2\Psi - l^2c^2\Psi = 0 \tag{A2.21}$$

e ricordando che $i^2 = -1$ e $(a-b)(a+b) = a^2 - b^2$, si ha che tale equazione può essere così riscritta:

$$\{[i\frac{\partial}{\partial t} - (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})][i\frac{\partial}{\partial t} + (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\}\Psi = 0\,, \tag{A2.22}$$

ossia anche:

$$[i\frac{\partial}{\partial t} - (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\Psi = 0$$

$$[i\frac{\partial}{\partial t} + (ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})]\Psi = 0 \tag{A2.23}$$

e la (A2.22) può essere così sviluppata:

$$[-\frac{\partial^2}{\partial t^2} + c^2(\vec{\alpha}\cdot\vec{\nabla})^2 + i\frac{m_0c^3}{\hbar}\beta(\vec{\alpha}\cdot\vec{\nabla}) + i\frac{m_0c^3}{\hbar}(\vec{\alpha}\cdot\vec{\nabla})\beta - \beta^2\frac{m_0^2c^4}{\hbar^2}]\Psi = 0 \tag{A2.24}$$

Quest'ultima equazione coincide con la (A2.21) se:

$\beta^2 = 1$, $\alpha\beta + \beta\alpha = 0$, $\alpha_i\alpha_j = 1^2$ se i=j e $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ se $i \neq j$

Le due ultime condizioni sugli alfa impongono che si ottenga proprio solo il ∇^2 e non termini misti in $\vec{\nabla}$. La (A2.23), che qui riscriviamo:

$$\boxed{(i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0} \tag{A2.25}$$

può essere considerata come l'**Equazione di Dirac**, che solitamente viene presentata nella seguente forma, in unità naturali ($\hbar = c = 1, \beta = 1$):

$$\boxed{(i\gamma^\mu\partial_\mu - m_0)]\Psi = 0}\,, \tag{A2.26}$$

dove $i\gamma^\mu\partial_\mu = i\gamma^\mu \partial/\partial x^\mu$, che contiene una sommatoria in convenzione di Einstein, fornisce, al variare di μ, la derivata sul tempo $\frac{\partial}{\partial t}$ e su x, y e z di $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$: $i\gamma^\mu\partial_\mu \to i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla}$.

Per ultimo, partendo dalla Equazione di Dirac semplice (A2.25), ossia dalla:

$(i\frac{\partial}{\partial t} + ic\alpha\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0$, possiamo giungere alla **Equazione Elettromagnetica di Dirac**:

$$\boxed{[(i\hbar\frac{\partial}{\partial t}-qV)-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}-q\vec{A})-\beta m_0c^2]\Psi=0} \qquad \text{(A2.27)}$$

semplicemente considerando che l'introduzione del fenomeno elettromagnetico comporta una aggiunta algebrica della quantità energetica (qV) all'operatore energia:

$$E\to i\hbar\frac{\partial}{\partial t} \qquad \to (i\hbar\frac{\partial}{\partial t}-qV)$$

ed una aggiunta algebrica della quantità di moto ($q\vec{A}$) all'operatore $\vec{p}$ appunto:

$$\vec{p}\to\frac{\hbar}{i}\vec{\nabla} \qquad \to(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}).$$

Tutti i dettagli in Appendice 6.

APPENDICE 3: LE ANTIPARTICELLE

Consideriamo l'Equazione Elettromagnetica di Dirac (A2.27):

$$[(i\hbar\frac{\partial}{\partial t}-qV)-c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}-q\vec{A})-\beta m_0c^2]\Psi=0 \qquad \text{(A3.1)}$$

(trattata in modo esaustivo al link sopra citato), unitamente all'Hamiltoniana di Dirac che è: $H_D=qV+c\vec{\alpha}(\vec{p}-q\vec{A})+\beta m_0c^2$, in quanto la (A3.1) possiamo così riscriverla: $[(H_D-qV)-c\vec{\alpha}\cdot(\vec{p}-q\vec{A})-\beta m_0c^2]\Psi=0$.

Bene, prendiamo tale Equazione Elettromagnetica di Dirac (A3.1) e riscriviamola nel seguente modo:

$$(\hbar\frac{\partial}{\partial t}+\hbar c\vec{\alpha}\cdot\vec{\nabla}+i\beta m_0c^2)\Psi=iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi\ . \qquad \text{(A3.2)}$$

E, con riferimento all'Appendice 2, ribadiamo le condizioni cui devono sottostare i vari α e β: $\beta^2=1$, $\alpha_i\beta+\beta\alpha_i=0$, $\alpha_i\alpha_i=1^2$, $\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}$ (i,j=1,2,3) e per lavorare solo con coefficienti in α , denominiamo: $\beta=\alpha_4$, da cui, in modo più compendioso, le condizioni diventano: $\alpha_i\alpha_j+\alpha_j\alpha_i=2\delta_{ij}$ (i,j=1,2,3,4) . Ora, magari non esistono dei numeri che soddisfano tali condizioni, ma sicuramente esistono delle matrici che lo fanno, tipo:

$$\sigma_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \qquad \text{(A3.3)}$$

$$\rho_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad \rho_2=\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix},\quad \rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix} \qquad \text{(A3.4)}$$

con: $\boxed{\alpha_1=\rho_1\sigma_1\ ,\ \ \alpha_2=\rho_3\ ,\ \ \alpha_3=\rho_1\sigma_3\ ,\ \ \alpha_4=\rho_1\sigma_2\ ;}$ basta infatti eseguire i vari prodotti matriciali,

come ad esempio:

$$\alpha_1=\rho_1\sigma_1=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$$ e verificare: →

$$\to\ \alpha_1\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\to|\alpha_1\alpha_1|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1$$, poi:

$$\alpha_2=\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\quad,\alpha_2\alpha_2=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ e:

$$|\alpha_2\alpha_2|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1$$, poi: $2=\alpha_1\alpha_1+\alpha_1\alpha_1=2\delta_{11}=2=\alpha_2\alpha_2+\alpha_2\alpha_2=2\delta_{22}=2$.

Poi: $$\alpha_3=\rho_1\sigma_3=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix},$$

$$\alpha_3\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ e: $$|\alpha_3\alpha_3|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

poi: $2=\alpha_3\alpha_3+\alpha_3\alpha_3=2\delta_{33}=2$, poi:

$$\alpha_4=\beta=\rho_1\sigma_2=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix},$$

$$\alpha_4\alpha_4=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$$ e: $$|\alpha_4\alpha_4|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

poi: $\alpha_4\alpha_4+\alpha_4\alpha_4=2\delta_{44}=2$. Poi: $\alpha_i\beta+\beta\alpha_i=(\alpha_i\alpha_4+\alpha_4\alpha_i)=0$, infatti:

$$\alpha_1\alpha_4+\alpha_4\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=$$

$$=\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}+\begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ ;$$

$$\alpha_2\alpha_4+\alpha_4\alpha_2=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ .$$

$$\alpha_3\alpha_4+\alpha_4\alpha_3=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}=$$

$$=\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}+\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\rightarrow 0\ .$$

Notiamo ora che solo α_4 è una matrice immaginaria, mentre α_1 , α_2 ed α_3 sono reali; bene, considerato ciò, se prendiamo la complessa coniugata della (A3.2), otteniamo la seguente: (cambiare le i in –i e le Ψ in Ψ*)

$$(\hbar\frac{\partial}{\partial t}+\hbar c\vec{\alpha}\cdot\vec{\nabla}+i\beta m_0c^2)\Psi^*=-iq(c\vec{\alpha}\cdot\vec{A}-V)\Psi^* \qquad \text{(A3.5)}$$

e vediamo che nell'ultimo termine a primo membro il +i è rimasto tale, poiché è vero che la i che lo accompagna diventerebbe –i , ma è anche vero che vi è la β che vale $\beta=\alpha_4$ ed abbiamo appena detto che la α_4 è una matrice immaginaria che vede tutte le sue i interne cambiate in –i (e viceversa), da cui non si ha nessun effetto finale di modifica appunto nell'ultimo termine a primo membro.
Ebbene, la (A3.5) ha la stessa forma della (A3.2), ma con la carica elettrica opposta (q→-q), **da cui l'ipotesi contemporanea del positrone (antimateria).**

APPENDICE 4: COULOMB DA HEISENBERG

Ricordiamo il Principio di Indeterminazione di Heisenberg:

$$\boxed{\begin{array}{l}\Delta p \ \cdot \Delta x \ \geq \hbar/2 \\ \Delta E \ \cdot \Delta t \geq \hbar/2\end{array}}$$

Considerando che esso sicuramente regola la comparsa (pre annichilazione) di una coppia particella (+) / antiparticella (-), tipo positrone/elettrone, segue che quando le due particelle compaiono, si separano (in un tempo Δt) alla velocità della luce c e fino a raggiungere la distanza reciproca Δx=r=cΔt e sappiamo altresì che in fisica la forza F vale: F=Δp/Δt, da cui:

$$\Delta p \ \cdot \Delta x \ \approx \hbar/2 = h/4\pi \approx F\Delta t \cdot r = F\frac{r}{c} \cdot r = F\frac{r^2}{c} \ \rightarrow$$

$$\boxed{\rightarrow F = (\frac{hc}{4\pi})\frac{1}{r^2} \propto \frac{1}{r^2}}$$

ossia proprio la stessa dipendenza di F da $1/r^2$, caratteristica della Legge di Coulomb (*):

$$\boxed{F = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r^2} \propto \frac{1}{r^2}} \quad .$$

(*): le due costanti non coincidono, ossia: $\frac{e^2}{4\pi\varepsilon_0} \neq \frac{hc}{4\pi}$, ma bensì: $\frac{e^2}{4\pi\varepsilon_0} = (2\alpha)\frac{hc}{4\pi} \cong (\frac{2}{137})\frac{hc}{4\pi}$,

dove α è la Costante di Struttura Fine; ciò (ossia il coefficiente aggiuntivo 2α) è causato dal fatto che il Principio di Indeterminazione di Heisenberg è calibrato sull'atomo e non su una coppia particella/antiparticella; infatti, mentre la particella e l'antiparticella, nel comparire, si separano alla distanza r percorrendo ciascuna la distanza r/2 dal centro di massa, nel caso di un atomo (di idrogeno H), il protone è così massivo rispetto all'elettrone (m_p=1836m_e) che sta praticamente fermo nel centro di massa, percorrendo così la distanza di 0r, mentre tutta la distanza r è percorsa dal leggero elettrone (1r), da cui la media (0r+1r)/2=r/2 ed il 2 è così giustificato.
Per giustificare invece il 137, basta osservare che, sempre data la differenza di massa tra protone ed elettrone, succede che la velocità di separazione non è c, ma bensì quella (unica, poichè il protone sta fermo nel centro di massa) che ha l'elettrone intorno al protone nell'atomo H, ossia notoriamente c/137.

APPENDICE 5: NEWTON DA HEISENBERG

In Appendice 4 si è dimostrato che Coulomb scaturisce da Heisenberg. Ora, qui dimostriamo che Newton scaturisce da Coulomb, da cui, infine, per la proprietà transitiva, anche Newton scaturirà da Heisenberg.

Dimostrazione:

UNA PROPOSTA DI UNIVERSO CHE EVIDENZIA COME, APPUNTO, NEWTON SCATURISCA DA COULOMB.

Questo è l'Universo in questione:

$$\boxed{\begin{aligned} M_{Univ} &= 1{,}59486 \cdot 10^{55} kg \\ R_{Univ} &= 1{,}17908 \cdot 10^{28} m \;. \\ T_{Univ} &= \frac{2\pi R_{Univ}}{c} = 2{,}47118 \cdot 10^{20} s \end{aligned}}$$

La massa è maggiore di quella visibile (come attualmente predicato) e la densità è quella osservata:

$$\rho = M_{Univ} / (\frac{4}{3}\pi \cdot R_{Univ}^3) = 2{,}32273 \cdot 10^{-30} kg / m^3$$

La composizione di tutte le piccole interazioni elettriche nell'Universo va a comporre l'Universo stesso; infatti, casualmente:

$$\boxed{\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_e} = G\frac{m_e M_{Univ}}{R_{Univ}}}$$

(UNIFICAZIONE GRAVITÀ ELETTROMAGNETISMO)

ossia Newton da Coulomb.

($r_e = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{m_e \cdot c^2} \cong 2{,}8179 \cdot 10^{-15} m$ è il raggio classico dell'elettrone). Inoltre, sempre casualmente:

$$m_e c^2 = G\frac{m_e M_{Univ}}{R_{Univ}}, \quad G\frac{M_{Univ}}{R_{Univ}^2} = G\frac{m_e}{r_e^2} = a_{Univ} = 7{,}62 \cdot 10^{-12} m/s^2 , \quad h = \frac{2m_e c^2}{T_{Univ}} \text{ (solo numericamente)}$$

$$R_{Univ} = \sqrt{N} r_e \quad \text{(essendo } N = M_{Univ} / m_e = 1{,}74 \cdot 10^{85}\text{)}, \quad T_{Univ}\frac{Gm_e^2}{hr_e} = \frac{1}{137} = \alpha \quad ,$$

$$T_{CMBR} = (\frac{h}{2m_e\sigma}\frac{M_{Univ}}{4\pi R_{Univ}^2})^{1/4} \cong 2{,}7K \;,$$

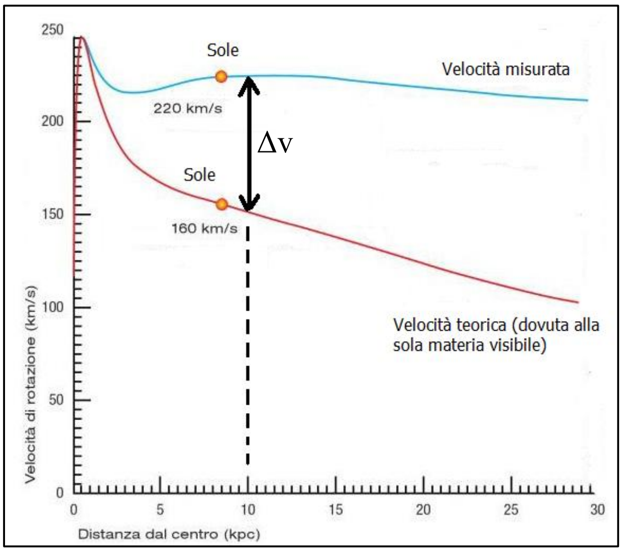
(dove $\sigma = 5{,}67 \cdot 10^{-8} W / m^2 K^4$ è la Costante di Stefan-Boltzmann). Inoltre:

$$T_{CMBR} = (\frac{3c^3}{80\pi\sigma}\frac{M_{Univ}}{R_{Univ}^3})^{1/4} = (\frac{72Gc^{11}h^3\varepsilon_0^4 m_e^6}{\pi^2 e^8 k^4})^{1/4} = 2{,}72846(02218319896)K \cong 2{,}72846K$$

-Essendo poi, per Heisenberg: $\Delta p \cdot \Delta x = \hbar / 2$

(con il segno di eguaglianza, per semplicità, e $\hbar / 2 = h / 4\pi = 0{,}527 \cdot 10^{-34} J \cdot s$), possiamo dire

che: $\Delta p \cdot \Delta x = m_e c \cdot \Delta x = m_e c \cdot [\frac{1}{(2\pi)^2} G\frac{M_{Univ}}{R_{Univ}^2}] = 0{,}527 \cdot 10^{-34}$ (numericamente)



-Nella nostra galassia (la Via Lattea) il Sole è ad una distanza di 8,5kpc dal centro e dovrebbe avere una velocità di rotazione di 160 km/s, se essa fosse dovuta solo alla materia barionica, cioè quella delle stelle e di tutta la materia visibile. Ma noi invece sappiamo che la velocità del Sole è 220 km/s. Dunque vi è una discrepanza Δv di 60 km/s: (Δv=220-160=60 km/s).(1kpc=1000pc ; 1pc=1 Parsec=$3,26\_a.l. = 3,08\cdot 10^{16} m$; 1 anno luce a.l.=$9,46\cdot 10^{15} m$) ($R_{Gal} = 8,5kpc = 27,71\cdot 10^{3}\_a.l. =$ $= 2,62\cdot 10^{20} m$ è la distanza del Sole dal centro della ViaLattea). Se il Sole fosse stato ad una distanza RGAL di 30 kpc, avrebbe avuto la stessa velocità di 220 km/s, ma la discrepanza Δv sarebbe stata più grande. In generale, sappiamo dalle curve di rotazione che:

$\Delta v = k\sqrt{R_{Gal}}$, dove k =costante. Ora, ci accorgiamo che, casualmente:

$$\boxed{k = \sqrt{2G\frac{m_e}{r_e^2}} = \sqrt{\frac{2GM_{Univ}}{R_{Univ}^2}}} \quad .$$

APPENDICE 6: SULLA EQUAZIONE ELETTROMAGNETICA DI DIRAC

$$[(i\hbar\frac{\partial}{\partial t} - qV) - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}) - \beta m_0 c^2]\Psi = 0$$

oppure:

$$[(E - qV) - c\vec{\alpha}\cdot(\vec{p} - q\vec{A}) - \beta m_0 c^2]\Psi = 0$$

Equazione Elettromagnetica di Dirac.

-<u>Premesse sulla Equazione di Dirac normale</u>:

$$[i\hbar\frac{\partial}{\partial t} - \vec{\alpha}\cdot c(\frac{\hbar}{i}\vec{\nabla}) - \beta m_0 c^2]\Psi = 0 \quad \textbf{oppure:} \quad [E - \vec{\alpha}\cdot c\vec{p} - \beta m_0 c^2]\Psi = 0$$

partiamo dalla Equazione di Dirac normale, per la quale si parte dalla Equazione di Klein-Gordon:

$$-\frac{\partial^2\Psi}{\partial t^2} + c^2\nabla^2\Psi - l^2c^2\Psi = 0 \quad (l = \frac{m_0 c}{\hbar}) \tag{A6.1}$$

e ricordando gli arcinoti operatori quantistici: $E \rightarrow i\hbar \frac{\partial}{\partial t}$ e $\vec{p} \rightarrow \frac{\hbar}{i} \vec{\nabla}$ e ricordando altresì che $i^2 = -1$ e: $(a-b)(a+b) = a^2 - b^2$, si ha che tale equazione può essere così riscritta:

$$\{[i\frac{\partial}{\partial t} - (ic\vec{\alpha} \cdot \vec{\nabla} - \beta \frac{m_0 c^2}{\hbar})][i\frac{\partial}{\partial t} + (ic\vec{\alpha} \cdot \vec{\nabla} - \beta \frac{m_0 c^2}{\hbar})]\}\Psi = 0, \tag{A6.2}$$

ossia anche così:

$$[i\frac{\partial}{\partial t} - (ic\vec{\alpha} \cdot \vec{\nabla} - \beta \frac{m_0 c^2}{\hbar})]\Psi = 0$$

$$[i\frac{\partial}{\partial t} + (ic\vec{\alpha} \cdot \vec{\nabla} - \beta \frac{m_0 c^2}{\hbar})]\Psi = 0 \tag{A6.3}$$

e la (A6.2) può essere così sviluppata:

$$[-\frac{\partial^2}{\partial t^2} + c^2(\vec{\alpha} \cdot \vec{\nabla})^2 + i\frac{m_0 c^3}{\hbar}\beta(\vec{\alpha} \cdot \vec{\nabla}) + i\frac{m_0 c^3}{\hbar}(\vec{\alpha} \cdot \vec{\nabla})\beta - \beta^2 \frac{m_0^2 c^4}{\hbar^2}]\Psi = 0 \tag{A6.4}$$

Quest'ultima equazione coincide con la (1) se:

$\beta^2 = 1$, $\alpha_i \beta + \beta \alpha_i = 0$, $\alpha_i \alpha_j = 1^2$ se i=j e $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$ se $i \neq j$

Le due ultime condizioni sugli alfa impongono che si ottenga proprio solo il ∇^2 e non termini misti in $\vec{\nabla}$. La (A6.3), che qui riscriviamo:

$$\boxed{(i\frac{\partial}{\partial t} + ic\vec{\alpha} \cdot \nabla - \beta \frac{m_0 c^2}{\hbar})\Psi = 0} \tag{A6.5}$$

può essere considerata come l'**Equazione di Dirac**, che solitamente viene presentata nella seguente forma, in unità naturali ($\hbar = c = 1, \beta = 1$):

$$\boxed{(i\gamma^\mu \partial_\mu - m_0)]\Psi = 0} \ , \tag{A6.6}$$

dove $i\gamma^\mu \partial_\mu = i\gamma^\mu \partial / \partial x^\mu$, che contiene una sommatoria in convenzione di Einstein, fornisce, al variare di μ, la derivata sul tempo $\frac{\partial}{\partial t}$ e su x, y e z di $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$: $i\gamma^\mu \partial_\mu \rightarrow i\frac{\partial}{\partial t} + ic\vec{\alpha} \cdot \vec{\nabla}$.

-<u>Premesse sull'elettromagnetismo</u>:

sappiamo che i campi elettrico e magnetico sono esprimibili in funzione dei potenziali elettrodinamici (vedere App. 8):

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V$$

e ricordiamo altresì la Forza di Lorentz: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. Bene, da queste si ottiene che:

$\vec{F} = q[\vec{E} + (\vec{v} \times \vec{B})] = q[-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V + (\vec{v} \times \vec{\nabla} \times \vec{A})]$; essendo ora che: $\vec{\nabla} V = \hat{i}\frac{\partial V}{\partial x} + \hat{j}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}$, segue che: $(\vec{\nabla} V)_x = \frac{\partial V}{\partial x}$; inoltre, dal momento che:

$$\vec{\nabla}\times\vec{A}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\hat{i}(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z})+\hat{j}(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})+\hat{k}(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})$$, segue che:

$$[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ v_x & v_y & v_z\\ (\vec{\nabla}\times\vec{A})_x & (\vec{\nabla}\times\vec{A})_y & (\vec{\nabla}\times\vec{A})_z\end{vmatrix}_x=v_y(\vec{\nabla}\times\vec{A})_z-v_z(\vec{\nabla}\times\vec{A})_y=$$

$$=v_y(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y})-v_z(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x})=v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}-v_y\frac{\partial A_x}{\partial y}-v_z\frac{\partial A_x}{\partial z} \qquad \text{(A6.7)}$$

inoltre, a livello di derivata temporale, si ha: $\frac{dA_x}{dt}=\frac{\partial A_x}{\partial t}+(\frac{\partial x}{\partial t}\frac{\partial A_x}{\partial x}+\frac{\partial y}{\partial t}\frac{\partial A_x}{\partial y}+\frac{\partial z}{\partial t}\frac{\partial A_x}{\partial z})=$ $=\frac{\partial A_x}{\partial t}+(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z})$, da cui: $-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}=-(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z})$ e allora, per la (A6.7), si ha: $[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x=\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})-(v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_x}{\partial z})=\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}$, ossia, infine: $[\vec{v}\times(\vec{\nabla}\times\vec{A})]_x=\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})-\frac{dA_x}{dt}+\frac{\partial A_x}{\partial t}$; in definitiva, per la componente x della forza di Lorentz, si ha:

$$(\vec{F})_x=F_x=q[\vec{E}+(\vec{v}\times\vec{B})]_x=q[(-\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}V)+(\vec{v}\times\vec{\nabla}\times\vec{A})]_x=q[-(\frac{\partial\vec{A}}{\partial t})_x-(\vec{\nabla}V)_x+(\vec{v}\times\vec{\nabla}\times\vec{A})_x]=$$

$$=q\{-\cancel{\frac{\partial A_x}{\partial t}}-\frac{\partial V}{\partial x}+[\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})-\frac{dA_x}{dt}+\cancel{\frac{\partial A_x}{\partial t}}]\}=q[-\frac{\partial V}{\partial x}+\frac{\partial}{\partial x}(\vec{v}\cdot\vec{A})-\frac{dA_x}{dt}]=q[-\frac{\partial}{\partial x}(V-\vec{v}\cdot\vec{A})-\frac{dA_x}{dt}]=$$

$$=q\{-\frac{\partial}{\partial x}(V-\vec{v}\cdot\vec{A})-\frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})]\}$$; infatti:

$$\frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})]=\frac{d}{dt}[\frac{\partial}{\partial v_x}(v_xA_x+v_yA_y+v_zA_z)]=\frac{d}{dt}[\frac{\partial}{\partial v_x}(v_xA_x)+\frac{\partial}{\partial v_x}(v_yA_y)+\frac{\partial}{\partial v_x}(v_zA_z)]=$$

$$=\frac{d}{dt}[A_x+0+0]=\frac{dA_x}{dt}$$. Perciò, in definitiva:

$$F_x=q\{-\frac{\partial}{\partial x}(V-\vec{v}\cdot\vec{A})-\frac{d}{dt}[\frac{\partial}{\partial v_x}(\vec{v}\cdot\vec{A})]\}. \qquad \text{(A6.8)}$$

Quest'ultima, ossia la (A6.8), equivale alla seguente: $F_x=-\frac{\partial U}{\partial x}+\frac{d}{dt}\frac{\partial U}{\partial v_x}$, con U che ha palesemente la valenza di un'energia: $U=qV-q(\vec{v}\cdot\vec{A})$; dunque, F_x dipende (ovviamente) anche da v ; inoltre, visto che in relatività abbiamo il quadrivettore potenziale $\underline{\Phi}=(A_x,A_y,A_z,\frac{V}{c})$ (vedere App. 9), segue che A si riferisce alla parte spaziale (S) del campo elettromagnetico, mentre V si riferisce a quella temporale (t); allora, anche per l'energia U possiamo scrivere: $U=qV-q(\vec{v}\cdot\vec{A})=U_t+U_S$.

Tornando ora alla (A6.5) e ricordando gli arcinoti operatori quantistici:

spaziale: $\vec{p}\rightarrow\frac{\hbar}{i}\vec{\nabla}$ e temporale: $E\rightarrow i\hbar\frac{\partial}{\partial t}$

(infatti, anche in meccanica quantistica l'energia E si correla col tempo: $\Delta E \cdot \Delta t \geq \hbar/2$, mentre l'impulso si correla con lo spazio: $\Delta p \cdot \Delta x \geq \hbar/2$)

ed essendo: $\boxed{U_t = qV}$ ed $U_S = -q(\vec{v}\cdot\vec{A})$ e, conseguentemente:

$$(U_S = -q(\vec{v}\cdot\vec{A}) \rightarrow p_{x\_S} = \int F_{x\_S}dt = -\int(\frac{d}{dt}\frac{\partial U}{\partial v_x})dt = -\int d(\frac{\partial U}{\partial v_x}) = -\frac{\partial U}{\partial v_x} \rightarrow$$

$\rightarrow \vec{p}_S = -\frac{\partial U}{\partial \vec{v}} = -\frac{\partial(-q\vec{v}\cdot\vec{A})}{\partial \vec{v}} = q\vec{A}$, ossia: $\boxed{\vec{p}_S = q\vec{A}}$), segue che abbiamo dei nuovi operatori quantistici: spaziale: $\vec{p} \rightarrow (\frac{\hbar}{i}\vec{\nabla} - q\vec{A})$ e temporale: $E \rightarrow (i\hbar\frac{\partial}{\partial t} - qV)$ e dunque dicevamo, per la (5) (-i=1/i):

$$(i\frac{\partial}{\partial t} + ic\vec{\alpha}\cdot\vec{\nabla} - \beta\frac{m_0c^2}{\hbar})\Psi = 0 = [i\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{1}{i}\vec{\nabla}) - \beta\frac{m_0c^2}{\hbar}]\Psi = \hbar\cdot 0 = \hbar[i\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{1}{i}\vec{\nabla}) - \beta\frac{m_0c^2}{\hbar}]\Psi =$$

$= [i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}) - \beta m_0c^2]\Psi = 0$ e con i nuovi operatori:

$$\boxed{[(i\hbar\frac{\partial}{\partial t} - qV) - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}) - \beta m_0c^2]\Psi = 0.} \qquad \text{(A6.9)}$$

e definiamo, tra l'altro, l'Hamiltoniana di Dirac: $H_D = qV + c\vec{\alpha}(\vec{p} - q\vec{A}) + \beta m_0c^2$, in quanto la (A6.9) possiamo così riscriverla: $[(H_D - qV) - c\vec{\alpha}\cdot(\vec{p} - q\vec{A}) - \beta m_0c^2]\Psi = 0$.

Dunque, mettendo l'una sopra l'altra la (A6.5) e la (A6.9), abbiamo:

$$\boxed{[i\hbar\frac{\partial}{\partial t} - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla}) - \beta m_0c^2]\Psi = [E - c\vec{\alpha}\cdot\vec{p} - \beta m_0c^2]\Psi = 0}$$

Eq. di Dirac normale

$$\boxed{[(i\hbar\frac{\partial}{\partial t} - qV) - c\vec{\alpha}\cdot(\frac{\hbar}{i}\vec{\nabla} - q\vec{A}) - \beta m_0c^2]\Psi = [(E - qV) - c\vec{\alpha}\cdot(\vec{p} - q\vec{A}) - \beta m_0c^2]\Psi = 0}$$

Eq. di Dirac EM.

-<u>Esistenza dello spin</u>:

ricordiamo ora, dalla meccanica quantistica, le equazioni agli autovalori dei momenti angolari: $L_z\Psi = l\Psi$, $L^2\Psi = \lambda\Psi$, $l = m\hbar$, $-l \leq m \leq l$, $\lambda = \hbar^2 l(l+1)$. Ora, detta $\Psi_\alpha(\vec{r})$ la funzione d'onda di una particella, allora la funzione d'onda della particella con spin è: $\Psi = \Psi_\alpha(\vec{r})\cdot\chi_s$.

Per particelle con spin 1/2 (s=+1/2 ed s=-1/2), si hanno due χ_s : $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$ e $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$; dunque, avendo due funzioni d'onda $|+\rangle$ e $|-\rangle$, che scritte in modo più compendioso sono: $|\pm\rangle$, allora la $L^2\Psi = \lambda\Psi = \hbar^2 l(l+1)\Psi$ riadattata dai momenti angolari d'orbitale L a quelli di spin S, fornisce:

$S^2\Psi = \lambda\Psi = \hbar^2 s(s+1)\Psi = \hbar^2\frac{1}{2}(\frac{1}{2}+1)\Psi = \frac{3}{4}\hbar^2\Psi$, da cui segue che: $S^2|\pm\rangle = \frac{3}{4}\hbar^2|\pm\rangle$ ed $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$; essendo ora che $|\pm\rangle$ rappresenta due funzioni d'onda, scriviamo il tutto in forma matriciale, con matrici 2x2 e ad ognuna delle due righe corrisponderà una funzione d'onda appunto:

$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $S^2 = \frac{3}{4}\hbar^2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ e si possono poi valutare anche le matrici S₊ ed S₋ : $S_\pm = S_x \pm iS_y$ da cui: $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ed $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; in modo più compendioso si può scrivere: $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$, dove: $\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; queste sono le **Matrici di Pauli** e godono delle seguenti proprietà, come si può verificare direttamente:

$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2\varepsilon_{ijk}\sigma_k$,

$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}$, ossia proprio come le matrici α di cui sopra, **da cui il primo indizio del fatto che queste ultime (le matrici α appunto) già preludono all'esistenza dello spin.**

Ora, teniamo conto di questa identità:

$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{N}) = [(\sigma_1\hat{1} + \sigma_2\hat{2} + \sigma_3\hat{3})(M_1\hat{1} + M_2\hat{2} + M_3\hat{3})][(\sigma_1\hat{1} + \sigma_2\hat{2} + \sigma_3\hat{3})(N_1\hat{1} + N_2\hat{2} + N_3\hat{3})] =$

$= (\sigma_1^2 M_1N_1 + \sigma_1\sigma_2 M_1N_2 + \sigma_2\sigma_1 M_2N_1) + (\sigma_2^2 M_2N_2 + \sigma_2\sigma_3 M_2N_3 + \sigma_3\sigma_2 M_3N_2) +$

$+ (\sigma_3^2 M_3N_3 + \sigma_1\sigma_3 M_1N_3 + \sigma_3\sigma_1 M_3N_1)$.

Ricordiamo ora che: $\sigma_1\sigma_2 = i\sigma_3$; infatti: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, nonché: $\sigma_2\sigma_1 = -i\sigma_3$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_3\sigma_2 = -i\sigma_1$, $\sigma_1\sigma_3 = i\sigma_2$, $\sigma_3\sigma_1 = -i\sigma_2$, mentre riguardo i quadrati:

$\sigma_1^2 = \sigma_1\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv 1 = \sigma_2^2 = \sigma_3^2$, allora:

$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{N}) =$

$= (\sigma_1^2 M_1N_1 + \sigma_1\sigma_2 M_1N_2 + \sigma_2\sigma_1 M_2N_1) + (\sigma_2^2 M_2N_2 + \sigma_2\sigma_3 M_2N_3 + \sigma_3\sigma_2 M_3N_2) +$

$+ (\sigma_3^2 M_3N_3 + \sigma_1\sigma_3 M_1N_3 + \sigma_3\sigma_1 M_3N_1) =$

$= (\sigma_1^2 M_1N_1 + i\sigma_3 M_1N_2 - i\sigma_3 M_2N_1) + (\sigma_2^2 M_2N_2 + i\sigma_1 M_2N_3 - i\sigma_1 M_3N_2) + (\sigma_3^2 M_3N_3 + i\sigma_2 M_1N_3 - i\sigma_2 M_3N_1) =$

$= (\sigma_1^2 M_1N_1 + \sigma_2^2 M_2N_2 + \sigma_3^2 M_3N_3) + i\sigma_3(M_1N_2 - M_2N_1) + i\sigma_1(M_2N_3 - M_3N_2) + i\sigma_2(M_1N_3 - M_3N_1) =$

$= (M_1N_1 + M_2N_2 + M_3N_3) + i\sigma_3(M_1N_2 - M_2N_1) + i\sigma_1(M_2N_3 - M_3N_2) + i\sigma_2(M_1N_3 - M_3N_1) =$

$= (M\cdot N) + i[\sigma\cdot(M\times N)] = (\sigma\cdot M)(\sigma\cdot N)$ e per $\vec{M} = \vec{N} \rightarrow$

$$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{M}) = (\vec{M}\cdot\vec{M}) + i[\vec{\sigma}\cdot(\vec{M}\times\vec{M})] \qquad \text{(A6.10)}$$

Ora, abbiamo già ricordato più sopra che:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} ,$$

e considerando ora un campo orientato verso z, si ha:

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} = 0 + 0 + B_z\hat{k} = B_z\hat{k} , \qquad \text{(A6.11)}$$

ossia: $B_x = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) = 0$, $B_y = (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) = 0$ e $B_z = (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$ (A6.12)

ed è quindi plausibile porre:

$A_y = \frac{1}{2}B_x$, $A_x = -\frac{1}{2}B_y$ ed $A_z = 0$ (A6.13)

infatti, le (A6.13), se inserite nelle (A6.12), confermano la (A6.11). Possiamo allora dire che:

$$\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} \qquad \text{(A6.14)}$$

infatti: $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & B_z \\ x & y & z \end{vmatrix} = \frac{1}{2}[\hat{i}(0-yB_z)+\hat{j}(xB_z-0)+\hat{k}(0-0)] =$

$= \hat{i}\frac{(-yB_z)}{2} + \hat{j}\frac{(xB_z)}{2} + \hat{k}(0) = \hat{i}A_x + \hat{j}A_y + \hat{k}A_z$ e, come controverifica:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{(-yB_z)}{2} & \frac{(xB_z)}{2} & 0 \end{vmatrix} = \hat{i}(0-0)+\hat{j}(0-0)+\hat{k}[\frac{\partial}{\partial x}(\frac{xB_z}{2}) - \frac{\partial}{\partial y}(\frac{-yB_z}{2})] = B_z$$

Ponendo ora nella (A6.10) $\vec{M} = (\vec{p}-q\vec{A})$, dal momento che: $\vec{M}\times\vec{M} = (\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A}) =$

$= \vec{p}\times\vec{p} - q(\vec{p}\times\vec{A}) - q(\vec{A}\times\vec{p}) + q^2(\vec{A}\times\vec{A}) = 0 - q(\vec{p}\times\vec{A}) - q(\vec{A}\times\vec{p}) + q^2(0) = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})]$

dunque $\vec{M}\times\vec{M} = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})]$ ed inserendo in questa la nota $\vec{p}\to -i\hbar\vec{\nabla}$, si ottiene:

$\vec{M}\times\vec{M} = -q[(\vec{p}\times\vec{A})+(\vec{A}\times\vec{p})] = -q\{(-i\hbar\vec{\nabla}\times\vec{A})+[\vec{A}\times(-i\hbar\vec{\nabla})]\}$ ed inserendo ora in quest'ultima la (A6.14) e la $\vec{B} = \vec{\nabla}\times\vec{A}$, si ottiene: $\vec{M}\times\vec{M} = i\hbar q(\vec{B}+\frac{1}{2}\vec{B}\times\vec{r}\times\vec{\nabla}) = i\hbar q(\vec{B}+\frac{1}{2}\vec{\nabla}\times\vec{B}\times\vec{r})$, ma:

$$\vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & B_z \end{vmatrix} = \hat{i}\frac{\partial B_z}{\partial y} - \hat{j}\frac{\partial B_z}{\partial x} + \hat{k}0 = 0+0+0 = 0$$, da cui:

$\vec{M}\times\vec{M} = i\hbar q(\vec{B}+\frac{1}{2}\vec{B}\times\vec{r}\times\vec{\nabla}) = i\hbar q(\vec{B}+0) = i\hbar q\vec{B}$; quindi, con riferimento alla (A6.10), si ha:

$$(\vec{\sigma}\cdot\vec{M})(\vec{\sigma}\cdot\vec{M}) = (\vec{\sigma}\cdot\vec{M})^2 = [\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 = (\vec{M}\cdot\vec{M}) + i[\sigma\cdot(\vec{M}\times\vec{M})] = (\vec{p}-q\vec{A})^2 + i(\vec{\sigma}\cdot i\hbar q\vec{B}) =$$
$$= (\vec{p}-q\vec{A})^2 - \hbar q(\vec{\sigma}\cdot\vec{B}) \qquad \text{(A6.15)}$$

Definimmo l'Hamiltoniana di Dirac: $H_D = qV + c\vec{\alpha}(\vec{p}-q\vec{A}) + \beta m_0c^2$, dunque, ricordando che, in relatività: $V=cA_0$ e ricordando altresì le matrici α_i: ($\alpha_1 = \rho_1\sigma_1$, $\alpha_2 = \rho_3$, $\alpha_3 = \rho_1\sigma_3$, $\alpha_4 = \rho_1\sigma_2$), nonché il fatto che se scambiamo α_2 con α_4 , tutte le proprietà delle matrici α_i continuano a sussistere (vedere Appendice 7) , abbiamo ora:

($\alpha_1 = \rho_1\sigma_1$, $\alpha_2 = \rho_1\sigma_2$, $\alpha_3 = \rho_1\sigma_3$, $\alpha_4 = \rho_3 = \beta$) e dunque:

$H = qV + c\vec{\alpha}(\vec{p}-q\vec{A}) + \beta m_0c^2 = +qcA_0 + [c\alpha_1(\vec{p}-q\vec{A})_1 + c\alpha_2(\vec{p}-q\vec{A})_2 + c\alpha_3(\vec{p}-q\vec{A})_3] + \rho_3 m_0c^2$,

in quanto, a causa dello scambio, β diventa, per ora $\alpha_4(=\rho_3)$. Continuando:

$$H = +qcA_0 + [c\alpha_1(\vec{p}-q\vec{A})_1 + c\alpha_2(\vec{p}-q\vec{A})_2 + c\alpha_3(\vec{p}-q\vec{A})_3] + \rho_3 m_0c^2 =$$
$$= +qcA_0 + [\rho_1\sigma_1 c(\vec{p}-q\vec{A})_1 + \rho_1\sigma_2 c(\vec{p}-q\vec{A})_2 + \rho_1\sigma_3 c(\vec{p}-q\vec{A})_3] + \rho_3 m_0c^2 = +qcA_0 + \rho_1 c[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] + \rho_3 m_0c^2$$

da cui: $(H-qcA_0)^2 = \{\rho_1 c[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] + \rho_3 m_0c^2\}^2 =$

$$= \rho_1^2 c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + \rho_3^2 m_0^2 c^4 + \rho_1\rho_3 m_0 c^3[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] + \rho_3\rho_1 m_0 c^3[\vec{\sigma}\cdot(\vec{p}-q\vec{A})] = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4$$

dunque: $(H - qcA_0)^2 = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4$, (A6.16)

poiché: $\rho_1^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv 1$,

$\rho_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv 1$ e:

$\rho_1\rho_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \to |\rho_1\rho_3| = \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = 1$.

$\rho_3\rho_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -\rho_1\rho_3$.

ed infine, utilizzando la (19) nella (20), otteniamo:

$(H - qcA_0)^2 = c^2[\vec{\sigma}\cdot(\vec{p}-q\vec{A})]^2 + m_0^2 c^4 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$ (A6.17)

Adesso, per un elettrone che si muove con un impulso piccolo, si ha: $H = H_1 + m_0 c^2$ ($H_1 << m_0 c^2$); dunque, con tale H nella (A6.17) otteniamo:

$(H - qcA_0)^2 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B}) = (H_1 + m_0 c^2 - qcA_0)^2 = c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$

ed approssimando il quadrato $(H_1 + m_0 c^2 - qcA_0)^2$ con $(H_1 + m_0 c^2 - qcA_0)$, visto l'H piccolo per ipotesi (e, intorno al vertice, la parabola la approssimiamo con la retta), si ottiene:

((($H_1 + m_0 c^2 - qcA_0 \to c^2(\vec{p}-q\vec{A})^2 + m_0^2 c^4 - \hbar c^2 q(\vec{\sigma}\cdot\vec{B})$))) , dove, però, a secondo membro bisogna effettuare un riequilibrio dimensionale, conseguente all'approssimazione fatta, ossia, riguardo il termine $c^2(\vec{p}-q\vec{A})^2$, esso ha appunto la valenza di un impulso quadro x c^2, allora per il fatto che in fisica classica (ossia non relativistica) si ha che:

$E_k = \frac{1}{2}m_0 v^2 = \frac{1}{2m_0}m_0^2 v^2 = \frac{1}{2m_0}p^2$, allora: $c^2(\vec{p}-q\vec{A})^2 \to \frac{1}{2m_0}(\vec{p}-q\vec{A})^2$ (cioè $[J^2] \to [J]$); riguardo poi il termine $-\hbar c^2 q(\vec{\sigma}\cdot\vec{B})$ si deve effettuare l'adattamento ($-[J^2] \to -[J]$) ossia si moltiplicherà anche qui per $\frac{1}{2m_0}$ e si dividerà poi per c^2: $\to -\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$; per ultimo, riguardo il termine rimasto: $m_0^2 c^4$, stessa cosa, ossia: ($[J^2] \to \sqrt{\ } \to [J]$) $\to m_0 c^2$;

in definitiva: $H_1 + m_0 c^2 - qcA_0 \approx \frac{1}{2m_0}(\vec{p}-q\vec{A})^2 + m_0 c^2 - \frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B}) \to$

$\to H_1 - qcA_0 \approx \frac{1}{2m_0}(\vec{p}-q\vec{A})^2 - \frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$; (A6.18) la quantità $-\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})$ è un'energia potenziale addizionale:

$$\left([\frac{\hbar q}{2m_0}(\vec{\sigma}\cdot\vec{B})] = [\frac{JsC}{kg}(1,T)] = [\frac{JsC}{kg}(\frac{Wb}{m^2})] = [\frac{JsC}{kg}(\frac{Vs}{m^2})] = [J(CV)\frac{1}{kg\frac{m^2}{s^2}}] = [J(J)\frac{1}{J}] = [J] \right)$$

ed essa ci dice che l'elettrone ha un momento magnetico intrinseco $\frac{\hbar q}{2m_0}\sigma \equiv \frac{\hbar q}{2m_0}$. Poniamo ora un atomo in un campo magnetico di bassa entità, di vettore induzione B, con tale B lungo z, come quello più sopra esaminato; bene, si avrà, per l'elettromagnetismo: $H = \frac{1}{2m_0}(\vec{p}-q\vec{A})^2 - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$,

ma considerando le componenti di $\vec{A}$ che abbiamo precedentemente trovato in occasione della dimostrazione della equazione (A6.14), si ha:

$$H = \frac{1}{2m_0}\{(\hat{i}p_x + \hat{j}p_y + \hat{k}p_z) - q[\hat{i}(-\frac{yB_z}{2}) + \hat{j}(\frac{xB_z}{2}) + \hat{k}0)]\}^2 - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r} =$$

$$= \frac{1}{2m_0}[(\hat{i}(p_x + \frac{qyB_z}{2}) + \hat{j}(p_y - \frac{qxB_z}{2}) + \hat{k}p_z]^2 - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r} =$$

$$= \frac{1}{2m_0}[(p_x + \frac{qy}{2}B_z)^2 + (p_y - \frac{qx}{2}B_z)^2 + p_z^2] - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r} = \frac{(p_x^2 + p_y^2 + p_z^2)}{2m_0} +$$

$$+ \frac{q}{2m_0}B_z(yp_x - xp_y) + \frac{1}{2m_0}(\frac{qx}{2})^2 B_z^2 + \frac{1}{2m_0}(\frac{qy}{2})^2 B_z^2 - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$$ ed essendo, per ipotesi, B di bassa entità, i termini in B_z^2 verranno trascurati rispetto a quelli in B_z , da cui:

$$H = \frac{p^2}{2m_0} + \frac{q}{2m_0}B_z(yp_x - xp_y) - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$$, ma dal momento che, in un atomo:

$$\vec{L} = \vec{r}\times\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = \hat{i}(yp_z - zp_y) + \hat{j}(zp_x - xp_z) + \hat{k}(xp_y - yp_x) \rightarrow L_z = -(yp_x - xp_y)$$, allora:

$H = \frac{p^2}{2m_0} - \frac{q}{2m_0}B_z L_z - \frac{1}{4\pi\varepsilon_0}\frac{Zq^2}{r}$; denominiamo ora: $H_P = -\frac{q}{2m_0}B_z L_z = -\frac{q}{2m_0}\vec{B}\cdot\vec{L}$, essendo che B ha solo componente B_z ; adesso, se oltre all'L della rotazione intorno al nucleo esiste anche un L'=S intrinseco (spin), si otterrà un H_p' tale che: (e ricordiamo che dimostrammo che $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$)

$$H'_P = -\frac{q}{2m_0}\vec{B}\cdot 2\vec{S} = -\frac{q\hbar}{2m_0}\vec{B}\cdot\vec{\sigma} \; ; \qquad (A6.19)$$

il 2 che moltiplica S è un fatto di evidenza sperimentale che si esprime dicendo che il rapporto giromagnetico di spin g_s risulta doppio di quello orbitale g_l :

$$g_l = \frac{\vec{\mu}_l}{\vec{L}} = +\frac{q}{2m_0} = \frac{g_s}{2} .$$

La (A6.19) ci ha portati ad ottenere lo stesso termine ($-\frac{\hbar q}{2m_0}(\sigma\cdot\vec{B})$) che è comparso nella (A6.18), che però ricavammo come conseguenza della Equazione Elettromagnetica di Dirac, potendo così definitivamente dire che quest'ultima, ossia appunto **l'Equazione Elettromagnetica di Dirac, dà la predizione dell'esistenza dello spin.**

APPENDICE 7: SULLA SUSSISTENZA DELLE PROPRIETÀ DELLE MATRICI $\vec{\alpha}$ ANCHE CON α₂ ED α₄ SCAMBIATE.

Ribadiamo le condizioni cui devono sottostare i vari α e β: $\beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i = 0$, $\alpha_i\alpha_i = 1^2$, $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ (i,j=1,2,3) e per lavorare solo con coefficienti in α , denominiamo: $\beta = \alpha_2$, da cui, in modo più compendioso, le condizioni diventano: $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$ (i,j=1,2,3,4) .

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} , \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} , \quad \rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} , \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

con: $\boxed{\alpha_1 = \rho_1\sigma_1 \ , \ \alpha_2 = \rho_1\sigma_2 \ , \ \alpha_3 = \rho_1\sigma_3 \ , \ \alpha_4 = \rho_3 \ ;}$ basta infatti eseguire i vari prodotti matriciali,

come ad esempio:

$$\alpha_1 = \rho_1\sigma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \text{ e verificare: } \rightarrow$$

$$\rightarrow \alpha_1\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow |\alpha_1\alpha_1| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \text{, poi:}$$

$$\alpha_2 = \beta = \rho_1\sigma_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} ,$$

$$\alpha_2\alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ e: } |\alpha_2\alpha_2| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 ,$$

poi: $2 = \alpha_1\alpha_1 + \alpha_1\alpha_1 = 2\delta_{11} = 2 = \alpha_2\alpha_2 + \alpha_2\alpha_2 = 2\delta_{22} = 2$, poi:

$$\alpha_3 = \rho_1\sigma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ,$$

$$\alpha_3\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\text{ e: }|\alpha_3\alpha_3|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1,$$

poi: $2=\alpha_3\alpha_3+\alpha_3\alpha_3=2\delta_{33}=2$, poi:

$$\alpha_4=\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},\rightarrow\alpha_4\alpha_4=\rho_3\rho_3=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\text{ e:}$$

$$|\alpha_4\alpha_4|=\begin{vmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}=1\quad\text{e:}\quad 2=\alpha_4\alpha_4+\alpha_4\alpha_4=2\delta_{44}=2\text{, poi:}$$

$\alpha_i\beta+\beta\alpha_i=(\alpha_i\alpha_2+\alpha_2\alpha_i)=0$, infatti:

$$\alpha_1\alpha_2+\alpha_2\alpha_1=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}i&0&0&0\\0&-i&0&0\\0&0&i&0\\0&0&0&-i\end{pmatrix}+\begin{pmatrix}-i&0&0&0\\0&i&0&0\\0&0&-i&0\\0&0&0&i\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\text{ ;}$$

$$\alpha_3\alpha_2+\alpha_2\alpha_3=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}=$$

$$=\begin{pmatrix}0&-i&0&0\\-i&0&0&0\\0&0&0&-i\\0&0&-i&0\end{pmatrix}+\begin{pmatrix}0&i&0&0\\i&0&0&0\\0&0&0&i\\0&0&i&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\text{ ,}$$

$$\alpha_4\alpha_2+\alpha_2\alpha_4=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}=$$

$$=\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&i\\0&0&-i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\rightarrow 0\text{ .}$$

APPENDICE 8 – POTENZIALI ELETTRODINAMICI

Definiamo un vettore $\vec{A}$ tale che:

$$\vec{\nabla} \times \vec{A} = \vec{B} \tag{A8.1}$$

Allora, per la Terza Equazione di Maxwell si ha: $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{A}) = -\vec{\nabla} \times (\frac{\partial \vec{A}}{\partial t})$, da cui:

$\vec{\nabla} \times (\vec{E} + \frac{\partial \vec{A}}{\partial t}) = 0$, dunque $\vec{E} + \frac{\partial \vec{A}}{\partial t}$ è irrotazionale e può essere scritto come gradiente di una funzione –V: $-\vec{\nabla} V = \vec{E} + \frac{\partial \vec{A}}{\partial t}$, ossia:

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} V \tag{A8.2}$$

Facendo ora sistema tra la (A8.1), la (A8.2), la prima e la quarta eq. di Maxwell e l'identità vettoriale:

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} + \vec{\nabla}(\vec{\nabla} \cdot \vec{A})$, si ottiene:

$$\nabla^2 V + \frac{\partial}{\partial t}(\vec{\nabla} \cdot \vec{A}) = -\rho/\varepsilon \tag{A8.3}$$

$$\nabla^2 \vec{A} - \varepsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}(\vec{\nabla} \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t}) = -\mu \vec{J} \tag{A8.4}$$

Queste due sono le equazioni elettrodinamiche non disaccoppiate. Disaccoppiamole ora tramite una trasformazione di gauge:

$$\vec{A}' = \vec{A} + \vec{\nabla}\varphi \qquad V' = V - \frac{\partial \varphi}{\partial t}$$

Infatti: $\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla}\varphi) = \vec{\nabla} \times \vec{A} + 0 = \vec{B}$ (rotore di gradiente = 0).

$$-\vec{\nabla} V' - \frac{\partial \vec{A}'}{\partial t} = -\vec{\nabla} V + \vec{\nabla}(\frac{\partial \varphi}{\partial t}) - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}(\vec{\nabla}\varphi) = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} = \vec{E}$$

cioè le espressioni per A e V valgono identiche anche per A' e V'.

La trasformazione di gauge vogliamo sceglierla in modo tale che venga soddisfatta la Condizione di Lorenz:

$$\vec{\nabla} \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t} = 0 . \tag{A8.5}$$

E infatti vediamo che questa vale anche per A' e V', ossia: $\vec{\nabla} \cdot \vec{A}' + \varepsilon\mu \frac{\partial V'}{\partial t} = 0$; infatti, esplicitando:

$0 = \vec{\nabla} \cdot \vec{A}' + \varepsilon\mu \frac{\partial V'}{\partial t} = \vec{\nabla} \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t} + (\nabla^2 \varphi - \varepsilon\mu \frac{\partial^2 \varphi}{\partial t^2})$, per cui φ deve soddisfare l'equazione:

$(\nabla^2 \varphi - \varepsilon\mu \frac{\partial^2 \varphi}{\partial t^2}) = -(\vec{\nabla} \cdot \vec{A} + \varepsilon\mu \frac{\partial V}{\partial t})$, ma, per ipotesi, il secondo membro è nullo, dunque anche il primo lo è e la Condizione di Lorenz è dunque sempre soddisfatta.

Sostituendo allora la (A8.5) nella (A8.4) e poi tenendo conto, nella (A8.3), che per la C. di Lorenz $\vec{\nabla} \cdot \vec{A} = -\varepsilon\mu \frac{\partial V}{\partial t}$, si ottengono le Equazioni Elettrodinamiche Disaccoppiate:

$$\nabla^2 \vec{A} - \varepsilon\mu \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J}$$

$$\nabla^2 V - \varepsilon\mu \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

Introducendo ora l'operatore d'Alembertiano $\square = \nabla^2(*) - \varepsilon\mu\frac{\partial^2(*)}{\partial t^2}$, si può scrivere:

$$\boxed{\begin{aligned}&\square\vec{A} = -\mu\vec{J} \qquad (A8.6)\\ &\square V = -\frac{\rho}{\varepsilon} \qquad (A8.7)\end{aligned}}$$

Si può verificare, per sostituzione diretta, che le soluzioni di queste due sono le seguenti:

$$\boxed{\begin{aligned}&\vec{A}(\vec{r},t) = \frac{\mu}{4\pi}\int_\tau \frac{\vec{J}(\vec{r}',t-\Delta r/V)}{\Delta r}d\tau' \qquad (A8.8)\\ &V(\vec{r},t) = \frac{1}{4\pi\varepsilon}\int_\tau \frac{\rho(\vec{r}',t-\Delta r/V)}{\Delta r}d\tau' \qquad (A8.9)\end{aligned}}$$

Infatti, $\nabla^2 V = \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\rho}{r})d\tau'] = \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \rho(\nabla\frac{1}{r})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \frac{1}{r}(\nabla\rho)d\tau'] =$

$= \nabla[\frac{1}{4\pi\varepsilon}\int_\tau \rho(-\frac{1}{r^2})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \frac{1}{r}(\nabla\rho)d\tau'] = -\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\rho}{r^2})d\tau' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' =$

$= -\frac{1}{4\pi\varepsilon}\int_{S'} \frac{\rho}{r^2}dS' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{1}{4\pi\varepsilon}\int_{S'} \rho d\Omega' + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{\rho}{\varepsilon} + \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'$

Allora: $\nabla^2 V - \frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau' = -\frac{\rho}{\varepsilon}$; a questo punto, se dimostriamo che

$\boxed{\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'} = \boxed{\varepsilon\mu\frac{\partial^2 V}{\partial t^2}}$, allora la (A8.9) è dimostrata. Verifichiamo:

$$\boxed{\frac{1}{4\pi\varepsilon}\int_\tau \nabla(\frac{\nabla\rho}{r})d\tau'} = \frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{\nabla c\rho}{r})d\tau' = \frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{\nabla J}{r})d\tau' = \boxed{-\frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau'}$$

$\boxed{\varepsilon\mu\frac{\partial^2 V}{\partial t^2}} = \varepsilon\mu\frac{\partial}{\partial t}(\frac{\partial V}{\partial t}) = \varepsilon\mu\frac{\partial}{\partial t}(-\frac{1}{\varepsilon\mu}\nabla A) =$ (dove abbiamo usato la (A8.5)); proseguendo:

$$= -\frac{\partial}{\partial t}(\nabla A) = -\frac{\partial}{\partial t}[\frac{\mu}{4\pi}\int_\tau \nabla(\frac{J}{r})d\tau'] = -\frac{\mu}{4\pi}\int_\tau \nabla(\frac{1}{r}\frac{\partial J}{\partial t})d\tau' = \qquad (A8.10)$$

$= -\frac{\mu c}{4\pi}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau' = \boxed{-\frac{1}{4\pi\varepsilon c}\int_\tau \nabla(\frac{1}{r}\frac{\partial\rho}{\partial t})d\tau'}$ cvd.

E visto che nel passaggio (A8.10) è stata cruciale la (A8.8), noi, in questa sede, ci limitiamo a ritenere ciò una prova della (A8.8) stessa, peraltro già simmetrica di suo, rispetto alla (A8.9).

APPENDICE 9: ELETTROMAGNETISMO E RELATIVITÀ.

Sappiamo che con una carica q abbiamo una forza su una carica test q_t (in P) pari a $\vec{F}_{orce} = \frac{1}{4\pi\varepsilon_0}\frac{qq_t}{(\Delta r)^2}\hat{\Delta r}$ ed un campo elettrico $\vec{E}(\vec{r}) = \frac{\vec{F}_{orce}}{q_t} = \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2}\hat{\Delta r}$ ed essendo che, per definizione di potenziale: $dV = -\vec{E}\cdot d\vec{l}$, si ha un potenziale: $V = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Delta r}$.

($V = \int dV = -\int \vec{E}\cdot d\vec{l} = -\int \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2}\hat{\Delta r}\cdot d\vec{l} = -\int \frac{1}{4\pi\varepsilon_0}\frac{q}{(\Delta r)^2} d(\Delta r) = -\frac{q}{4\pi\varepsilon_0}\int \frac{d(\Delta r)}{(\Delta r)^2} = \frac{q}{4\pi\varepsilon_0}\frac{1}{\Delta r}$).

Sappiamo poi dalla fisica che la densità di corrente vale $\vec{j} = \rho\vec{v}$.

Viene allora spontaneo definire il quadrivettore densità di corrente nel seguente modo:

$\underline{j} = (\vec{j}, \rho c) = (j_x, j_y, j_z, \rho c) = (\gamma\rho_0\vec{v}, \gamma\rho_0 c)$; si noti la similitudine con il quadrivettore momento-energia: $\underline{p} = (\vec{p}, mc) = (\gamma m_0\vec{v}, \gamma m_0 c)$. E si ha poi: $|\underline{j}|^2 = |\vec{j}|^2 - j_4^2 = -\rho_0^2 c^2$.

Sappiamo poi dall'elettromagnetismo che i campi elettrico e magnetico (vettore induzione B) possono essere espressi in funzione dei potenziali elettrodinamici V ed A:

$\vec{E} = -\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}$ e $\vec{B} = \vec{\nabla}\times\vec{A}$ e conosciamo anche la Condizione di Lorenz:

$\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = \vec{\nabla}\cdot\vec{A} + \varepsilon\mu\frac{\partial V}{\partial t} = 0$, nonché l'Eq. di Continuità: $\vec{\nabla}\cdot\vec{j} + \frac{\partial\rho}{\partial t} = 0$.

Sempre dall'elettromagnetismo sappiamo inoltre che:

$\Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon} = \Box V$ e $\Delta\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu\vec{j} = \Box\vec{A}$.

Viene ora spontaneo definire il Quadrivettore Potenziale o Quadripotenziale $\underline{\Phi}$:

$\underline{\Phi} = (A_x, A_y, A_z, \frac{V}{c}) = (A_x, A_y, A_z, A_t) = (A_x, A_y, A_z, A_0)$; infatti, così facendo, le precedenti si possono così riassumere: $\Box\Phi_k = -\mu j_k$ e definendo la quadri divergenza:

$div_4 = \vec{\nabla}_4 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial x} + \frac{\partial}{\partial(ct)} = (\vec{\nabla} + \frac{\partial}{\partial(ct)})$ si ha anche, banalmente: $\vec{\nabla}_4\underline{j} = 0$ per l'Equazione di Continuità e $\vec{\nabla}_4\underline{\Phi} = 0$ per la Condizione di Lorenz.